SEMISIMPLICITY OF 2-GRADED LIE ALGEBRAS, II

BY

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1. Introduction

As in [2], we consider finite-dimensional graded Lie algebras over a field F of characteristic 0. The grading is by integers mod 2 and enters into the defining identities as follows. Let L_0 and L_1 be the homogeneous components of the 2-graded Lie algebra L, let $x \in L_\alpha$, $y \in L_\beta$, $z \in L$. Then $[x, y]$ lies in $L_{\alpha+\beta}$,

$$[x, y] = -(-1)^{\alpha\beta}[y, x]$$

and

$$[x, [y, z]] = [[x, y], z] + (-1)^{\alpha\beta}[y, [x, z]].$$

We call such a Lie algebra *semisimple* if all its finite-dimensional 2-graded modules are semisimple, i.e., have the property that every homogeneous submodule has a homogeneous module complement.

It has been made apparent in [2] that the requirement that a 2-graded Lie algebra be semisimple in this representation-theoretical sense is a very severe restriction, ruling out all the examples that come to mind first, excepting, of course, the ordinary semisimple Lie algebras L with $L_1 = (0)$. The main result we obtain here is the complete determination of all semisimple 2-graded Lie algebras over an algebraically closed field F of characteristic 0. It turns out that the sole example given in [2] is the first member of an infinite sequence of semisimple, oddly generated and simple 2-graded Lie algebras $L(n)$, which are obtained in a natural way from the ordinary simple Lie algebras of type C_n. This is the *symplectic sequence* given in Section 4 below. In the algebraically closed case, every semisimple 2-graded Lie algebra is a direct sum of members of this sequence and ordinary semisimple Lie algebras. In view of this result, the general theory given in Sections 2 and 3 below essentially completes its life cycle right here.

From the viewpoint of classical representation theory of Lie algebras, the feature singling out the type C_n as the only possibility in the above is that it is the only type in which the extremal (highest) roots are divisible by 2 in the group of weights. This is seen quite clearly in the proof of Proposition 3.2.

It should be emphasized that almost all questions concerning simple, not necessarily semisimple, oddly generated 2-graded Lie algebras are still open. We merely exhibit two interesting new sequences of examples in Section 5.

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2. A semisimplicity criterion

Let F be a field of characteristic 0, and let $R = R_0 + R_1$ be a 2-graded F-algebra. Let A and B be 2-graded R-modules, and let $\mathrm{Hom}_F(A, B)$ denote the F-space of all F-linear maps from A to B. The 2-gradings of A and B define a 2-grading of $\mathrm{Hom}_F(A, B)$, where $\mathrm{Hom}_F(A, B)_0$ consists of the degree preserving maps (i.e., of the morphisms of the category of 2-graded F-spaces), while $\mathrm{Hom}_F(A, B)_1$ consists of the F-linear maps sending A_0 into B_1 and A_1 into B_0.

Suppose that $S = S_0$ is an F-subalgebra of R_0. We denote by $\mathrm{Hom}_S(A, B)$ the homogeneous subspace of $\mathrm{Hom}_F(A, B)$ consisting of the elements f such that $f(s \cdot a) = s \cdot f(a)$ for every s in S and every a in A.

Finally, we define the homogeneous F-subspace $\mathrm{Hom}_R(A, B)$ of $\mathrm{Hom}_F(A, B)$ as follows. The component $\mathrm{Hom}_R(A, B)_\eta$ consists of the elements f of $\mathrm{Hom}_F(A, B)_\eta$ such that

$$f(r \cdot a) = (-1)^{\eta\rho} r \cdot f(a)$$

for all elements a of A and all elements r of R_ρ. Clearly, $\mathrm{Hom}_R(A, B) \subset \mathrm{Hom}_S(A, B)$, and the morphisms of the category of 2-graded R-modules are the elements of $\mathrm{Hom}_R(A, B)_0$.

