SOME COUNTABILITY CONDITIONS IN A COMMUTATIVE RING

BY

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Introduction

Let R denote a commutative ring with identity and let $R[[X]]$ denote the ring of formal power series over R in an indeterminate X. If A is an ideal of R, then two naturally associated ideals of $R[[X]]$ are (1) $AR[[X]]$, the ideal of $R[[X]]$ generated by A, and (2) $A[[X]]$, the set of power series of $R[[X]]$ with coefficients all in A. It is clear that $AR[[X]]$ is contained in $A[[X]]$ and that if A is a finitely generated ideal, then equality holds. In general, however, $AR[[X]]$ may not be equal to $A[[X]]$, and it is noted in [19, p. 386] that the equality $AR[[X]] = A[[X]]$ holds precisely if the ideal A satisfies the following condition:

($*$) *If B is a countably generated ideal contained in A, then there exists a finitely generated ideal containing B and contained in A.*

We shall say that A is a ($*$)-*ideal* if the preceding condition is satisfied. It is clear that finitely generated ideals are ($*$)-ideals and that a countably generated ($*$)-ideal is finitely generated. An example in [19, p. 386, footnote 2] shows, however, that a ($*$)-ideal need not be finitely generated. If A is a ($*$)-ideal, then it can be shown that A satisfies the following condition (see Proposition 1.1).

($**$) *If $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending sequence of ideals of R such that $\bigcup_{i=1}^{\infty} A_i = A$, then $A = A_i$ for some i.*

If A satisfies ($**$), then we call A a ($**$)-*ideal*. It is easy to see that R satisfies the ascending chain condition on ideals (that is, R is Noetherian) if and only if every ideal of R is a ($*$)-ideal if and only if every ideal of R is a ($**$)-ideal (see Proposition 1.2).

In analogy with a theorem of I. S. Cohen [10, Theorem 2] to the effect that R is Noetherian if each prime ideal of R is finitely generated, Arnold in [4, p. 20] asked if R is Noetherian, provided that each prime ideal is a ($*$)-ideal. Theorem 2.3 shows that the answer to Arnold's question is affirmative. We prove in Example 2.4 that the corresponding question for ($**$)-ideals has a negative answer—that is, there exist non-Noetherian rings in which each prime ideal is a ($**$)-ideal. Thus, in particular, there exist ($**$)-ideals that are not

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(*)-ideals. In order to examine the possibility of a local version of our result on (*)-ideals being finitely generated, we examine in Section 3 (*)- and (**)-ideals of a ring with a linearly ordered ideal system and show, for example, that a prime (*)-ideal may be the radical of a finitely generated ideal and yet not be finitely generated.

In order to facilitate the proofs of our main results concerning (*)- and (**)-ideals, it is useful to define the (*)- and (**)-concepts for modules; this we do in Section 1, where several preliminary results are stated as well.

The concluding section of the paper, Section 4, is primarily concerned with the problem of determining what rings R have the property that each (**)-module over R is finitely generated. Our main results, Theorems 4.2 and 4.10, show that Noetherian rings and finite-dimensional valuation rings have the property.

All rings considered in this paper are assumed to be commutative and to contain an identity element; this is true even in our remarks in the introduction to Section 3, where a number of the sources cited consider noncommutative rings as well. All modules are assumed to be unitary, and if R is a subring of the ring S, then we assume that R and S have the same identity element.

1. Preliminaries

Let M be a module over the ring R. Extending the definitions of the Introduction, we say that M is a *(*)-module* if each countably generated submodule of M is contained in a finitely generated submodule of M; similarly, M is a *(**)-module* if M cannot be expressed as the union of a countably infinite strictly ascending sequence $M_1 < M_2 < \cdots$ of submodules of M. Our first result is that the second of these notions is implied by the other.

