ANNIHILATOR IDEALS AND PRIMITIVE ELEMENTS IN COMPLEX BORDISM

BY

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Abstract

We determine the prime ideals in the complex bordism ring $\pi_*(MU)$ which can be the annihilator ideals of elements in the complex bordism of finite complexes. Such a prime ideal must be finitely generated and invariant under all stable MU-cohomology operations. We go on to determine the invariant prime ideals in $\pi_*(MU)$.

1. Introduction

Let MU denote the unitary Thom spectrum. Thus MU is a ring spectrum, and $\pi_*(MU)$ is isomorphic to the cobordism ring of manifolds whose stable normal bundle has a complex structure. Algebraically

$$\pi_*(MU) = Z[x_1, x_2, \cdots, x_n, \cdots]$$

where $x_n = [M^{2n}]$ has degree $2n$. (See [2] or [15].)

For each based CW-complex X the complex bordism groups $MU_n(X) = \pi_n(MU \wedge X)$ yield a graded left module $MU_*(X) = \oplus_{n \geq 0} MU_n(X)$ over $\pi_*(MU)$. For $\alpha \in MU_*(X)$ there is the annihilator ideal

$$A(\alpha) = \{\lambda \in \pi_*(MU); \lambda\alpha = 0\}.$$

Much of the study of the complex bordism of finite complexes carried out by P. E. Conner and L. Smith [4], [10], [11] depends on knowledge of annihilator ideals of spherical bordism classes.

In this paper we determine the prime ideals that can be annihilator ideals for a finite complex. Such a prime ideal must be finitely generated and invariant under all stable MU-cohomology operations. We go on to determine the invariant prime ideals in $\pi_*(MU)$. We refer to §2 for more precise information.

We remark that for each finite complex X, $MU_*(X)$ has only a finite number of prime annihilator ideals. However, it is not an easy matter to decide if a given annihilator ideal $A(\alpha)$ is a prime ideal, for $\alpha \in MU_*(X)$.

The results are collected in §2, as well as a number of easy arguments. In §3 we show that prime annihilator ideals are invariant; the proof uses the primary decomposition of ideals in a commutative Noetherian ring. Then in §4 we prove an algebraic generalization of the Hattori-Stong theorem [6, 14] which permits us to determine the invariant prime ideals in $\pi_*(MU)$.

Received February 11, 1971.

[1] Research supported in part by a National Science Foundation grant.

I would like to acknowledge the benefit of a preliminary version of J. Morava's paper [7]. In particular the determination of the invariant prime ideals in $\pi_*(MU)$ is implicitly carried out in [7].

2. Statement of results

We first define a family of ideals in $\pi_*(MU)$, as in [13]. All ideals and modules that we consider are understood to be graded.

Recall the Hurewicz homomorphism

$$(i^H)_* : \pi_*(MU) \to H_*(MU)$$

induced by the unit of the Eilenberg-MacLane spectrum H [2, §2]. Now it is possible to choose the polynomial generators x_n for $\pi_*(MU)$ so that p divides $(i^H)_*(x_n)$ if $i+1$ is a power of a prime p. We then put

$$I(p, h) = (p, x_{p-1}, \cdots, x_{p^{h-1}-1})$$

and $I(p) = (p, x_{p-1}, \cdots) = \bigcup_h I(p, h)$, with the convention that $I(p, 1) = (p)$. In fact $I(p)$ is the kernel of

$$\pi_*(MU) \xrightarrow{(i^H)_*} H_*(MU) \xrightarrow{\rho} H_*(MU; Z_p),$$

i.e. the ideal of all cobordism classes $[M^{2n}]$ of stably complex manifolds all of whose Chern numbers are divisible by p. We now state the main theorem.

THEOREM 2.1. *If P is a prime ideal in $\pi_*(MU)$ and $P = A(\alpha)$ for a bordism class $\alpha \in MU_*(X)$, X a finite complex, then $P = 0$ or $P = I(p, h)$ for some prime p and integer $h = 1, 2, \cdots$.*

We may instead place the finiteness assumption on the ideal.

