ON THE ALGEBRAIC STRUCTURE OF THE K-THEORY OF $\frac{G_2}{SU(3)}$ AND $\frac{F_4}{Spin(9)}$

BY

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This paper is an extension of the results of [8] to the exceptional Lie groups G_2 *and* F_4. In [8] we discussed the following situation. Suppose G is a compact connected Lie group and H is a subgroup of maximal rank. We let $R(G)$ and $R(H)$ denote the *complex representation rings* of G and H respectively [1], [6]. We can think of $R(G)$ as a subring of $R(H)$ [6] making $R(H)$ an $R(G)$ module.

An extension of the Weyl character formula yields a *duality homomorphism*,

$$F : R(H) \to \mathrm{Hom}_{R(G)}(R(H), R(G)),$$

and this was shown in [8] to be an isomorphism for a large number of cases involving the classical groups.

Among the corollaries of this theorem is a new proof of the conjecture by Atiyah-Hirzebruch [2] that $\alpha : R(H) \to K(G/H)$ is onto. We are also able to derive an explicit free basis for generating $R(H)$ over $R(G)$. This in turn yields an explicit basis for the free abelian group $K(G/H)$ [8, §9].

For those more familiar with equivariant K-theory we know that $R(H) \cong K_G(G/H)$, $R(G) \cong K_G(\text{point})$ [7]. The theorem can then be thought of as a Poincare duality result for this cohomology theory.

1. Let G be a compact connected Lie group and H a subgroup of maximal rank. That is H contains a maximal torus, T, of the group G. We can form the complex representation ring of G, denoted $R(G)$ [1], [6]. As a group $R(G)$ is the free abelian group on the set of isomorphism classes of irreducible complex representations of G, with the ring structure induced by the tensor product of representations. Restriction of representations makes $R(G)$ in a natural way a subring of $R(H)$, and $R(H)$ a subring of $R(T)$ [6]. We also think of each ring as a module over its subrings.

If $T \cong S^1 \times \cdots \times S^1$ (n times) then $R(T) \cong Z[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}]$ the polynomial ring over the integers in n indeterminates and their inverses [1]. To each group G, H is associated a group of automorphisms of T (hence of $R(T)$) called the Weyl group, denoted $W(G)$, $W(H)$ respectively. A major theorem in the representation theory of Lie groups asserts that $R(G)$ is the fixed subring of $R(T)$ under the action of $W(G)$ [1].

It is well known [1] that each element of the Weyl group can be given a sign, $(-1)^\sigma = \pm 1$, for $\sigma \in W(G)$. An alternating operator, A, can then be defined for all $x \in R(T)$ by $A(x) = \sum_{\sigma \in W(G)} (-1)^\sigma \sigma(x)$ [1], [8].

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The irreducible representations of T are known as *weights*. To each group G, H is associated a subset of weights called the *roots* of the group. With some choice of "orientation" a subset of roots called the *positive roots* can be singled out [1]. If G is simply connected then the product of positive roots (a monomial) raised to the one half power is a well defined weight denoted $\beta(G)$ [1], [8]. Here we are translating the usual additive notation of [1] into multiplicative notation. $\beta(G)$ is usually referred to as "one half the sum of the positive roots" (see [8] for correspondence).

These notions can be extended to the case where $\pi_1(G)$ has no two torsion but we will only need the simply connected case for this paper (see [8]).

Using a generalization of the Weyl character formula discussed in [4] we can define an $R(G)$ module map $f : R(H) \to R(G)$ [8, Proposition 3]. If G and H are simply connected then $f(x)$ is defined as $A(\beta(H)\cdot x)/A(\beta(G))$ for $x \in R(H)$. For the case $H = T$ we let $\beta(T) = 1$ and we get the usual Weyl character formula.

f induces a *duality homomorphism* $F : R(H) \to \operatorname{Hom}_{R(G)}(R(H), R(G))$ by $F(x)(y) = f(x\cdot y)$, $x, y \in R(H)$. We can also associate to F a bilinear form

$$\bar{F} : R(H) \times R(H) \to R(G)$$

defined by $\bar{F}(x, y) = f(x\cdot y)$. In [8] we showed that F was an isomorphism for the case G a classical group and H a suitable subgroup of maximal rank. In this paper we will extend this result to the cases $G = G_2$, $H = SU(3)$ and $G = F_4$, $H = Spin(9)$. Let us first recall a number of lemmas and corollaries from [8].