PROPOSITION 1.1. *A (*)-module is a (**)-module.*

Proof. If M is not a (**)-module, then let $\{M_i\}_{i=1}^{\infty}$ be a strictly ascending sequence of submodules of M such that $M = \bigcup_{i=1}^{\infty} M_i$. Choose $m_i \in M_{i+1} - M_i$ for each i. The countably generated submodule $N = (\{m_i\}_{i=1}^{\infty})$ of M is contained in no M_k, but each finitely generated submodule of M is contained in some M_k. Therefore N is not contained in a finitely generated submodule of M, and M is not a (*)-module.

It is clear that a countably generated (**)-module is finitely generated. This observation immediately implies that the conditions (*), (**), and Noetherian are globally equivalent.

PROPOSITION 1.2. *The following conditions are equivalent for an R-module M.*

(1) *Each submodule of M is a (*)-module.*
(2) *Each submodule of M is a (**)-module.*
(3) *M is Noetherian.*

Since, for R Noetherian, finite generation of M is equivalent to the condition that M is Noetherian, the next result follows from Proposition 1.2.

COROLLARY 1.3. *Let M be an R-module, where R is a Noetherian ring.*

(1) *M is a $(*)$-module if and only if M is finitely generated, in which case each submodule of M is also a $(*)$-module.*
(2) *Each submodule of M is a $(**)$-module if and only if M is finitely generated.*

One of the main results of Section 4 is the analogue of (1) of Corollary 1.3 for the $(**)$-condition (Theorem 4.2)—that is, a $(**)$-module over a Noetherian ring is finitely generated; the proof is considerably more complicated for the $(**)$-condition.

We record in the next result some basic properties of the $(*)$- and $(**)$-conditions with respect to exact sequences and quotient ring formation. The proof of Proposition 1.4 is routine and will be omitted.

PROPOSITION 1.4. *Let M be an R-module, let N be a submodule of M, and let S be a multiplicative system in R.*

(1) *If M is a $(*)$-module, then so is M/N; the analogous statement for $(**)$-modules is also valid.*
(2) *If N is finitely generated and if M/N is a $(*)$-module, then M is a $(*)$-module.*
(3) *If N and M/N are $(**)$-modules, then so is M. Thus if M is the direct sum of a finite family $\{M_i\}_{i=1}^k$ of submodules, then M is a $(**)$-module if and only if each M_i is a $(**)$-module.*
(4) *Conditions $(*)$ and $(**)$ are inherited by the R_S-module M_S from the R-module M.*

Additional questions arise from the statement of Proposition 1.4, and we have considered some of these questions. But a detailed investigation of the conditions $(*)$ and $(**)$ for modules is largely irrelevant to our two primary questions of concern, as indicated in the Introduction:

(1) *If each prime ideal of R is a $(*)$-ideal, is R Noetherian?*
(2) *What rings R are such that each $(*)$-module (or $(**)$-module) over R is finitely generated?*

We turn our attention to the first of these questions in the next section.

2. R is Noetherian if each prime ideal of R is a $(*)$-ideal

In Theorem 2.3 we establish the result that is the title of this section; we subsequently prove in Example 2.4 that the analogous statement for $(**)$-ideals is false. The proof of Theorem 2.3 uses two lemmas.

LEMMA 2.1. *Assume that A and B are ideals of R, that B is a $(*)$-ideal, and that B is contained in* rad A. *Then $B^n \subseteq A$ for some positive integer n.*

Proof. Suppose that $B^n \nsubseteq A$ for each positive integer n. Then there exist elements $b_{1n}, \ldots, b_{nn} \in B$ such that the product $b_{1n} \cdots b_{nn}$ is not in A. Let B' be the ideal of R generated by all the elements b_{ij}. Then B' is a countably generated ideal of R contained in B and no power of B' is contained in A. But B is a $(*)$-ideal, so there exists a finitely generated ideal C such that $B' \subseteq C \subseteq B$; and C finitely generated with $C \subseteq \operatorname{rad} A$ implies that some power of C, say C^n, is contained in A. Hence $(B')^n \subseteq C^n \subseteq A$, a contradiction.