The 2-graded R-module B may be viewed naturally as a 2-graded S-module. If K is any 2-graded S-module, we have the functor $\mathrm{Hom}_S(K, \quad)$ from the category of 2-graded R-modules to the category of 2-graded F-spaces. On the other hand, we consider the 2-graded R-module $R \otimes_S K$ and the functor $\mathrm{Hom}_R(R \otimes_S K, \quad)$. As in the usual ungraded case, these two functors are naturally equivalent. The isomorphism $\mathrm{Hom}_S(K, B) \to \mathrm{Hom}_R(R \otimes_S K, B)$ is as follows. If f belongs to $\mathrm{Hom}_S(K, B)_\eta$ then the corresponding element f' of $\mathrm{Hom}_R(R \otimes_S K, B)_\eta$ is characterized by $f'(r \otimes k) = (-1)^{\eta\rho} r \cdot f(k)$ for every k in K and every r in R_ρ. The inverse map is obtained in the evident way from the canonical map $K \to R \otimes_S K$.

Now let us consider an exact sequence

$$(0) \to U \to V \to W \to (0)$$

in the category of 2-graded R-modules. Assume that this sequence is split when viewed as an exact sequence in the category of 2-graded S-modules. Then the induced sequence

$$(0) \to \mathrm{Hom}_S(K, U) \to \mathrm{Hom}_S(K, V) \to \mathrm{Hom}_S(K, W) \to (0)$$

in the category of 2-graded F-spaces is exact. Because of the above natural equivalence of functors, this implies that the sequence

$$(0) \to \mathrm{Hom}_R(R \otimes_S K, U) \to \mathrm{Hom}_R(R \otimes_S K, V) \to \mathrm{Hom}_R(R \otimes_S K, W) \to (0)$$

is exact.

Now let L be a 2-graded Lie algebra over F. Let R be the universal enveloping algebra $\mathcal{U}(L)$, and let S be the universal enveloping algebra $\mathcal{U}(L_0)$. As usual, we identify 2-graded L-modules with 2-graded $\mathcal{U}(L)$-modules. Let A and B be

2-graded L-modules. Then $\mathrm{Hom}_F(A, B)$ has a 2-graded L-module structure, as follows. For x in L_ρ and f in $\mathrm{Hom}_F(A, B)_\eta$, the transform $x \cdot f$ in $\mathrm{Hom}_F(A, B)_{\eta+\rho}$ is given by

$$(x \cdot f)(a) = x \cdot f(a) - (-1)^{\eta\rho} f(x \cdot a).$$

If y is an element of L_σ, one must verify that

$$x \cdot (y \cdot f) - (-1)^{\rho\sigma} y \cdot (x \cdot f) = [x, y] \cdot f.$$

We leave this verification to the reader. It is clear from the definitions that $\mathrm{Hom}_R(A, B)$ coincides with the L-annihilated part $\mathrm{Hom}_F(A, B)^L$ of $\mathrm{Hom}_F(A, B)$, and that $\mathrm{Hom}_S(A, B) = \mathrm{Hom}_F(A, B)^{L_0}$.

We regard F as a trivial L-module, with $L \cdot F = (0)$, choosing the 2-grading such that $F = F_0$. As in [2], we let $\otimes_0$ indicate tensoring with respect to $S = \mathscr{U}(L_0)$. Now we are fully prepared for the following semisimplicity criterion.

THEOREM 2.1. *Let L be a finite-dimensional 2-graded Lie algebra over the field F of characteristic* 0. *Then L is semisimple if and only if the following two conditions are satisfied.* (1) *L_0 is semisimple.* (2) *There is an element u_0 in $(\mathscr{U}(L)_0 \otimes_0 F)^L$ whose canonical image in F is not* 0.