LEMMA 1. *If x, a weight, is left fixed by any element of the Weyl group then $A(x) = 0$.*

Proof. See [1, 6.12].

COROLLARY 1. (i) *Suppose $S_n \leq W(G)$ where S_n denotes the symmetric group on $\{x_1, \cdots, x_n\}$. Then if $x = \prod_{i=1}^n x_i^{m_i} \in R(T)$ and $m_i = m_j$, $i \neq j$, then $A(x) = 0$.*

(ii) *Suppose $W(G)$ contains the group generated by S_n and the maps $x_i \leftrightarrow x_i^{-1}$, $i = 1, \cdots, n$. Then*

$$A(x) = 0, \qquad x = \prod_{i=1}^n x_i^{m_i} \quad \text{if} \quad |m_i| = |m_j|.$$

In either case we say x is symmetric in some (i, j).

LEMMA 2. *Suppose $\{a_i\}_{i=1}^N$, $\{b_j\}_{j=1}^N$ are 2 sets of elements from $R(H)$, where $N = |W(G)|/|W(H)|$, and suppose that the determinant of the matrix $((\bar{F}(a_i, b_j)))$ is a unit of $R(G)$. Then $R(H)$ is a free module over $R(G)$ of rank N, freely generated by either $\{a_i\}_{i=1}^N$ or $\{b_j\}_{j=1}^N$. Furthermore F is then an isomorphism,*

$$F : R(H) \to \operatorname{Hom}_{R(G)}(R(H), R(G)).$$

Proof. See [8], §2.

Remark. If the hypothesis of Lemma 2 are fulfilled we call $\bar{F}$ *strongly non-singular* (often *s.n.s.*). We will show that F is an isomorphism by showing that $\bar{F}$ is s.n.s. in each case.

LEMMA 3 (Inductive Lemma). *If both*

$$\bar{F} : R(T) \times R(T) \to R(H) \quad \textit{and} \quad \bar{F} : R(H) \times R(H) \to R(G)$$

are s.n.s. then so is $\bar{F} : R(T) \times R(T) \to R(G)$.

Proof. See [8, §3].

In [2] it was conjectured that $\alpha : R(H) \to K(G/H)$ is onto for suitable $H \leq G$ (see [2] for details). This conjecture was proved there for a number of cases including those discussed in this paper. The results here will yield another proof and will in fact give a specific set of generators for the free abelian group $K(G/H)$.

COROLLARY 2. *If* $\bar{F} : R(H) \times R(H) \to R(G)$ *is strongly non-singular then*

$$\alpha : R(H) \to K(G/H)$$

is onto. Furthermore $\{\alpha(a_i)\}_{i=1}^{N}(\{\alpha(b_j)\}_{j=1}^{N})$ *provides a basis for the free abelian group* $K(G/H)$.

Proof. See [8, §9].

2. Let G_2 be the simply connected compact Lie group representing the local structure G_2. G_2 contains $SU(3)$ as a subgroup of maximal rank [3]. Let T be a maximal torus for $SU(3)$ and G_2, $R(T) \cong Z[x_1^{\pm 1}, x_2^{\pm 1}]$. The positive roots of G_2 can be chosen to be [3]

$$\{x_1, x_2, x_1 \cdot x_2, x_1 \cdot x_2^{-1}, x_1^2 \cdot x_2, x_1 x_2^2\}.$$

The last three represent a choice of positive roots for the maximal subgroup $SU(3)$ [3]. It follows that $\beta(G_2) = x_1^3 x_2^2$ and $\beta(SU(3)) = x_1^2 x_2$.

It is well known [1] that to each root there corresponds an element of the Weyl group, usually referred to as "reflection in the plane perpendicular to the root" (see [1] where an explicit formula is given to determine the action of this element). For example, to the root x_1 of G_2 there corresponds $\varphi_1 \in W(G_2)$ sending $x_1 \leftrightarrow x_1^{-1}$ and leaving everything else fixed. Similarly for x_2.

The Weyl group of $SU(3)$ acts on $R(T)$ as the group of permutations on the set $\{x_1, x_2, x_3\}$, where $x_3 \equiv x_1^{-1} x_2^{-1}$ [1]. The index of $W(SU(3))$ in $W(G_2)$ is 2 [5], and φ_1 is a representative for the non-trivial left coset.

