A MONOTONE INTERSECTION PROPERTY FOR MANIFOLDS

BY

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1. Introduction

Brown [1] has established that if N is an open n-cell, then N has the monotone union property. Kwun [2] proved that if N is a closed PL manifold whose dimension is not four, then $M - p$ has the monotone union property where p is any point of M. In this paper we establish conditions which tell us when a manifold has the monotone union property. We define a monotone intersection property and indicate ways that it is related to the monotone union property. The principal results established are:

THEOREM 2.3. *Let $\{N_i\}$ be a sequence of manifolds such that for each i ($i = 1, 2, \ldots$), N_i is trivially embedded in N_{i+1}. Then $\bigcup_{i=1}^{\infty} N_i$ is homeomorphic to N_1°.*

THEOREM 3.5. *A manifold N has the monotone intersection property if and only if whenever $N \subset^\circ N_1$ where N_1 is homeomorphic to N, then N is trivially embedded in N_1.*

COROLLARY 3.6. *If a manifold has the monotone intersection property, then it also has the monotone union property.*

Theorem 2.3 generalizes the following result of C. H. Edwards [3] to all compact manifolds with boundary. Let N be a compact 3-manifold with boundary B and spine K and for each integer n let $h_n(N)$ be a homeomorphic image of N. Edwards has shown that if $X = \bigcup_{i=1}^{\infty} h_n(N)$ where for each n, $h_n(N) \subset^\circ h_{n+1}(N)$ and $h_n(K) = h_{n+1}(K)$, then X is homeomorphic to N°. Husch [4] stated that this result could be extended to all PL manifolds by using the regular neighborhood annulus conjecture.

The following question is raised by this paper. Are the monotone union and monotone intersection properties equivalent for compact topological manifolds with boundary?

2. Definitions and proof of Theorem 2.3

Throughout this paper we assume all manifolds are compact topological manifolds with boundary and all homeomorphisms are topological. A compact

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manifold N with boundary has the *monotone union property* provided that whenever $\{N_i\}$ is a sequence of manifolds such that for each i,

(a) N_i is homeomorphic to N and
(b) N_i is contained in the interior of N_{i+1} ($N_i \subset^\circ N_{i+1}$),

then $\bigcup_{i=1}^{\infty} N_i$ is homeomorphic to the interior of N (N°). A set D contained in the interior of a manifold M is a *hub* of M if $M - D$ is an open collar on the boundary of M ($M - D$ is homeomorphic to $\dot{M} \times [0, 1)$). Let D_1 be a hub of the manifold M_1. M_1 is *trivially embedded* in a manifold M_2 with respect to D_1 provided:

(a) $M_1 \subset^\circ M_2$,
(b) M_2 is homeomorphic to M_1, and
(c) D_1 is also a hub of M_2.

LEMMA 2.1. *If M_1 is trivially embedded in M_2 with respect to the hub D_1 and if D'_1 is any hub of M_1, then M_1 is trivially embedded in M_2 with respect to D'_1.*

Proof. Since D_1 is a hub of M_2 there is a homeomorphism g_2 of $\dot{M}_2 \times [0, 1)$ onto $M_2 - D_1$. Also since D'_1 is a hub of M_1, there is a homeomorphism g_1 of $M_1 - D_1$ onto $M_1 - D'_1$ which is the identity on $\dot{M}_1$. Extend g_1 by the identity to $M_2 - M_1$. Then the composite $g_1 g_2$ is a homeomorphism of $\dot{M}_2 \times [0, 1)$ onto $M_2 - D'_1$.

The following lemma is needed to establish Theorem 2.3 and it is proved using a well-known technique.