Proof. Condition (2) is evidently equivalent to the condition that the exact sequence

$$(0) \to L\mathscr{U}(L) \otimes_0 F \to \mathscr{U}(L) \otimes_0 F \to F \to (0)$$

coming from the trivial $\mathscr{U}(L)$-module structure of F be split as a sequence in the category of 2-graded L-modules. This makes it evident that condition (2) is necessary. We know from Theorem 4.3 of [2] that condition (1) is necessary.

Now suppose that conditions (1) and (2) are satisfied. Let $(0) \to A \to B \to C \to (0)$ be an exact sequence of finite-dimensional 2-graded L-modules. It is clear that our definition of $\mathrm{Hom}_F(A, B)$ as a 2-graded L-module makes $\mathrm{Hom}_F(C, \ )$ a functor from the category of 2-graded L-modules to itself. Since F is a field, this functor is exact. Therefore, applying $\mathrm{Hom}_F(C, \ )$ to our above sequence, we obtain the following exact sequence in the category of 2-graded L-modules

$$(0) \to \mathrm{Hom}_F(C, A) \to \mathrm{Hom}_F(C, B) \to \mathrm{Hom}_F(C, C) \to (0).$$

Since L_0 is semisimple, this sequence is split as a sequence of L_0-modules. In other words, it is split as a sequence in the category of 2-graded S-modules. From our introductory discussion in this section, we know that therefore the sequence obtained by applying the functor $\mathrm{Hom}_R(R \otimes_0 F, \ )$ is exact. Since condition (2) is satisfied, we may identify the trivial L-module F with a direct 2-graded R-module summand of $R \otimes_0 F$. This implies that the functor $\mathrm{Hom}_R(F, \ )$ has the same exactness property as the functor $\mathrm{Hom}_R(R \otimes_0 F, \ )$. Clearly, for every 2-graded L-module U, we have $\mathrm{Hom}_R(F, U) \approx U^L$. Hence, applying the functor $\mathrm{Hom}_R(F, \ )$ to our above sequence, we find that the sequence

$$(0) \to \mathrm{Hom}_F(C, A)^L \to \mathrm{Hom}_F(C, B)^L \to \mathrm{Hom}_F(C, C)^L \to (0)$$

is exact. In particular, the map $\mathrm{Hom}_F(C, B)^L \to \mathrm{Hom}_F(C, C)^L$ is surjective. Let I denote identity map $C \to C$. This is evidently an element of $\mathrm{Hom}_F(C, C)_0^L$, and therefore is the image of an element f of $\mathrm{Hom}_F(C, B)_0^L$. Thus, f is a morphism of 2-graded L-modules $C \to B$ whose composite with the given morphism $B \to C$ is the identity map $C \to C$. The existence of such a morphism f means precisely that the given sequence $(0) \to A \to B \to C \to (0)$ is split as a sequence of 2-graded L-modules. We have shown that conditions (1) and (2) imply that L is semisimple, so that Theorem 2.1 is now established.

The trivial part of Theorem 2.1, namely, the necessity of condition (2) gives the following very useful necessary condition for semisimplicity.

PROPOSITION 2.2. *Let L be a semisimple 2-graded Lie algebra over the field F, and let a be a nonzero element of L_1. Then $[a, a] \neq 0$.*

Proof. Suppose that $0 \neq a_1 \in L_1$ and $[a_1, a_1] = 0$. Choose elements $a_2, \ldots, a_n$ in L_1 so that $(a_1, \ldots, a_n)$ is an F-basis of L_1. Then 1 and the monomials $a_{i_1} \cdots a_{i_q}$ with $i_1 < \cdots < i_q$ constitute a free right $\mathcal{U}(L_0)$-basis of $\mathcal{U}(L)$ (cf. [2, Section 2]). Let u be an element of $\mathcal{U}(L) \otimes_0 F$ whose canonical image in F is 1. Then u is the canonical image of an element v of $\mathcal{U}(L)$ that has the form