LEMMA 2.2. *Assume that B is a $(*)$-ideal of the ring R and that $(*)$-modules over the ring R/B are finitely generated. If there exists a countably generated ideal A contained in B such that $B \subseteq \operatorname{rad} A$, then B is finitely generated.*

Proof. Since B is a $(*)$-ideal, we may assume that the ideal A is actually finitely generated. By Lemma 2.1, the ideal B^n is contained in A for some positive integer n. Also, B/B^2 is a $(*)$-module over R/B, and hence is finitely generated. Therefore there exists a finitely generated ideal C of R such that $B = B^2 + C$, from which it follows that $B = B^k + C$ for each positive integer k. In particular, $B = B^n + C \subseteq A + C \subseteq B$, and $B = A + C$ is finitely generated.

THEOREM 2.3. *If each prime ideal of R is a $(*)$-ideal, then R is Noetherian.*

Proof. The proof is conveniently divided into two parts, the first part being to show that each prime ideal of R is the radical of a finitely generated ideal. This is a condition equivalent to R satisfying the ascending chain condition on radical ideals (that is, to R having Noetherian spectrum) [36, Corollary 2.4]. Here by a radical ideal we mean an ideal that is equal to its radical. A proof of this result can be obtained from a careful reading of Arnold's proof of Theorem 1 of [3]; we give here a self-contained proof for the sake of completeness. Suppose that in R there exists an infinite strictly ascending sequence $A_1 < A_2 < \cdots$ of radical ideals. We show that this implies the existence of a prime ideal of R that is not a $(*)$-ideal. Let $A = \bigcup_{i=1}^{\infty} A_i$ and choose $a_i \in A - A_i$ for each i. If $\alpha = \sum_{i=1}^{\infty} a_i X^{i!}$ and if k is a positive integer, then the coefficient of $X^{k(i!)}$ in α^k, for each $i \geq k$, is a_i^k; consequently, α is an element of $A[[X]]$ such that no power of α is in $AR[[X]] \subseteq \bigcup_{i=1}^{\infty} A_i[[X]]$. Hence there exists a prime ideal Q of $R[[X]]$ such that $AR[[X]] \subseteq Q$, but $A[[X]] \nsubseteq Q$. Let $P = Q \cap R$. Then P is a prime ideal of R with $A \subseteq P$, so $A[[X]] \subseteq P[[X]]$. Since $PR[[X]] \subseteq Q$, we conclude that $PR[[X]] \neq P[[X]]$ and P is not a $(*)$-ideal. We have thus shown that each prime ideal of R is the radical of a finitely generated ideal.

Assume that R is not Noetherian. Among the prime ideals of R that are not finitely generated we may choose a prime P maximal with this property. Then R/P is a Noetherian ring since prime ideals of R/P are finitely generated. Hence $(*)$-modules over R/P are finitely generated, and since P is the radical of a finitely generated ideal, Lemma 2.2 implies that P is finitely generated. This contradiction completes the proof of Theorem 2.3.

We remind the reader that Arnold's phrasing of the question of whether R is Noetherian if each prime ideal of R is a $(*)$-ideal was the following: If $PR[[X]] = P[[X]]$ for each prime ideal P of R, is R Noetherian? In view of this, the presence of power series in the proof of Theorem 2.3 may seem more natural than it otherwise would on the surface.

We next present an example of a ring R_1 that is not Noetherian (in fact, R_1 does not have Noetherian spectrum), although each prime ideal of R_1 is a $(**)$-ideal.