COROLLARY 2.2 *If P is a finitely generated prime ideal in $\pi_*(MU)$ and $P = A(\alpha)$ for a bordism class in $MU_*(X)$, X a CW-complex, then $P = 0$ or $P = I(p, h)$ for some prime p and integer $h = 1, 2, \cdots$.*

For it is easy to see that there is a finite subcomplex X' of X and $\alpha' \in MU_*(X')$ so that $P = A(\alpha')$. We remark that it would be desirable to have an effective criterion for a CW-complex X to have the property that all annihilator ideals of $MU_*(X)$ are finitely generated.

We now recall the pertinent facts about the stable operations and comodule structure on $MU_*(X)$. There are the classes

$$b_i = b_i^{MU} \in MU_{2i}(MU) \, (i = 1, 2, \cdots)$$

of [2, §4] and then

$$MU_*(MU) = \pi_*(MU)[b_1, b_2, \cdots].$$

We put

$$S_* = Z[b_1, b_2, \cdots]$$

and recall from [2, §11] that $MU_*(MU)$ can be made a Hopf algebra and then S_* is a sub Hopf algebra.

For each "exponent sequence" $E = (e_1, e_2, \cdots)$ in which $e_i \geqq 0$ and almost all e_i vanish we form the monomial $b^E = \prod b_i^{e_i}$ in S_* (put $b^0 = 1$). Let $\{s^E\}$ be the dual basis in the cobordism module $MU^*(MU)$, so that for each CW-complex X we receive operations

$$s^E : MU_*(X) \to MU_*(X)$$

which lower degrees by $2\sum ie_i$ (see [1, p. 72]). We now define

$$\psi_X : MU_*(X) \to MU_*(X) \otimes S_*$$

by putting

$$\psi_X(\alpha) = \sum s^E(\alpha) \otimes b^E. \tag{2.3}$$

In particular if $X = S^0$ we obtain a homomorphism

$$\psi : \pi_*(MU) \to \pi_*(MU) \otimes S_* = \pi_*(MU)[b_1, b_2, \cdots] \tag{2.4}$$

which can be identified with the Hurewicz homomorphism

$$\eta_R : \pi_*(MU) \to MU_*(MU)$$

in view of Proposition 11.2 of [2]. Then ψ_X is a homomorphism of left $\pi_*(MU)$ modules, where $\pi_*(MU)$ acts on $MU_*(X) \otimes S_*$ via ψ, and ψ_X makes $MU_*(X)$ a comodule over the Hopf algebra S_*.

Remark. It is easy to compare ψ_X with the comodule structure map

$$\psi'_X : MU_*(X) \to MU_*(MU) \otimes_{\pi_*(MU)} MU_*(X)$$

of [1, Lecture 3] which makes $MU_*(X)$ a comodule over $MU_*(MU)$. Let c denote the canonical antiautomorphism of $MU_*(MU)$. If $\psi_X(\alpha)$ is given by (2.3) then Proposition 2 on p. 72 of [1] shows that

$$\psi'_X(\alpha) = \sum c(b^E) \otimes s^E(\alpha).$$

DEFINITION 2.5. An ideal $I \subset \pi_*(MU)$ is *invariant* if $\psi(I) \subset I \otimes S^*$. Equivalently, $s^E(I) \subset I$ for all exponent sequences.

The main theorem now breaks into two parts

THEOREM 2.6. *If P is a prime annihilator ideal on a finite complex, then P is also invariant.*

The proof is given in §3.