$$v = 1 + x + a_1 y$$

where x is a linear combination of basis elements $a_{i_1} \cdots a_{i_q}$ with $1 < i_1$, and y is such a linear combination plus an element of F. Since $[a_1, a_1] = 0$, we have $a_1 a_1 = 0$ in $\mathcal{U}(L)$, whence $a_1 v = a_1 + a_1 x$. This is a nonzero F-linear combination of elements of our $\mathcal{U}(L_0)$-basis of $\mathcal{U}(L)$, whence $a_1 \cdot u \neq 0$. Thus, condition (2) of Theorem 2.1 is not satisfied, contradicting the assumption that L is semisimple. This proves Proposition 2.2.

3. Implications of simplicity

PROPOSITION 3.1. *Suppose that L is a semisimple 2-graded F-Lie algebra having no homogeneous ideals other than* (0) *and L. Then L_1 is simple* (*or* (0)) *as an L_0-module, and L_0 is simple* (*or* (0)).

Proof. By [2, Theorem 4.3], L_0 is semisimple as an ordinary Lie algebra, and $[L_0, L_1] = L_1$. We assume that $L_1 \neq (0)$, because otherwise there is nothing to prove. Then we have also $L_0 \neq (0)$. Now $L_1 + [L_1, L_1]$ is clearly a nonzero homogeneous ideal of L, whence $[L_1, L_1] = L_0$.

Let U be any nonzero ideal of L_0, and put $A = L_1^U$. First, we show that $A = (0)$. Clearly, A is an L_0-submodule of L_1, so that L_1 is a direct L_0-module sum $A + M$, with $[U, M] = M$. We have

$$[A, M] = [A, [U, M]] = [U, [A, M]] \subset U$$

whence $[[A, M], A] = (0)$. On the other hand, $[[M, M], A] = (0)$, because it is contained in both A and M. Since

$$L_0 = [L_1, L_1] = [A, A] + [M, M] + [A, M]$$

it follows that

$$A = [L_0, A] = [[A, A], A]$$

Now $[[A, A], M] = (0)$, because it is contained in both M and A. Hence we have

$$[A, M] = [[A, A], [A, M]] \subset [A, A]$$

and it is now clear that $[A, A] + A$ is a homogeneous ideal of L. If this coincided with L, we would get the contradiction $U = [L_0, U] = [[A, A], U] = (0)$. Therefore, we must have $A = (0)$, i.e., $L_1^U = (0)$.

Now let S be any nonzero simple L_0-submodule of L_1. Make a direct L_0-module decomposition $L_1 = S + T$. As above, $[[S, S], T] = (0)$. By Proposition 2.2, $[S, S] \neq (0)$. By the above, with $U = [S, S]$, we have $T = (0)$ so that $L_1 = S$. Thus, we have shown that L_1 is simple as an L_0-module.

In showing that L_0 is simple, let us first deal with the case where F is algebraically closed. Suppose that L_0 is the direct sum $X + Y$ of two nonzero ideals X and Y. Since L_1 is simple as an L_0-module, with $L_1^X = (0) = L_1^Y$, and F is algebraically closed, it follows from standard basic theory of semisimple F-algebra modules that L_1 is a tensor product module $A \otimes B$, where $Y \cdot A = (0) = A^X$ and $X \cdot B = (0) = B^Y$. By decomposing A and B into weight spaces with respect to Cartan subalgebras of X and Y, respectively, we see that there are nonzero elements a in A, b in B, x in X, y in Y, and α, β in F, such that $x \cdot a = \alpha a$ and $y \cdot b = \beta b$. Let u be the element $a \otimes b$ of L_1. By Proposition 2.2, we have $[u, u] \neq 0$. On the other hand, $[x, [u, u]] = 2\alpha[u, u] \in X$, whence $[u, u]$ is a nonzero element of X. Similarly, operating with y, we see that $[u, u]$ is a nonzero element of Y. This contradicts the assumption $X \cap Y = (0)$. Therefore, L_0 is simple.