LEMMA 2.2. *Suppose $D_1 \subset^\circ M_1 \subset^\circ M_2$ where D_1 is a hub of both M_1 and M_2. Let g_i be a homeomorphism of $\dot{M}_i \times [0, 1)$ onto $M_i - D_1$ where $i = 1, 2$. Then given numbers a, b, and c where $0 < a < b < 1$ and $0 < c < 1$, there is a homeomorphism H of M_2 such that:*

(a) *$H \mid \dot{M}_2 \cup D_1 \cup g_1(\dot{M}_1 \times [b, 1))$ is the identity and*
(b) *$H(g_1(\dot{M}_1 \times [a, 1)) \supset g_2(\dot{M}_2 \times [c, 1))$.*

Proof. Since D_1 is a hub of both M_1 and M_2 and M_2 is compact, there is a t_1 where $0 < t_1 < 1$ such that

$$g_2(\dot{M}_2 \times [t_1, 1)) \subset g_1(\dot{M}_1 \times [b, 1)).$$

By the same reason there is a number t'_1 where $b < t'_1 < 1$ and

$$g_1(\dot{M}_1 \times [t'_1, 1) \subset g_2(\dot{M}_2 \times ((t + 1)/2, 1)).$$

Let h_1 be a homeomorphism of $[0, 1)$ such that $h_1(b) = t'_1$ and

$$h_1 \mid [0, a) \cup ((t'_1 + 1)/2, 1)$$

is the identity. Similarly let h_2 be a homeomorphism of $[0, 1)$ such that $h_2(t_1) = c_1$ and $h_2 \mid ((t_1 + 1)/2, 1)$ is the identity. Now let h'_1 be a homeomorphism of M_1 defined as follows:

(a) $h'_1(m, t) = g_1(m, h_1(t))$ for (m, t) in $\dot{M}_1 \times [0, 1)$ and
(b) $h'_1(x) = x$ for x in D_1.

Clearly h'_1 is a homeomorphism of M_1 which is the identity on $\dot{M}_1$. Hence we can extend h'_1 to a homeomorphism $\bar{h}_1$ of M_2 by letting it be the identity on $M_2 - M_1$.

Let $\bar{h}_2$ be the homeomorphism of M_2 defined as follows:

(a) $\bar{h}_2(m, t) = g_2(m, h_2(t))$ for (m, t) in $\dot{M}_2 \times [0, 1)$ and
(b) $\bar{h}_2(x) = x$ for x in D_1.

The composite $\bar{h}_1^{-1}\bar{h}_2\bar{h}_1$ is the desired homeomorphism of M_2.

THEOREM 2.3. *Let $\{M_i\}$ be a sequence of manifolds such that for each i; $i = 1, 2, \ldots$; M_i is trivially embedded in M_{i+1}. Then $\bigcup_{i=1}^{\infty} M_i$ is homeomorphic to M_1°.*

Proof. A homeomorphism H will be constructed from M_1° onto $\bigcup_{i=1}^{\infty} M_i$. The homeomorphism H will be defined as the limit of a sequence $\{h_i\}$ of homeomorphism where each h_i is a homeomorphism from M_1° into $\bigcup_{i=1}^{\infty} M_i$. Let D_1 be a hub of M_1. Since M_i is trivially embedded in M_{i+1} for each i, it is clear by Lemma 2.1 that D_1 is a hub of M_i for every i. Hence for each i, there is a homeomorphism g_i from $\dot{M}_i \times [0, 1)$ onto $M_i - D_1$.

Since M_2 is compact and $M_1 \subset M_2^\circ$, there is a number c such that $0 < c < 1$ and $g_2(\dot{M}_2 \times (c, 1)) \supset M_1 - D_1$. We apply Lemma 2.2 with $a = \frac{1}{4}$, $b = \frac{1}{2}$, and c as given above to obtain a homeomorphism h'_1 of M_2 such that $h'_1 \mid \dot{M}_2 \cup D_1 \cup g_1(\dot{M}_1 \times [\frac{1}{2}, 1))$ is the identity and

$$M_1 \subset h'_1 g_1(\dot{M}_1 \times [\tfrac{1}{4}, 1)) \cup D_1.$$

Let $h_1 = h'_1 \mid M_1^\circ$.