THEOREM 2.7 *If P is an invariant prime ideal in $\pi_*(MU)$, then P is one of the ideals 0, $I(p, h)$ for $h = 1, 2, \cdots$ or $I(p)$, where p is a prime.*

From now on, by a *comodule* we mean a pair (M, ψ_M) where M is a left $\pi_*(MU)$-module and $\psi_M : M \to M \otimes S_*$ is a left $\pi_*(MU)$ homomorphism making M a comodule over S_* (where $\pi_*(MU)$ acts on $M \otimes S_*$ via ψ). We often drop ψ_M from the notation. We call a comodule *coherent* if it is a coherent $\pi_*(MU)$-module. If X is a CW-complex then $MU_*(X)$ is a comodule in this sense, and if X is a finite complex then $MU_*(X)$ is a coherent comodule [4, §1]. If M is a coherent comodule and $\alpha \in M$ then the annihilator ideal $A(\alpha) \subset \pi_*(MU)$ is finitely generated [4, §5]. We use (2.3) to define operations s^E on each comodule.

Definition 2.8. An element α of a comodule M is *primitive* if $\psi_M(\alpha) = \alpha \otimes 1$. Equivalently, $S^E(\alpha) = 0$ if $E \neq 0$. We write $Pr\{M\}$ for the primitive elements in M. The utility of this concept is shown by the following elementary results.

Lemma 2.9. *If α is a primitive element of a comodule M then $A(\alpha)$ is an invariant ideal.*

Proof. If $\psi_M(\alpha) = \alpha \otimes 1$ and $\lambda\alpha = 0$, suppose that

$$\psi(\lambda) = \sum s^E(\lambda) \otimes b^E.$$

Then $0 = \psi_M(\lambda\alpha) = \psi(\lambda)\psi_M(\alpha) = \sum s^E(\lambda)\cdot\alpha \otimes b^E$ and so $s^E(\lambda)\cdot\alpha = 0$ for all E, hence $s^E(\lambda) \in A(\alpha)$ for all E. Thus $A(\alpha)$ is invariant.

Lemma 2.10. *If M is a non-negatively graded comodule, then each element in M of lowest degree is primitive. Thus $M \neq 0$ if and only if $Pr\{M\} \neq 0$.*

This is clear since for $E \neq 0$, s^E lowers degrees in M.

We now turn to invariant prime ideals in $\pi_*(MU)$. Notice that the Hurewicz homomorphism $(i^H)_* : \pi_*(MU) \to H_*(MU)$ can be identified with the composition

$$\pi_*(MU) \xrightarrow{\psi} \pi_*(MU) \otimes S_* \xrightarrow{\varepsilon \otimes 1} S_*$$

where $\varepsilon : \pi_*(MU) \to Z$ is the augmentation. Thus an element $\lambda \in \pi_{2n}(MU)$ lies in $I(p)$ if and only if $s^E(\lambda) \in \pi_0(MU) \cong Z$ is divisible by p whenever $E = (e_1, e_2, \cdots)$ satisfies $\sum ie_i = n$. It now follows easily that $I(p)$ is invariant. This implies that

$$s^E x_{p^h-1} \in I(p, h) \quad \text{if } E \neq 0$$

and then induction shows that each $I(p, h)$ is an invariant ideal. At the same time we observe that $\pi_*(MU)/I(p, h)$ is a coherent comodule and that in this quotient x_{p^h-1} is a primitive element. Then all the powers of x_{p^h-1} are also primitive.

Proposition 2.11. *The only primitive elements in $\pi_*(MU)/I(p, h)$ are the integral multiples of the powers of x_{p^h-1}, i.e.,*

$$Pr\{\pi_*(MU)/I(p, h)\} = Z_p[X_{p^h-1}].$$

We defer the proof of this result to §4, but it is now an easy matter to deduce Theorem 2.7 from Proposition 2.11. Thus let P be an invariant prime ideal in $\pi_*(MU)$. If $P \neq 0$ then the fact that

$$(i^H)_* : \pi_*(MU) \to H_*(MU)$$

is monic [2, §8; 15] easily implies that $P \cap \pi_0(MU) = (p)$ for some prime p, and then

$$(p) \subset P \subset I(p).$$

So either $P = I(p)$ or for some h we have

$$I(p, h) \subset P, \ x_{p^h-1} \notin P.$$

But then $P/I(p, h) \subset \pi_*(MU)/I(p, h)$ has no primitive elements by Proposition 2.11 and the fact that $P/I(p, h)$ is a prime ideal, hence $P = I(p, h)$ by Lemma 2.10. This proves Theorem 2.7.