Now let us consider the general case. Assume that $L_0 = X + Y$, as above. Let T be an algebraically closed field containing F. Since $L_0 \otimes_F T$ is semisimple, it therefore follows from Theorem 2.1 that $L \otimes_F T$ is semisimple as a 2-graded Lie algebra over T. Clearly, $(L \otimes_F T)_0$ is the direct sum of the two nonzero ideals $X \otimes_F T$ and $Y \otimes_F T$. By the above, the simple components of $L \otimes_F T$ have the simple components of $(L \otimes_F T)_0$ as their degree 0 parts. Therefore, $L \otimes_F T$ is a direct 2-graded Lie algebra sum $U + V$, where $U_0 = X \otimes_F T$ and $V_0 = Y \otimes_F T$. If both U_1 and V_1 are (0) then $L_1 = (0)$. Therefore, we may suppose that $U_1 \neq (0)$. Now we have $[V_0, U_1] = (0)$, whence $(L_1 \otimes_F T)^Y \neq (0)$. Clearly, this implies that $L_1^Y \neq (0)$, which contradicts what we have found in proving the first part of our proposition. The proof of Proposition 3.1 is now complete.

PROPOSITION 3.2. *If the base field F is algebraically closed, then the simple Lie algebra L_0 of Proposition* 3.1 *is of the symplectic type C_n ($n = 1, 2, \ldots$).*

Proof. Let μ denote the highest weight of the simple L_0-module L_1, and let u be a nonzero element belonging to the weight subspace $(L_1)_\mu$ of L_1. By Proposition 2.2, $[u, u]$ is a nonzero element of L_0. Clearly, it belongs to the root subspace $(L_0)_{2\mu}$ of L_0. Since $[L_1, L_1] = L_0$, it is clear that 2μ is therefore the largest root of L_0. Thus, *a necessary condition for L_0 is that its largest root be divisible by* 2 *in the group of weights*, for any choice of a Cartan subalgebra and ordering of the roots.

The following facts are easily collected from the tables given at the end of [1]. In all of the exceptional types G_2, F_4, E_6, E_7, E_8, in B_n for $n > 2$, and in D_n for $n > 3$, the largest root is listed as one of the fundamental weights. In A_n for $n > 1$, the largest root is the sum of the first and the last fundamental weights.

Since the fundamental weights constitute a free basis of the group of weights, all these types are thus ruled out. This leaves only C_n for $n = 1, 2, \ldots$ (note that $A_1 = B_1 = C_1$, $B_2 = C_2$).

4. The symplectic sequence

The standard representations of the ordinary simple Lie algebras of type C_n give rise to an infinite sequence of semisimple (and simple) 2-graded Lie algebras $L(n)$ such that $L(n)_0$ is the ordinary simple Lie algebra of type C_n. Let us recall the standard representation of C_n.

Let V be an F-space of dimension $2n$ ($n = 1, 2, \ldots$). Choose an F-basis $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ of V, and let π be the skew symmetric nondegenerate bilinear form on $V \times V$ such that $\pi(a_i, a_j) = 0 = \pi(b_i, b_j)$ for all i and j, while $\pi(a_i, b_j)$ is equal to 1 if $i = j$ and equal to 0 otherwise. Let L_0 be the Lie algebra of all those linear endomorphisms of V which annihilate π, i.e., the elements of L_0 are the linear endomorphisms e such that $\pi(e(u), v) + \pi(u, e(v)) = 0$ for all elements u and v of V. Then L_0 is a simple Lie algebra of type C_n, and V is the standard simple L_0-module. We define L_1 to be the L_0-module V. Thus, for x in L_0 and v in V, the Lie product $[x, v]$ is defined as $x(v)$.