We consider the following inductive statement. There is a homeomorphism h'_i of M_{i+1} such that:

(a) $h'_i \mid \dot{M}_{i+1} \cup D_1 \cup h'_{i-1}h'_{i-2} \cdots h'_1(g_1(\dot{M}_1 \times [\frac{1}{2}^i, 1))$ is the identity and
(b) $h'_i h'_{i-1} \cdots h'_1(g_1(\dot{M}_1 \times [\frac{1}{2}^{i+1}, 1)) \cup D_1 \supset M_i$.

Clearly h'_1 satisfies this statement.

We assume the inductive statement is true for all $k \leq n - 1$ and we now establish it for $k = n$. We apply Lemma 2.2 with

$$M_1 = h'_{n-1}h'_{n-2} \cdots h'_1(M_1), \quad M_2 = M_{n+1},$$

$a = \frac{1}{2}^{n+1}$, $b = \frac{1}{2}^n$ and c a number such that

$$M_n \subset g_{n+1}(\dot{M}_{n+1} \times [c, 1)) \cup D_1.$$

By Lemma 2.2, there exists a homeomorphism h'_n such that:

(a) $h'_n \mid \dot{M}_{n+1} \cup D_1 \cup h'_{n-1}h'_{n-2} \cdots h'_1(g_1(\dot{M}_1 \times [\frac{1}{2}^n, 1)))$ is the identity, and
(b) $h'_n h'_{n-1} \cdots h'_1(g_1(\dot{M}_1 \times [\frac{1}{2}^{n+1}, 1))) \cup D_1 \supset M_n$.

This establishes the inductive statement. For each i, we let

$$h_i = h'_i h'_{i-1} \cdots h'_1(M_1^\circ).$$

Let $H(x) = \lim_{n\to\infty} h_n(x)$ for x in M_1°. Since h_r for $r \geq n$ is the identity on

$$g_1(\dot{M}_1 \times [\tfrac{1}{2}^n, 1)),$$

we see that $H(x)$ is well defined on M_1°. Furthermore since

$$h_n g_1(\dot{M}_1 \times [\tfrac{1}{2}^{n+1}, 1)) \cup D_1 \supset M_n,$$

we see that the image of M_1° under H is $\bigcup_{i=1}^\infty M_i$. Hence H is the desired homeomorphism.

3. Definitions and proof of Theorem 3.2

A set X is *N-ular* (N is an n-manifold) in the manifold M^n if there is a sequence of manifolds $\{N_i\}$ such that:

(a) $M \supset^\circ N_1 \supset^\circ N_2 \supset^\circ \cdots$;
(b) for each i, N_i is homeomorphic to N;
(c) for each i, N_{i+1} is trivially embedded in N_i;
(d) $X = \bigcap_{i=1}^\infty N_i$.

A compact manifold N with boundary has the *monotone intersection property* provided that whenever $\{N_i\}$ is a sequence of manifolds such that:

(a) $N_1 \supset^\circ N_2 \supset^\circ \cdots$;
(b) for each i, N_i is homeomorphic to N, then $N_1 - \bigcap_{i=1}^\infty N_i$ is homeomorphic to $\dot{N}_1 \times [0, 1)$.

The following lemma will be useful in the proof of Theorem 3.2.

Lemma 3.1. *Suppose A and B are hubs of the manifold M with boundary and K is a compact set in M°. Let g be a homeomorphism of $\dot{M} \times [0, 1)$ onto $M - A$ and let $t \in (0, 1)$. Then there is a homeomorphism h of $M - A$ onto $M - B$ such that $hg(\dot{M} \times [0, t)) \cap K = \emptyset$ and $h \mid \dot{M}$ is the identity.*

Proof. Since A and B are hubs of M, it is easy to construct a homeomorphism k_1 of $M - A$ onto $M - B$ which is the identity on $\dot{M}$. Since K is compact and is contained in M°, there is a number s where $0 < s < 1$ such that

$$k_1 g(\dot{M} \times [0, s]) \cap K = \emptyset.$$

Let θ be a homeomorphism of $[0, 1)$ which carries t onto s. Let k_2 be the homeomorphism of $\dot{M} \times [0, 1)$ defined by $k_2(m, t) = (m, \theta(t))$.