Example. We are almost in a position to determine the annihilator of the spherical bordism class $\sigma_n \in MU_n(K(Z_p, n))$, studied in [13]. It is shown there that

$$I(p, n) \subset A(\sigma_n), \ x_{p^n-1} \notin A(\sigma_n).$$

Since σ_n is spherical it is primitive and $A(\sigma_n)$ is invariant by Lemma 2.9. If the argument of [13] can be extended to show that no power of x_{p^n-1} lies in $A(\sigma_n)$, then Proposition 2.11 implies that $A(\sigma_n) = I(p, n)$.

3. Prime annihilator ideals are invariant

In this section we prove a slight generalization of Theorem 2.6. Recall that for a finite complex $MU_*(X)$ is a coherent comodule in the sense of §2.

PROPOSITION 3.1. *If M is a coherent comodule and P is a prime annihilator ideal of an element in M, then P is invariant.*

We now develop the commutative algebra upon which the proof rests. For each comodule M, let $\mathrm{Ass}(M)$ denote the prime annihilators of elements of M, the associated prime ideals of M.

LEMMA 3.2. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of comodules then*

$$\mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

For example see [3, §1, n^{0}1, Prop. 3].

The following result allows us to assume that $M = \pi_*(MU)/I$ where I is an invariant finitely generated ideal in $\pi_*(MU)$. This is not an essential step, but we prefer to employ the primary decomposition for ideals rather than modules because it is more familiar and because our main interest is in ideals in $\pi_*(MU)$. Call graded modules *stably* isomorphic if they become isomorphic after shifting degrees.

LEMMA. 3.3. *If M is a coherent comodule then there is a sequence*

$$0 = M^0 \subset M^1 \subset \cdots \subset M^n = M$$

of coherent comodules so that for $1 \leqq i \leqq n$ M^i/M^{i-1} is stably isomorphic to $\pi_*(MU)/I_i$ where I_i is an invariant finitely generated ideal.

Proof. Since M is a finitely generated $\pi_*(MU)$-module, $Z \otimes_{\pi_*(MU)} M$ is a finitely generated abelian group. If $M_r \neq 0$ but $M_s = 0$ for $s < r$, then each element of M_r is primitive by Lemma 2.10. Choose a minimal set of generators in M_r and let α be one of the generators. By Lemma 2.9, $A(\alpha)$ is an invariant ideal, and it is finitely generated since M is coherent. Take $M^1 = \pi_*(MU) \cdot \alpha$, so that M^1 is a coherent comodule and M^1 is stably isomorphic to $\pi_*(MU)/A(\alpha)$. Moreover M/M^1 is a coherent comodule and $Z \otimes_{\pi_*(MU)} M/M^1$ requires one less generator than $Z \otimes_{\pi_*(MU)} M$, so we can continue by induction.

It remains to show that if I is a finitely generated invariant ideal in $\pi_*(MU)$ then $\mathrm{Ass}(\pi_*(MU)/I)$ consists of invariant ideals. Now suppose that

$$I = Q_1 \cap \cdots \cap Q_r$$

is a "normal" primary decomposition of I ([8, p. 104]; the Q_i are primary ideals whose radicals $\sqrt{Q_i} = P_i$ are distinct prime ideals, and no Q_i can be deleted). We also suppose that the prime ideals $P_i = \sqrt{Q_i}$ are finitely generated. In this case Theorem 19 on p. 110 of [8] asserts that

$$\{P_1, \cdots, P_r\} = \mathrm{Ass}(\pi_*(MU)/I).$$

Thus our problem is to prove the following result.

PROPOSITION 3.4. *Let I be a finitely generated invariant ideal in $\pi_*(MU)$. Then I has a normal primary decomposition*

$$I = Q_1 \cap \cdots \cap Q_r$$

for which each $P_i = \sqrt{Q_i}$ is an invariant and finitely generated prime ideal.