Now let u and v be elements of V. We must define $[u, v]$ as an element of L_0. The definition is actually obtained in the usual way, using the isomorphism between V and its dual coming from π. Explicitly, we define $[u, v]$ to be the linear endomorphism of V given by

$$[u, v](w) = \pi(v, w)u + \pi(u, w)v.$$

A direct check shows that $[u, v]$ indeed belongs to L_0 (i.e., annihilates π). Since $[u, v] = [v, u]$, there is an F-linear map $\eta\colon S^2(L_1) \to L_0$, where $S^2(L_1)$ denotes the homogeneous component of degree 2 of the symmetric algebra built over L_1, such that $\eta(uv) = [u, v]$ for all elements u and v of L_1. A part of the Jacobi identity for 2-graded Lie algebras says that η is a homomorphism of L_0-modules. This is verified directly, as follows. Let x be an element of L_0. Then, in $S^2(L_1)$, we have $x \cdot (uv) = x(u)v + ux(v)$. Hence, with w in V,

$$\begin{aligned}\eta(x \cdot (uv))(w) &= \pi(v, w)x(u) + \pi(x(u), w)v + \pi(x(v), w)u + \pi(u, w)x(v) \\ &= x(\pi(v, w)u + \pi(u, w)v) + \pi(x(v), w)u + \pi(x(u), w)v \\ &= x([u, v](w)) - \pi(v, x(w))u - \pi(u, x(w))v \\ &= x([u, v](w)) - [u, v](x(w)) \\ &= [x, [u, v]](w) \\ &= [x, \eta(uv)](w).\end{aligned}$$

Thus we have, indeed, $\eta(x \cdot (uv)) = [x, \eta(uv)]$.

The remaining part of the Jacobi identity says that, for u, v and w in L_1, we should have

$$[\eta(uv), w] + [\eta(vw), u] + [\eta(wu), v] = 0$$

(cf. [2, Section 4]). This is seen immediately from the definitions, using that π is skew symmetric. Now we have established that L is a 2-graded Lie algebra. Since η is a nonzero L_0-module homomorphism and since L_0 is simple as an L_0-module, η is surjective. The dimensions of $S^2(L_1)$ and L_0 are both equal to $n(2n + 1)$. Therefore, η is actually an isomorphism.

An ideal of L is an L_0-submodule of L. Since the L_0-module L is the direct sum of the two nonisomorphic simple L_0-modules L_0 and L_1, an ideal must therefore be one of (0), L, L_0, L_1. Clearly, L_0 and L_1 are not ideals of L. Therefore, L is simple, in the sense that its only ideals (homogeneous or not) are (0) and L. As we know from [2, Section 5], this does *not* imply that L is semisimple (in our representation-theoretical sense).

We shall now use the criterion of Theorem 2.1 in order to prove that L is semisimple. It suffices to exhibit an element u_0, as in condition (2) of Theorem 2.1. Working in $\mathcal{U}(L)$, put $t_i = a_i b_i \in \mathcal{U}(L)_0$. Let u_0 be the canonical image in $\mathcal{U}(L) \otimes_0 F$ of the element

$$(1 - t_1)(3 - t_2)\cdots(2n - 1 - t_n)$$

of $\mathcal{U}(L)_0$. Since the canonical image of u_0 in F is not zero (being the product of the odd integers from 1 to $2n - 1$), it remains only to show that u_0 is annihilated by every element of L. In order to see this, we examine some commutation relations in $\mathcal{U}(L)$, as follows.