Example 2.4. Let k be a field and let N be an infinite set. Let $R_1 = k^N$ denote the ring of all functions from N to k, with addition and multiplication defined coordinatewise.[3] For example, for $f, g \in R_1$, $(f \cdot g)(n)$ is defined to be $f(n) \cdot g(n)$ for each $n \in N$. Since N is infinite, it is clear that R_1 is not Noetherian. For any $f \in R_1$, let

$$\mathcal{N}(f) = \{n \in N \mid f(n) \neq 0\}$$

be the *support* of f. Note that $f \in gR_1$ if and only if $\mathcal{N}(f) \subseteq \mathcal{N}(g)$. It is well known and easily seen that every finitely generated ideal of R_1 is principal. Indeed, $(f, g) = (h)$ if and only if $\mathcal{N}(h) = \mathcal{N}(f) \cup \mathcal{N}(g)$. It follows that if B is an ideal of R_1, if $f \in B$, and if $g \notin B$, then $h \notin B$ for any h such that $\mathcal{N}(h)$ contains $\mathcal{N}(g) - \mathcal{N}(f)$. Let $A_1 < A_2 < \cdots$ be a countably infinite strictly ascending sequence of ideals of R_1. We wish to show that $A = \bigcup_{i=1}^{\infty} A_i$ is not prime. There exist elements $f_i \in A_{i+1} - A_i$ for each positive integer i, and we may choose f_i with this property so that also $f_j \in f_i R_1$ for $j \leq i$. Thus $\mathcal{N}(f_j) \subseteq \mathcal{N}(f_i)$ for $j \leq i$. For notational convenience we write $0 = f_0$ and define functions g and h as follows: g restricted to $\mathcal{N}(f_{2n}) - \mathcal{N}(f_{2n-1})$ agrees with f_{2n}, $n = 1, 2, \ldots$; g restricted to $\mathcal{N}(f_{2n-1}) - \mathcal{N}(f_{2n-2})$ is 0, $n = 1, 2, \ldots$; h restricted to $\mathcal{N}(f_{2n}) - \mathcal{N}(f_{2n-1})$ is 0, $n = 1, 2, \ldots$; h restricted to $\mathcal{N}(f_{2n-1}) - \mathcal{N}(f_{2n-2})$ agrees with f_{2n-1}, $n = 1, 2, \ldots$. We define both g and h to be zero at each element of $N - (\bigcup_{i=1}^{\infty} \mathcal{N}(f_i))$. By our construction $gh = 0 \in A$, while $g \notin A_{2n}$ and $h \notin A_{2n-1}$ for each n. Thus, neither g nor h is in A; hence A is not prime. We conclude that every prime ideal of R_1 is a $(**)$-ideal. Since R_1 is not Noetherian, Theorem 2.3 implies there exist in R_1 prime ideals that are not $(*)$-ideals, and hence examples of $(**)$-ideals that are not $(*)$-ideals.

Rings such as the ring $R_1 = k^N$ in Example 2.4 belong to a well-known class of rings called *absolutely flat rings* (or *von Neumann regular rings*). Such rings are characterized by the property that localization at any prime ideal yields a field. We remark that an absolutely flat ring need not have the property that its prime ideals are $(**)$-ideals, for there exist countable non-Noetherian absolutely flat rings; such a ring has a prime ideal P that is countably generated

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[3] In the terminology of [7, Definition 9, p. 131], R_1 is the *product* of the family $\{k_\alpha\}_{\alpha \in N}$ of rings, where $k_\alpha = k$ for each α; with the same notation, the terminology of [15, Exercise 13, p. 23] for R_1 *is the complete direct sum* of the family $\{k_\alpha\}_{\alpha \in N}$.

but not finitely generated, and such a P is not a $(**)$-ideal. (In Example 2.4, if the set N is countable, then the k-subalgebra of R_1 generated by the identity element of R_1, together with the set of all functions f such that $\mathcal{N}(f)$ is finite, provides an example of a countable non-Noetherian absolutely flat ring.)

If each prime ideal of the ring R is a $(**)$-ideal, then clearly R satisfies the ascending chain condition for prime ideals; on the other hand, Example 2.4 shows that R need not have Noetherian spectrum.[4] In this connection we pose two questions.