First we show that I has a normal primary decomposition with the Q_i and P_i finitely generated, and then we modify it to a decomposition

$$I = Q_1^* \cap \cdots \cap Q_r^*$$

with the Q_i^* and $P_i^* = \sqrt{Q_i^*}$ invariant. Then the uniqueness of the associated prime ideals (Theorem 13 on p. 105 of [8]) implies that each P_i is invariant as well as finitely generated, and this will prove Proposition 3.4 and with it also Proposition 3.1.

We establish the existence of a primary decomposition as follows. Recall that $\pi_*(MU) = Z[x_1, \cdots, x_n, \cdots]$. I is finitely generated, so for some n the Noetherian ring $Z[x_1, \cdots, x_n]$ contains a set of generators of I. Then $J = I \cap A[z_1, \cdots, x_n]$ has a normal primary decomposition in $Z[x_1, \cdots, x_n]$. By extending scalars from $Z[x_1, \cdots, x_n]$ to $\pi_*(MU)$ we obtain a normal primary decomposition

$$I = Q_1 \cap \cdots \cap Q_r$$

with the Q_i and $P_i = \sqrt{Q_i}$ all finitely generated, in view of Proposition 9 on p. 265 of [8].

To convert this to a primary decomposition with all ideals invariant, we associate to each ideal A in $\pi_*(MU)$ the ideal

$$A^* = \psi^{-1}(A[b_1, b_2, \cdots])$$

where $\psi : \pi_*(MU) \to \pi_*(MU)[b_1, b_2, \cdots]$. Thus $\lambda \in A^*$ if and only if $s^E(\lambda) \in A$ for all E. This construction has the following properties, all easily verified.

(1) A^* is an ideal, $A^* \subset A$.
(2) A is invariant $\Leftrightarrow A = A^*$.
(3) A^* is an invariant ideal.
(4) $(A \cap B)^* = A^* \cap B^*$.
(5) $(\sqrt{A})^* = \sqrt{(A^*)}$.
(6) P prime $\Rightarrow p^*$ prime.
(7) Q primary $\Rightarrow Q^*$ primary.

We return to the proof of the proposition, and notice that our construction provides a primary decomposition

$$I = Q_1^* \cap \cdots \cap Q_r^*$$

with associated prime ideals P_i^*. This new decomposition must also be normal, since otherwise we could refine it to a normal decomposition of length $<r$. Hence $\{P_1^*, \cdots, P_r^*\} = \{P_1, \cdots, P_r\}$ by the uniqueness theorem cited above, so the P_i are invariant as well as finitely generated (in fact $P_i^* = P_i$). This proves both Propositions of this section, and also Theorem 2.6.

Remark. We are not claiming that each $P \in \mathrm{Ass}(M)$ is the annihilator of a primitive element, although this might be the case and this is how one most easily obtains invariant ideals. The argument does contain the finiteness of $\mathrm{Ass}(M)$ for each coherent comodule M.

4. Algebraic Hattori-Stong theorems

The aim of this section is to prove Proposition 2.11, which locates the primitive elements in $\pi_*(MU)/I(p, h)$. This is a non-trivial matter. For example if $h = 1$ then $I(p, h) = (p)$ and we may take $x_{p-1} = [CP^{p-1}]$. So we are claiming that the only primitive elements in $\pi_*(MU)/(p)$ are integral multiples of the powers of $[CP^{p-1}]$. This case was proved by L. Smith; see the proof of Proposition 1.1 of [12]. The proof depends on the Hattori-Stong theorem, so in general we will need an algebraic replacement for the Hattori-Stong theorem.

We recall the Hattori-Stong theorem in a convenient form. Once again we shall use the homomorphism

$$\psi : \pi_*(MU) \to \pi_*(MU) \otimes S_* .$$

From the Todd genus $Td : \pi_*(MU) \to Z$ we manufacture a homomorphism

$$\tilde{T}d : \pi_*(MU) \to Z[t], \quad \deg t = 2,$$

by putting $\tilde{T}d([M^{2n}]) = (Td[M^{2n}])t^n$. Then we follow ψ by $\tilde{T}d \otimes 1$ to obtain a homomorphism

$$B : \pi_*(MU) \to Z[t][b_1, b_2, \cdots]$$

which records K-theory characteristic numbers.