First, let us note that if u, v and w are elements of L_1 then, in $\mathcal{U}(L)$, we have $uv + vu = [u, v]$, etc., whence

$$uvw - vwu = [u, v]w - v[u, w].$$

In particular,

$$ut_j - t_j u = [u, a_j]b_j - a_j[u, b_j].$$

We have $[u, a_j]b_j = [[u, a_j], b_j] + b_j[u, a_j]$. Hence we have

$$ut_j = t_j u + u + \pi(u, b_j)a_j + b_j[u, a_j] - a_j[u, b_j].$$

Now let i be an index other than j. Then this gives

$$a_i t_j = t_j a_i + a_i + b_j[a_i, a_j] - a_j[a_i, b_j]$$

and

$$b_i t_j = t_j b_i + b_i + b_j[b_i, a_j] - a_j[b_i, b_j].$$

Multiplying the second relation by a_i from the left and then substituting for the resulting $a_i t_j$ the right-hand side of the last equation but one, we obtain

$$t_i t_j = t_j t_i + t_i + b_j[a_i, a_j]b_i - a_j[a_i, b_j]b_i + t_i + a_i b_j[b_i, a_j] - a_i a_j[b_i, b_j]$$

Next, we note that

$$[a_i, a_j]b_i = a_j + b_i[a_i, a_j] \quad \text{and} \quad [a_i, b_j]b_i = b_j + b_i[a_i, b_j].$$

Hence we have

$$b_j[a_i, a_j]b_i - a_j[a_i, b_j]b_i = b_j a_j + b_j b_i[a_i, a_j] - t_j - a_j b_i[a_i, b_j]$$

and

$$t_i t_j = t_j t_i + 2(t_i - t_j) + d_{ij}$$

where d_{ij} lies in $\mathscr{U}(L)L_0$. We shall not need the precise expression for d_{ij} (as obtained from the above), but only the following fact. Let V_k be the F-subspace $Fa_k + Fb_k$ of L_1. Let $[V_i, V_j]$ be the F-subspace of L_0 spanned by the elements $[u, v]$ with u in V_i and v in V_j. Then d_{ij} lies in $[a_j, b_j] + \mathscr{U}(L)[V_i, V_j]$.

It follows immediately from this last result that, for every q in F, and in particular for every integer q, we have

$$(q - t_i)(q + 2 - t_j) - (q - t_j)(q + 2 - t_i) \in [a_j, b_j] + \mathscr{U}(L)[V_i, V_j].$$

Now observe that if neither i nor j is equal to k then, in $\mathscr{U}(L)$, every element of $[V_i, V_j]$ commutes with every element of V_k. It follows from this and the last result that, if σ is any permutation of $(1, \ldots, n)$, the image in $\mathscr{U}(L) \otimes_0 F$ of

$$(1 - t_{\sigma(1)}) \cdots (2n - 1 - t_{\sigma(n)})$$

coincides with u_0.

Since $[L_1, L_1] = L_0$, it suffices to prove that u_0 is annihilated by every element of L_1. Therefore, it suffices to show that $a_i \cdot u_0 = 0 = b_i \cdot u_0$ for every i. Because of the above symmetry with respect to permutations of the indices, it is clear that it suffices to prove that u_0 is annihilated by a_1 and b_1. It is easy to verify directly that both $a_1(1 - t_1)$ and $b_1(1 - t_1)$ lie in $\mathscr{U}(L)[V_1, V_1]$. Since the elements of $[V_1, V_1]$ commute with the elements of every V_k with $k > 1$, it follows immediately that

$$V_1(1 - t_1) \cdots (2n - 1 - t_n) \subset \mathscr{U}(L)[V_1, V_1] \subset \mathscr{U}(L)L_0,$$

whence $V_1 \cdot u_0 = (0)$. This completes the proof that L is semisimple.

We note that the case $n = 1$ is the unique lowest dimensional odd (i.e., generated by L_1) semisimple 2-graded Lie algebra, whose simple modules have been determined explicitly in [2, Section 6].