If we let

$$\rho_1 = x_1 + x_2 + x_1^{-1} x_2^{-1} \quad \text{and} \quad \rho_2 = x_1^{-1} + x_2^{-1} + x_1 \cdot x_2$$

then $R(SU(3)) \cong Z[\rho_1, \rho_2]$, the polynomial ring on 2 indeterminates [1].

THEOREM 1. *$F : R(SU(3)) \to \mathrm{Hom}_{R(G_2)}(R(SU(3)), R(G_2))$ is an isomorphism.*

Proof. If we let $\bar{F}$ be the induced bilinear form then we will show that $\bar{F}$ is s.n.s. Let $\{a_i\} = \{1, \rho_2\}$ and $\{b_j\} = \{\rho_2, 1\}$. The result will follow if

$$\bar{F}(a_i, b_j) \begin{array}{l} = +1, \quad i = j. \\ = \quad 0, \quad i < j. \end{array}$$

This in turn will follow if

(1) $A(\beta(SU(3))) = 0$ and

(2) $A(\rho_2 \beta(SU(3)) = A(\beta(G_2))$.

$\beta(SU(3)) = x_1^2 x_2$ which under the action of the map sending $x_1 x_2 \to x_1^{-1} x_2^{-1}$ followed by the map $x_1 \leftrightarrow x_3 = x_1^{-1} x_2^{-1}$ is left fixed. Lemma 1 then implies that $A(x_1^2 x_2) = 0$.

$A(\beta(SU(3)) \cdot \rho_2) = A(x_1 x_2) + A(x_1^2) + A(x_1^3 x_2^2)$. The first two summands are zero by Lemma 1 and the last is $A(\beta(G_2))$.

COROLLARY. *$R(SU(3))$ is freely generated over $R(G_2)$ by the set $\{1, \rho_2\}$. (See Lemma 2.)*

Note. This corollary together with [8, §4], provides a free basis for $R(T)$ as an $R(G_2)$ module.

COROLLARY. *$\alpha : R(SU(3)) \to K(G_2/SU(3))$ is onto and $K(G_2/SU(3))$ is a free abelian group with $\{\alpha(1), \alpha(\rho_2)\}$ providing a free basis.*

3. Let F_4 be the simply connected compact Lie group representing the local structure F_4. F_4 contains *Spin*(9) as a subgroup of maximal rank. Let H = *Spin*(9) for the rest of this section, and let T be a maximal torus for F_4 and H. In order to get a reasonable model for the action of $W(H)$ on $R(T)$ we must use the method of [6] and view

$$R(T) \cong Z[x_1^{\pm 1}, \cdots, x_4^{\pm 1}, x_1^{1/2} x_2^{1/2} x_3^{1/2} x_4^{1/2}].$$

With this description $W(H)$ acts on $R(T)$ as the group generated by S_4, acting on $\{x_1, \cdots, x_4\}$, together with the maps sending $x_i \leftrightarrow x_i^{-1}$, $i = 1, \cdots, 4$ [6].

The positive roots of F_4 can be chosen to be [5]

$$\{x_i\} \cup \{x_i x_j^{\pm 1}\} \cup \{x_1^{1/2} x_2^{\pm 1/2} x_3^{\pm 1/2} x_4^{\pm 1/2}\}, \qquad 1 \le i < j \le 4.$$

The first two sets represents a choice for the positive roots of H [5]. Accordingly

$$\beta(F_4) = x_1^{11/2} x_2^{5/2} x_3^{3/2} x_4^{1/2} \quad \text{and} \quad \beta(H) = x_1^{7/2} x_2^{5/2} x_3^{3/2} x_4^{1/2}.$$

Let $\gamma \in W(F_4)$ be the element corresponding to the root $x_1^{1/2} x_2^{-1/2} x_3^{-1/2} x_4^{-1/2}$. An elementary calculation using the formula in [1] yields the following action for γ:

$$\gamma(x_1) = x_1^{1/2}x_2^{1/2}x_3^{1/2}x_4^{1/2},$$

$$\gamma(x_i) = x_1^{1/2}x_2^{\varepsilon_2/2}x_3^{\varepsilon_3/2}x_4^{\varepsilon_4/2} \quad \text{where} \quad \epsilon_i = 1, \quad \epsilon_j = -1 \quad \text{for} \quad j \neq i, \quad i = 2, 3, 4.$$

In this calculation we are using the formula in [1] with $\langle x_i, x_j \rangle = \delta_{ij}$. Let us choose γ as a representative for a non-trivial element of $W(F_4)/W(H)$.