(Q1) *If R has Noetherian spectrum and if each prime ideal of R is a $(**)$-ideal, is R Noetherian?*

(Q2) *If P and A are ideals of R such that P is a prime $(**)$-ideal and $P \subseteq$ rad A, does A contain a power of P?*

It is clear that questions (Q1) and (Q2) are related; in fact (Q1) has an affirmative answer if the answer to (Q2) is affirmative. To see this we need to know that $(**)$-modules over a Noetherian ring are finitely generated (Theorem 4.2). An affirmative answer to (Q2) implies that Lemma 2.2 generalizes to the case where $(*)$ is replaced throughout by $(**)$ and B is assumed to be prime; using this generalized version of Lemma 2.2 and Theorem 4.2, an examination of the proof of Theorem 2.3 then yields an affirmative answer to (Q1). A case of (Q1) of particular interest is that in which R is a one-dimensional quasi-local domain; we recast the special case in slightly different phraseology.

(Q1)′ *Assume that D is a one-dimensional quasi-local domain with maximal ideal M. If M is a $(**)$-ideal, is M finitely generated?*

3. The case of a chained ring

As an illustrative example, we consider in this section $(*)$-ideals and $(**)$-ideals of rings in the class $\mathscr{C}$ of rings in which the ideals are linearly ordered. Since such rings have arisen fairly frequently in the recent literature (the subsequent discussion will confirm this), and since we were aware of certain conflicts in the terminology applied to elements of $\mathscr{C}$, it became a point of interest to us to determine as accurately as possible what the situation in this regard is. Kaplansky in [21, p. 479] and [22, p. 35] calls the elements of $\mathscr{C}$ *valuation rings*, and other recent papers [41, p. 1277], [32, p. 232], [40, p. 890], [39, p. 405] have used this terminology, citing [21, p. 479] as a reference. This use of the term valuation ring conflicts with the meaning of the term (*Bewertungsring*, in German) introduced by Krull [24, pp. 164–5] (see also [25, Section 40]) to mean an integral domain in the class $\mathscr{C}$ (for integral domains in $\mathscr{C}$, Kaplansky uses the term *valuation domain* in [22, p. 35]). For commutative rings, Krull's

[4] If R satisfies the ascending chain condition for prime ideals, then R has Noetherian spectrum if and only if each ideal of R has only finitely many minimal prime ideals [34, Sätze 15 and 16].

definition has been the traditional meaning of the term valuation ring; on the other hand, Kaplansky does not restrict to commutative rings in his definition in [21, p. 479], and Schilling [37, p. 298] had previously used the term to include certain noncommutative rings (see also [38, p. 9]). Citing Skornjakov [42], Clark and Drake in [8, pp. 148–9] and Clark and Liang in [9, p. 445] call the elements of $\mathscr{C}$ *chain rings*[5]; *chained ring* is the terminology of Gilmer in [15, Exercise 8, p. 184]. Finally, Warfield refers to the elements of $\mathscr{C}$ as *generalized valuation rings* in [46, p. 167] and as *serial rings* in [47, p. 167].[6]

The class $\mathscr{C}^*$ consisting of elements of $\mathscr{C}$ that are principal ideal rings has also been considered in several contexts during the last sixty years, again with no uniformity in terminology. The integral domains in $\mathscr{C}^*$ are, of course, the fields and discrete valuation rings[7] of rank 1. Working chronologically, and considering only rings in $\mathscr{C}^*$ with zero divisors, Fränkel used the term *einfach zerlegbar Ring* for such an element of $\mathscr{C}^*$ in [12, p. 172]; Krull used *Ring von spezillen Typ* in [26, pp. 16–17], *spezieller zerlegbarer Ring* in [28, p. 15] and in [27, p. 186], then switched to *primärer zerlegbarer Ring* in [25, Section 30]; Köthe's term in [23, pp. 33, 39] is *primärer einreihiger Ring*, while Snapper, in a paper based largely on [27], used the term *completely primary* PIR (PIR is an abbreviation for principal ideal ring) in [45, Section 2]; to Zariski and Samuel [48, p. 245] an element of $\mathscr{C}^*$ is a *special* PIR,[8] and to Ayoub it is a *homogeneous ring of type p* in [5, p. 249] and a *primary homogeneous p-ring* in [6, p. 383].[9]