HATTORI-STONG THEOREM [6, 14]. *B is a split monomorphism. Equivalently, for each prime p, B induces a monomorphism*

$$B_1 : \pi_*(MU)/(p) \to Z_p[t][b_1, b_2, \cdots].$$

For completeness we now present the argument in [12] to show that $Pr\{\pi_*(MU)/(p)\} = Z_p[x_{p-1}]$. Let $y \in \pi_{2n}(MU)/(p)$ be a primitive element. Then y^{p-1} and x_{p-1}^n are both primitive of degree $2n(p-1)$. Then

$$y^{p-1} - Td(y^{p-1})x_{p-1}^n$$

has Todd genus zero, since $x_{p-1} = [CP^{p-1}]$ has Todd genus one. Hence B_1 sends this element to zero, and then the Hattori-Stong theorem implies that

$$y^{p-1} = Td(y^{p-1})x_{p-1}^n$$

in $\pi_*(MU)/(p)$. If $y \neq 0$ then $Td(y^{p-1}) \not\equiv 0 \bmod p$. Since $\pi_*(MU)/(p)$ is a unique factorization domain, y must be a multiple of some power of the prime element x_{p-1}. This completes the proof.

Here is the appropriate generalization of the Hattori-Stong theorem:

PROPOSITION 4.1. *For each integer $h \geqq 1$ there is a homomorphism*

$$T : \pi_*(MU) \to Z_p$$

which vanishes on $I(p, h)$, which takes the value 1 on the generator x_{p^h-1} and which yields a monomorphism

$$B_h : \frac{\pi_*(MU)}{I(p,h)} \xrightarrow{\psi} \frac{\pi_*(MU)}{I(p,h)} \otimes S_* \xrightarrow{\tilde{T} \otimes 1} Z_p[t] \otimes S_*$$

where $\tilde{T} : \pi_*(MU) \to Z_p[t]$ is given by $\tilde{T}([M^{2n}]) = T(M^{2n}])t^n$, $\deg t = 2$.

Clearly we must first produce a homomorphism $T : \pi_*(MU) \to Z_p$ with good properties. Then we shall show that the derived homomorphism B_h is monic by what amounts to the algebraic part of Stong's argument in [14]. It is clear that Proposition 2.11 follows from Proposition 4.1 by the argument given above in the case $h = 1$.

In order to construct $T : \pi_*(MU) \to Z_p$ and establish its properties we shall use formal group laws. We refer to [2] and [5] for details and to [7] for an exposition of an extremely suggestive point of view.

By a group law μ over a commutative ring R we mean a power series

$$\mu(x_1, x_2) \in R[[x_1, x_2]]$$

satisfying

(1) $\mu(x, 0) = \mu(0, x) = x$
(2) $\mu(x_1, \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), x_3)$
(3) $\mu(x_2, x_1) = \mu(x_1, x_2)$.

The main example is the formal group law μ^{MU} over $\pi_*(MU)$. There is a canonical generator $x^{MU} \in MU^2(CP^\infty)$. For a complex line bundle $L \to X$ classified by a map $f : X \to CP^\infty$ put

$$c_1(L) = f^*(x^{MU}) \in MU^2(X).$$

The there is a universal formula

$$c_1(L_1 \otimes L_2) = \mu^{MU}(c_1(L_1), c_1(L_2))$$

which defines μ^{MU}. We now have Quillen's theorem on the universal nature of μ^{MU}.