THEOREM 4.1. *Let F be an algebraically closed field of characteristic* 0, *and let L be a finite-dimensional* 2*-graded Lie algebra over F. Then L is semisimple if and only if it is a direct sum of* 2*-graded Lie algebras each of which is either a member of the symplectic sequence or an ordinary simple Lie algebra.*

Proof. All that remains to be shown is that if L is as in Proposition 3.2 then it is a member of the symplectic sequence (the sufficiency of our condition is clear from Theorem 4.1 of [2]). Let $L(n)$ be the member of the symplectic sequence such that $L(n)_0 = L_0$. The proof of Proposition 3.2 has shown that, as an L_0-module, L_1 is determined up to isomorphisms by L_0. Therefore, we may identify L_1 with $L(n)_1$. Let η_n and η denote the L_0-module homomorphisms $S^2(L_1) \to L_0$ of $L(n)$ and L, respectively. Since each of these is an isomorphism and since L_0 is simple, we must have $\eta = c\eta_n$, where c is a nonzero element of F. Choose an element d in F such that $d^2 = c$. Then the map $L \to L(n)$ that coincides with the identity map on L_0 and with the scalar multiplication by d on L_1 is clearly an isomorphism of 2-graded Lie algebras. This establishes Theorem 4.1.

5. Other simple 2-graded Lie algebras

Let us call a 2-graded Lie algebra L simple if its only homogeneous ideals are (0) and L. The classification of these is probably quite difficult. The most natural family of such 2-graded Lie algebras has been briefly discussed in [2, Section 5]. The fact that they are *not* semisimple is now seen immediately from Proposition 2.2.

We shall describe two sequences of simple 2-graded Lie algebras that arise in an interesting way from the classical type A_n. Let n be a positive integer, and let V be an $(n + 1)$-dimensional vector space over the field F of characteristic 0. Let L_0 be the simple Lie algebra of all linear endomorphisms of trace 0 of V. Let $S^2(V)$ and $E^2(V)$ denote the homogeneous components of degree 2 of the symmetric and exterior, respectively, algebras built on V. We regard these as L_0-modules in the natural way. Let ° indicate dual space (and L_0-module), and let L_1 be the direct sum of the L_0-modules $S^2(V)$ and $E^2(V)^\circ$. Define the linear map $\eta\colon S^2(L_1) \to L_0$, indicated also by writing $\eta(uv) = [u, v]$, as follows:

$$[S^2(V), S^2(V)] = (0) = [E^2(V)^\circ, E^2(V)^\circ]$$

Next, let f be an element of $E^2(V)^\circ$, and let a, b, x be elements of V. Let ab denote the canonical image of $a \otimes b$ in $S^2(V)$, and let $a * x$ and $b * x$ denote the canonical images of $a \otimes x$ and $b \otimes x$ in $E^2(V)$. Then the bracketing with f is defined so that

$$[ab, f](x) = f(a * x)b + f(b * x)a = [f, ab](x).$$

It is easy to verify that the map η so defined is indeed an L_0-module homomorphism $S^2(L_1) \to L_0$. Since L_0 is simple, it follows from the evident fact that $\eta \neq 0$ that η is surjective. In order to verify that we have now the structure of an odd 2-graded Lie algebra, it suffices to show that, for all elements u, v, w in L_1, one has

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

This verification is somewhat lengthy, but automatic. The fact that L is simple is easily established, using that L_0 is simple and that $S^2(V)$ and $E^2(V)$ are simple L_0-modules.

The other sequence of simple 2-graded Lie algebras is obtained from the same V and L_0, but with L_1 the direct sum of $S^2(V)^\circ$ and $E^2(V)$. As above, only the mixed brackets are different from 0, and the critical part of the definition of η is as follows. Let g be an element of $S^2(V)^\circ$, and let a, b, x be elements of V. Then

$$[g, a * b](x) = g(ax)b - g(bx)a = [a * b, g](x).$$

The required verifications are very similar, in the two cases.

REFERENCES

1. N. BOURBAKI, *Groupes et algèbres de Lie*, Ch. 6, Hermann, Paris, 1968.
2. G. HOCHSCHILD, *Semisimplicity of 2-graded Lie algebras*, Illinois J. Math., vol. 20 (1976), pp. 107–123 (this issue).

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