If $\varphi_i \in W(H) \leq W(F_4)$ is the element permuting $\{x_i, x_i^{-1}\}$ then

$$\varphi_1 \circ \gamma(x_1) = x_1^{-1/2}x_2^{1/2}x_3^{1/2}x_4^{1/2}.$$

Since $\gamma \circ \sigma(x_1) \neq x_1^{-1/2}x_2^{1/2}x_3^{1/2}x_4^{1/2}$ for any $\sigma \in W(H)$, $\varphi_1 \circ \gamma$ represents a second non-trivial element of $W(F_4)/W(H)$. Being that $|W(F_4)|/|W(H)| = 3$ [2], $\{\gamma, \varphi_1 \circ \gamma\}$ together with $W(H)$ describes completely the action of $W(F_4)$ on $R(T)$.

$R(H) \cong Z[\rho_1, \rho_2, \rho_3, \Delta]$ where ρ_i are the ith elementary symmetric functions on the set $\{x_1, \cdots, x_4, x_1^{-1} \cdots, x_4^{-1}, 1\}$ (e.g. $\rho_1 = x_1 + \cdots + x_4 + x_1^{-1} + \cdots + x_4^{-1} + 1$) and Δ is the "Spinor representation" $\sum_{\epsilon_i=\pm1} x_1^{\varepsilon_1/2}x_2^{\varepsilon_2/2}x_3^{\varepsilon_3/2}x_4^{\varepsilon_4/2}$ [6].

THEOREM 2. *$F : R(Spin(9)) \to \mathrm{Hom}_{R(F_4)}(R(Spin(9)), R(F_4))$ is an isomorphism.*

Proof. Let $\bar{F}$ represent the induced bilinear form. We will show that $\bar{F}$ is s.n.s. If we let $\{a_i\} = \{1, \Delta, \Delta^2\}$ and $\{b_j\} = \{\Delta^2, \Delta, 1\}$ then we claim that

$$F(a_i, b_j) \begin{array}{l} = +1, \quad i = j. \\ = \quad 0, \quad i < j \end{array}$$

The claim will follow provided we can show:

(1) $A(\beta(H)) = 0$,
(2) $A(\beta(H) \cdot \Delta) = 0$,
(3) $A(\beta(H) \cdot \Delta^2) = +A(\beta(F_4))$.

Recall that $\beta(H) = x_1^{7/2}x_2^{5/2}x_3^{3/2}x_4^{1/2}$. $\gamma(\beta(H)) = x_1^4\, x_2^2\, x_3$ which is left fixed by φ_4. Lemma 1 therefore implies that $A(\beta(H)) = 0$.

$\beta(H) \cdot \Delta$ is a sum of monomials of the form $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ where the m_i are non-negative integers ≤ 4. If any $m_i = 0$ then the monomial is left fixed by φ_i. If $m_i = m_j$, $i \neq j$, then it is left fixed by the element permuting x_i and x_j. In either case the alternating sum of these terms is zero by Lemma 1. It follows that $A(\beta(H) \cdot \Delta) = A(x_1^4\, x_2^2\, x_3^2\, x_4)$. $\gamma(x_1^4\, x_2^3\, x_3^2\, x_4) = x_1^5\, x_2^2\, x_3$ which is left fixed by φ_4. Therefore $A(\beta(H) \cdot \Delta) = 0$.

In $R(H)$ we have the identity $\Delta^2 = \rho_4 + \rho_3 + \rho_2 + \rho_1 + 1$ [6]. To conclude the proof we will show

$$A(\beta(H)\rho_i) = 0, \quad i \neq 3, \quad \text{and} \quad A(\beta(H) \cdot \rho_3) = +A(\beta(F_4)).$$

An analogous argument to the one used for $\beta(H)\cdot\Delta$ shows that

$$A(\beta(H)\cdot\rho_1) = A(x_1^{9/2}x_2^{5/2}x_3^{3/2}x_4^{1/2}).$$

All other summands are zero (e.g. most are symmetric in some (i, j) (see Corollary 1). γ fixes $x_1^{9/2}x_2^{5/2}x_3^{3/2}x_4^{1/2}$ implying $A(\beta(H)\rho_1) = 0$.