QUILLEN'S THEOREM [2, 9]. *If μ is a group law over R then there is a unique ring homomorphism $g : \pi_*(MU) \to R$ which carried μ^{MU} to μ.*

Next we recall some notation. If μ is a group law over R and $f, g \in R[[x]]$ are power series without constant terms, we define

$$(f +_\mu g)(x) = \mu(f(x), g(x)).$$

The operation $f +_\mu g$ is commutative and associative, and allows us to define a power series

$$[n]_\mu(x) = x +_\mu \cdots +_\mu x \quad (n \text{ times})$$

for each positive integer n.

THEOREM 4.2. *Let p be a prime and h a positive integer. Then there exists a group law μ over Z_p so that $[p]_\mu(x) = x^{p^h}$.*

We refer to pp. 72–76 of [5] for the proof of this result. We are not asserting that such a group law is unique, but can still use to advantage the simple form of the power series $[p]_\mu(x)$.

Let $T : \pi_*(MU) \to Z_p$ be the homomorphism carrying μ^{MU} to μ, and introduce the homomorphism $\tilde{T} : \pi_*(MU) \to Z_p[t]$ of graded rings by putting $\tilde{T}(\lambda) = T(\lambda)t^n$ if $\lambda \in \pi_{2n}(MU)$.

LEMMA 4.3 *$\tilde{T}$ carries μ^{MU} to the group law $\tilde{\mu}$ over $Z_p[t]$ defined by $\tilde{\mu}(x_1, x_2) = (1/t)\mu(tx_1, tx_2)$. In particular*

$$[p]_{\tilde{\mu}}(x) = t^{p^h-1}x^{p^h}.$$

We write $\mu^{MU}(x_1, x_2) = \sum a_{ij}^{MU} x_1^i x_2^j$ so that $\mu(x_1, x_2) = \sum T(a_{ij}^{MU}) x_1^i x_2^j$.

Then
$$(1/t)\mu(tx_1, tx_2) = \sum T(a_{ij}^{MU})t^{i+j-1}x_1^i x_2^j = \sum \tilde{T}(a_{ij}^{MU})x_1^i x_2^j$$
since $a_{ij}^{MU} \in \pi_{2(i+j-1)}(MU)$. The last assertion now follows easily.

Our next task is to locate the group law over $Z_p[t][b_1, b_2, \cdots]$ to which μ^{MU} is carried by
$$B_h = (\tilde{T} \otimes 1)\psi : \pi_*(MU) \to Z_p[t][b_1, b_2, \cdots].$$
For this we consider an arbitrary group law μ over R and let
$$g : \pi_*(MU) \to R$$
be the homomorphism that carries μ^{MU} to μ. Following [2, §7] we introduce the power series
$$\exp(y) = \textstyle\sum_{i\geqq 0} b_i y^{i+1} \quad (b_0 = 1)$$
over S_* and let $\log(x) = \sum_{i\geqq 0} m_i x^{i+1}$ denote the inverse series under composition. We now consider the homomorphism
$$\pi_*(MU) \xrightarrow{\psi} \pi_*(MU) \otimes S_* \xrightarrow{g \otimes 1} R \otimes S_*.$$

PROPOSITION 4.4. *This homomorphism carries μ^{MU} to the group law*
$$\exp(\mu(\log x_1, \log x_2))$$
over $R \otimes S_* = R[b_1, b_2, \cdots]$.

Proof. It suffices to prove this in the universal case clearly. Thus we take $\mu = \mu^{MU}$ and let g be the identity on $\pi_*(MU)$, and our aim is to show that
$$\psi : \pi_*(MU) \to \pi_*(MU) \otimes S_*$$
carries μ^{MU} to $\exp(\mu^{MU}(\log x_1, \log x_2))$. We recall from §2 that ψ is to be identified with the Hurewicz homomorphism $\eta_R : \pi_*(MU) \to MU_*(MU)$, and then the result follows immediately from Corollary 6.8 of [2].