The composite $k_1 g k_2 g^{-1}$ gives the desired homeomorphism of $M - A$.

THEOREM 3.2. *Suppose that X is N-ular in the manifold M where $X = \bigcap_{i=1}^{\infty} N_i$. Then X is a hub of N_1.*

Proof. We need to establish that $N_1 - X$ is homeomorphic to $\dot{N} \times [0, 1)$. Let D_1 be a hub of N_1. A homeomorphism H will be constructed from $N_1 - D_1$ onto $N_1 - X$. The homeomorphism H will be defined as the limit of a sequence $\{h_i\}$ of homeomorphisms.

To obtain h_1, we apply Lemma 3.1 with $M = N_1$, $K = N_2$, $t = \frac{1}{2}$, $A = D_1$, and $B = D_2$ where D_2 is a hub of N_2. Clearly B is a hub of N_1 by Lemma 2.1 since N_2 is trivially embedded in N_1. We let h_1 be the homeomorphism given by Lemma 3.1.

To obtain h_2, we apply Lemma 3.1 with $M = N_2$, $K = N_3$, $A = D_2$, and $B = D_3$ where D_3 is a hub of N_3. Let g_1 be the homeomorphism identified in Lemma 3.1 from $\dot{N}_1 \times [0, 1)$ onto $N_1 - D_1$. To obtain t we note that since $h_1 g_1(\dot{N}_1 \times [0, 1)) = N_1 - D_2$ and $\dot{N}_1$ is compact, there is a t_1 where $0 < t_1 < 1$ such that

$$g_2(\dot{N}_2 \times (t_1, 1)) \cap h_1 g_1(\dot{N}_1 \times [0, \tfrac{2}{3})) = \emptyset$$

where g_2 is the homeomorphism from $\dot{N}_2 \times [0, 1)$ onto $N_2 - D_2$. Let $t = t_1$ and let h_2 be the homeomorphism given by Lemma 3.1. Since $h_2 \mid \dot{N}_2$ is the identity, we can extend h_2 to $N_1 - N_2$ by the identity. Also since

$$h_1 g_1(\dot{N}_1 \times [0, \tfrac{2}{3})) \subset g_2(\dot{N} \times [0, t_1]),$$

then

$$h_2 h_1 g_1(\dot{N}_1 \times [0, \tfrac{2}{3})) \cap N_3 = \emptyset.$$

Inductively we assume h_{n-1} has been constructed so that

$$h_{n-1} h_{n-2} \cdots h_1 g_1(\dot{N}_1 \times [0, 1)) = N_1 - D_n$$

and $h_{n-1} \mid N_1 - N_{n-1}$ is the identity. Also

$$h_{n-1} h_{n-2} \cdots h_1 g_1(\dot{N}_1 \times [0, n - 1/n)) \cap N_n = \emptyset.$$

To obtain h_n we apply Lemma 3.1 with $M = N_n$, $A = D_n$, $B = D_{n+1}$ where D_{n+1} is a hub of N_{n+1} and $K = N_{n+1}$. Let g_n be a homeomorphism from $\dot{N}_n \times [0, 1)$ onto $N_n - D_n$. To obtain t_n, we note that since

$$h_{n-1} h_{n-2} \cdots h_1 g_1(\dot{N}_1 \times [0, 1)) = N_1 - D_n,$$

then there is a t_n where $0 < t_n < 1$ such that

$$g_n(\dot{N}_n \times (t_n, 1)) \cap h_{n-1} h_{n-2} \cdots h_1 g_1(\dot{N}_1 \times [0, n/n + 1)) = \emptyset.$$

Let $t = t_n$. We then let h_n be the homeomorphism given by Lemma 3.1. Since $h_n \mid \dot{N}_n$ is the identity, we can extend h_n to $N_1 - N_n$ by the identity.