ρ_2 contains $\rho_1 + 3$ as a summand. The compliment of $\rho_1 + 3$ in ρ_2 is the sum $\sum_{i<j} x_i^{\varepsilon_i}x_j^{\varepsilon_j}, \epsilon_i, \epsilon_j = \pm 1$. Once more all terms but one are trivially in the kernel of A and we get

$$A(\beta(H)\cdot\rho_2) = A(x_1^{9/2}x_2^{7/2}x_3^{3/2}x_4^{1/2}).$$

$\gamma(x_1^{9/2}x_2^{7/2}x_3^{3/2}x_4^{1/2}) = x_1^5\, x_2^3\, x_3$ which is left fixed by φ_4 proving that $A(\beta(H)\cdot\rho_2) = 0$.

It follows from the previous remarks that we can drop all the summands of ρ_3 which are either integers or summands of $\rho_2 + \rho_1$. The complement consists of

$$x = \sum_{i<j<k} x_i^{\varepsilon_i}x_j^{\varepsilon_j}x_k^{\varepsilon_k}, \quad \epsilon_t = \pm 1.$$

If we check $\beta(H)\cdot x$ we will see that all but one term is either symmetric in some (i, j) or is in the orbit of a monomial previously shown to have alternating sum zero (e.g. $\beta(H)\cdot x_1 x_3^{-1} x_4 = x_1^{9/2}x_2^{5/2}x_3^{1/2}x_4^{3/2}$ which is in the orbit of $x_1^{9/2}x_2^{5/2}x_3^{3/2}x_4^{1/2}$). The exception is $\beta(H)\cdot x_1 x_2 x_3 = x_1^{9/2}x_2^{7/2}x_3^{5/2}x_4^{1/2}$. Under $\varphi_4 \circ \gamma$ this monomial is mapped to $\beta(F_4)$ implying that $A(\beta(H)\cdot\rho_3) = +A(\beta(F_4))$.

$$\rho_4 = \sum_{\epsilon_i=\pm 1} x_1^{\varepsilon_1}x_2^{\varepsilon_2}x_3^{\varepsilon_3}x_4^{\varepsilon_4} + \sum_{i<j<k} x_i^{\varepsilon_i}x_j^{\varepsilon_j}x_k^{\varepsilon_k} + C$$

where $\epsilon_t = \pm 1$ and C is a term made up of monomials appearing as summands in $\rho_2 + \rho_1$. A monotonous repetition of the previous arguments shows that

$$\begin{aligned} A(\beta(H)\varphi_4) &= A[\beta(H)(x_1\cdot x_2\cdot x_3\cdot x_4 + x_1\cdot x_2\cdot x_3\, x_4^{-1} + x_1\cdot x_2\cdot x_3)] \\ &= A(x_1^{9/2}x_2^{7/2}x_3^{5/2}x_4^{3/2} + x_1^{9/2}x_2^{7/2}x_3^{5/2}x_4^{-1/2} + x_1^{9/2}x_2^{7/2}x_3^{5/2}x_4^{1/2}). \end{aligned}$$

The last two terms are images of each other under the action of φ_4 ($(-1)^{\varphi_4} = -1$) and therefore cancel in the alternating sum.

$$\gamma(x_1^{9/2}x_2^{7/2}x_3^{5/2}x_4^{3/2}) = x_1^6\, x_2^2\, x_3$$

which is left fixed by φ_4. Therefore $A(\beta(H)\cdot\rho_4) = 0$ completing the proof of the theorem.

COROLLARY. *$R(\mathrm{Spin}(9))$ is a free module over $R(F_4)$ with $\{1, \Delta, \Delta^2\}$ as a set of free generators.* (Using the results of [8, §8], we can get a basis for $R(T)$ over $R(F_4)$.)

COROLLARY.

$$\alpha : R(Spin(9)) \to K(F_4/\mathrm{Spin}(9))$$

is onto and $K(F_4/\mathrm{Spin}(9))$ is a free abelian group of rank 3 freely generated by the set $\{\alpha(1), \alpha(\Delta), \alpha(\Delta^2)\}$.

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