We now write out the series $[p]_{\mu^{MU}}(x)$ as $\sum_{i\geqq 0} a_i x^{i+1}$ and recall that $a_0 = p$ and that in $\pi_*(MU)/(p)$ the coefficients a_{p^n-1} are acceptable polynomial generators for $n = 1, 2, \cdots$. In fact if we apply Proposition 4.4 with $g : \pi_*(MU) \to Z$ the augmentation then μ^{MU} is carried to $\exp(\log x_1 + \log x_2)$ over S_* and so $[p]_{\mu^{MU}}$ is carried to $\exp(p \log x)$. Recall that one writes $\Delta_i = (e_1, e_2, \cdots)$ where $e_j = \delta_{ij}$. For convenience let $q = p^n - 1$. Then it is easy to see that
$$S^{\Delta_q}(a_q) = p(p^q - 1)$$
and from this our assertion is immediate; we refer to [2, §§7–8] for further details.

It follows that the ideal $I(p, h)$ is generated by $p, a_{p-1}, \cdots, a_{p^{h-1}-1}$. From now on we shall choose the generators x_i of $\pi_*(MU)/I(p, h)$ so that $x_{p^n-1} = a_{p^n-1}$ for $n \geqq h$.

We next apply Proposition 4.4 with g the homomorphism

$$\tilde{T} : \pi_*(MU) \to Z_p[t]$$

of Lemma 4.3. It follows that the homomorphism

$$B_h : \pi_*(MU) \to Z_p[t][b_1, b_2, \cdots]$$

carries $[p]_{\mu MU}(x)$ to the power series

$$\exp\,(t^{p^h-1}(\log x)^{p^h}).$$

For convenience we put $\tau = t^{p^h-1}$. Since $Z_p[t][b_1, b_2, \cdots]$ has characteristic p we see that B_h carries $[p]_{\mu MU}(x)$ to

$$\sum_{i \geqq 0} b_i \tau^{i+1}(x^{p^h} + (m_1 x^2)^{p^h} + \cdots)^{i+1}.$$

From this we conclude by inspecting coefficients that B_h kills the ideal $I(p, h)$, $B_h(a_{p^h-1}) = \tau$ and that for $n > 0$,

$$B_h(a_{p^{h+n}-1}) \equiv \tau(m_{p^n-1})^{p^h} \mod \tau^2.$$

From [2, §7] we recall that the coefficients m_i of $\log(x)$ may be used as polynomial generators for S_* in place of the b_i.

We now choose the generators x_i for $\pi_*(MU)/I(p, h)$ so that $x_{p^n-1} = a_{p^n-1}$ for $n \geqq h$ and so that $B_h(x_i) \equiv m_i$ modulo decomposables if $i + 1$ is not a power of p. In order to show that

$$B_h : \pi_*(MU)/I(p, h) \to Z_p[t] \otimes S_*$$

is monic it is only necessary to show that the elements $B_h(x_i)$ are algebraically independent over Z_p. We do this by putting a partial ordering on monomials in $Z_p[t, m_1, m_2, \cdots]$ of the same total degree. If

$$t^\varepsilon m^E = t^\varepsilon m_1^{e_1} m_2^{e_2} \cdots \quad \text{and} \quad t^{\varepsilon'} m^{E'} = t^{\varepsilon'} m_1^{e_1'} m_2^{e_1'}$$

are monomials of the same total degree, put $t^\varepsilon m^E > t^{\varepsilon'} m^{E'}$ if $\varepsilon < \varepsilon'$ or if $\varepsilon = \varepsilon'$ and $\sum e_i < \sum e_i'$. Thus the highest monomial appearing in $B_h(x_{p^{h+n}-1})$ is $t^{p^h-1}(m_{p^n-1})^{p^h}$ for $n \geqq 0$ and the highest monomial appearing in $B_h(x_i)$ is m_i if $i + 1$ is not a power of p. From this it is clear that the image under B_h of each monomial in the x_i contains a highest monomial with non-zero coefficient, and that these highest monomials in $Z_p[t, m_1, m_2, \cdots]$ are distinct. Thus the images under B_h of the monomials in the x_i are linearly independent over Z_p, which proves that

$$B_h : \pi_*(MU)/I(p, h) \to Z_p[t] \otimes S_*$$

is a monomorphism. This completes the proof of Proposition 4.1.

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