For $x \in N_1 - D_1$, we let $H(x) = \lim_{n\to\infty} h_n h_{n-1} \cdots h_1(x)$. If $x \in N_1 - D_1$, there is an integer n such that $x \in g_1(\dot{N}_1 \times [0, n/n+1))$. Hence

$$H(x) = h_n h_{n-1} \cdots h_1(x)$$

since $h_n h_{n-1} \cdots h_1(x) \cap N_{r+1} = \emptyset$ and $h_r \mid N_1 - N_n$ is the identity for $r \geq n$. Thus H is well defined and the image of $N_1 - D_1$ under H is

$$N_1 - \bigcap_{i=1}^{\infty} N_i = N_1 - X.$$

Therefore H is the desired homeomorphism.

The following corollaries follow from Theorem 3.2.

COROLLARY 3.3. *A set X is N-ular in a manifold M if and only if there is a manifold N' which is homeomorphic to N and is such that $X \subset^\circ N' \subset^\circ M$ and $N' - X$ is homeomorphic to $\dot{N}' \times [0, 1)$.*

Proof. Assume N' is homeomorphic to N and $X \subset^\circ N' \subset^\circ M$ with $N' - X$ homeomorphic to $\dot{N}' \times [0, 1)$. We can express X as the intersection of a nested sequence of manifolds $\{N_i\}$ where each N_i is homeomorphic to N and for each i, N_{i+1} is trivially embedded in N_i. The N_i are obtained by shrinking in on the collar of N' which is the complement of X in N'. Hence X is N-ular in M.

If X is N-ular in M, then by Theorem 3.2 X is a hub of N_1 where N_1 is homeomorphic to N and the result follows.

COROLLARY 3.4. *Let N be a manifold such that whenever $N \subset^\circ N_1$, where N_1 is homeomorphic to N, N is trivially embedded in N_1. Then N has the monotone intersection property.*

Proof. If $\{N_i\}$ is a sequence of manifolds such that $N_{i+1} \subset^\circ N_i$ where each N_i is homeomorphic to N, then the sequence $\{N_i\}$ satisfies the hypothesis of Theorem 3.2 and so $N_1 - \bigcap_{i=1}^{\infty} N_i$ is homeomorphic to $\dot{N}_1 \times [0, 1)$. Hence N has the monotone intersection property.

THEOREM 3.5. *A manifold N has the monotone intersection property if and only if whenever $N \subset^\circ N_1$ where N_1 is homeomorphic to N, then N is trivially embedded in N_1.*

Proof. Corollary 3.4 states that this condition is sufficient.

Let N be a manifold which has the monotone intersection property and let $N \subset^\circ N_1$ where N_1 is homeomorphic to N. Let D be a hub of N. Then $N - D$ is homeomorphic to $\dot{N} \times [0, 1)$. D can be expressed as the intersection of a nested sequence of manifolds $\{N_i\}$ where each N_i is homeomorphic to N. The N_i are obtained by shrinking in on the collar of $\dot{N}$ which is the complement of D in N. Since N has the monotone intersection property, then $N_1 - \bigcap_{i=1}^{\infty} N_i$ is homeomorphic to $\dot{N}_1 \times [0, 1)$ and since $N_1 - \bigcap_{i=1}^{\infty} N_i = N_1 - D_1$ it follows that N is trivially embedded in M_1.

COROLLARY 3.6. *If a manifold N has the monotone intersection property then it also has the monotone union property.*

Proof. This follows from Theorem 3.5 and Theorem 2.3.

Question. Are the monotone union and monotone intersection properties equivalent for compact topological manifolds with boundary?

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