In tracking down the information above we ran across a few papers in which modules M with a linearly ordered family of submodules are considered. For the reader's benefit, the MR reviewer of Skornjakov's article [44] on this topic uses the term *chained module* for M (see MR 39, No. 1500), while the Zbl. reviewer's translation of Skornjakov's term is *chainlike module* (see Zbl., Volume 174, p. 331).[10] Following Warfield (who considers modules over

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[5] We do not have access to [42], but the *Mathematical Reviews* reviewer of [42] uses *chained ring*, rather than chain ring, in the review. We find no review of [42] in *Zentralblatt für Mathematik*, but for the closely related paper [43] of Skornjakov, both the review in MR 34, no. 190 and in Zbl., Volume 199, p. 77 use the term chained ring for an element of $\mathscr{C}$. Skornjakov considers noncommutative rings in both [42] and [43], as do Clark and Drake in [8]; all rings considered by Clark and Liang in [9] are assumed to be commutative.

[6] Precisely, Warfield in [47] defines a (possibly noncommutative ring) E to be a *serial ring* if each left ideal and each right ideal of E is a finite direct sum of submodules each having a linearly ordered system of submodules.

[7] Even in the terminology of [22], where a valuation ring need not be an integral domain, the term *discrete valuation ring* means a one-dimensional principal ideal domain in the class $\mathscr{C}$—see page 67 of [22].

[8] This term for the elements of $\mathscr{C}^*$ has probably been used more frequently than any other; however, we will not attempt to document this statement.

[9] Ayoub considers only finite rings in [5], but in [6] infinite rings are also considered.

[10] Each of these reviews is written in English, while the paper [44] is in Russian; there is an inference from the two reviews of [44] that Skornjakov would use the term *semichained module* (in English), or *halbkettig Modul* (in German), for a module that is a finite direct sum of submodules, each with a linearly ordered family of submodules. It must be apparent to the reader by now that none of the authors reads Russian.

(possibly) noncommutative rings in [47]), Lewis and Shores in [40, p. 889] call a module like M a *serial module*; they indicate that Albu and Nastasescu consider such modules in [1], which is the preprint form of [2]. Albu and Nastasescu use no special term in [2] for a module with a linearly ordered family of submodules.

In view of the above, or perhaps because Krull remains a mathematical idol of the present authors, we choose to call elements of $\mathscr{C}$ *chained rings*, and we use the term *valuation ring* as Krull used it—to mean an integral domain in the class $\mathscr{C}$. Given the choice of the term chained ring for elements of $\mathscr{C}$, consistency dictates our choice of the term *chained module* for a module with a linearly ordered family of submodules.

The proof of the first result of the section is straightforward and will be omitted.

PROPOSITION 3.1. *Let M be a chained module over a ring R. Then M is a $(*)$-module if and only if M is a $(**)$-module if and only if M is either cyclic or not countably generated.*

PROPOSITION 3.2. *Let R be a chained ring and let P be a prime ideal of R that is not principal. Then P is a $(*)$-ideal if and only if either* (1) *the set of prime ideals of R properly contained in P, ordered by inclusion, is nonempty and has no countable cofinal subset, or* (2) *PR_P is principal and the set of prime ideals of R properly containing P, ordered under reverse containment, has no countable cofinal subset.*

There are two points in the proof of Proposition 3.2 at which we need to know that a containment relation of the form $uvR \subseteq vR$ is proper; this fact follows in each case from a basic lemma.

LEMMA 3.3. *Assume that u and v are nonunits of the chained ring R and that v is nonzero. Then $uvR < vR$.*

Proof of Lemma 3.3. The proof uses only the fact that u is in the Jacobson radical of R. Thus, if v is in uvR, then $v = ruv$ for some r in R, whence $v(1 - ru) = 0$ and $v = 0$ since $1 - ru$ is a unit of R. This contradiction establishes the lemma.

Proof of Proposition 3.2. Assume that (1) is satisfied. If $\{x_i\}_{i=1}^{\infty}$ is a countable subset of P, then the ideal

$$B_k = (x_1, x_2, \ldots, x_k)$$

is principal for each k, and hence $B_k < P$. Let $P_k = \operatorname{rad} B_k$. We observe that P is not the radical of a principal ideal xR; otherwise the set $\{Q\}$, where Q is the (unique) prime ideal of R maximal with respect to not containing x, is a countable cofinal family of the primes of R properly contained in P. Therefore $P_k < P$ for each k, and (1) implies that $(\bigcup_{i=1}^{\infty} P_k) < P$. In particular,

$(\{x_i\}_1^\infty) < P$ and P is not countably generated. Therefore P is a $(*)$-ideal by Proposition 3.1, and (1) implies that P is a $(*)$-ideal.

If P satisfies (2), then since P is not principal, P is nonzero and we can choose a nonzero element $x \in P$ such that $xR_P = PR_P$. To show that P is not countably generated it will suffice to show that if $\{y_i\}_{i=1}^\infty$ is a countable subset of P such that $xR < y_iR$ for each i, then $(y_1, y_2, \ldots)$ is properly contained in P. Since $xR < y_iR$, there exists $a_i \in R$ with $x = a_iy_i$. We note that $a_i \notin P$, for $xR_P = y_iR_P$ implies there exists $s_i \in R - P$ with $s_iy_i \in xR$, and $a_i \in P$ would imply, since the ideals of R are linearly ordered, that $a_i = s_ir_i$ with $r_i \in P$. Thus we would have $xR = a_iy_iR = s_ir_iy_iR \subseteq r_ixR$, while Lemma 3.3 implies that the inclusion $r_ixR \subseteq xR$ is strict. Hence $a_i \notin P$ for each i. We prove, using (2), that $P < (\bigcap_1^\infty a_iR)$. Thus, let Q_i be the prime of R maximal with respect to not containing a_i. Then $P \subseteq Q_i < a_iR$ and the inclusion $P \subseteq Q_i$ is strict, for if not, then $\{\text{rad } a_iR\}$ is a countable cofinal family of the set of primes properly containing P, ordered under reverse containment. Thus $P < Q_i$ for each i, and (2) implies that $\bigcap_1^\infty Q_i$ properly contains P; in particular $P < \bigcap_1^\infty a_iR$ and we can choose $s \in (\bigcap_1^\infty a_iR) - P$. Then $s^2 \notin P$ so $x \in s^2R$, say $x = s^2y$. Since P is prime, $y \in P$. Moreover, y is not in the ideal $(y_1, y_2, \ldots)$, for $y \in y_iR$ would imply $xR = s^2yR < syR \subseteq a_iy_iR = xR$, which would say that the ideal xR is properly contained in itself. We conclude that (2) also implies that P is a $(*)$-ideal.

Conversely, if P is a $(*)$-ideal, then PR_P is a $(*)$-ideal of R_P. Hence if P does not satisfy (1), then an argument similar to that used in the first paragraph of the proof shows that PR_P is the radical of a countably generated ideal of R_P. Lemma 2.2 then implies that PR_P is principal, say $PR_P = xR_P$, where x is nonzero. If P does not satisfy (2), then there exists a countable subset $\{s_i\}_{i=1}^\infty$ of $R - P$ such that $\bigcap_1^\infty s_iR = P$. We show that this implies that P is countably generated. But this leads to the contradiction that P is principal since P is assumed to be a $(*)$-ideal. Since the ideals of R are linearly ordered $x \in s_iR$ for each i, say $x = s_ix_i$; and since P is prime, $x_i \in P$ for each i. We claim that $P = (x_1, x_2 \ldots)$. If $y \in P$ and $y \notin x_iR$ for all i, then $y \notin xR$ so $x = yr$ with r a nonunit of R. Moreover, $xR_P = yR_P$ implies as above that $r \notin P$. Hence $r^2 \notin P$ and $s_i \in r^2R$ for some i. This yields $xR = s_ix_iR \subseteq r^2yR < ryR = xR$, a contradiction. This completes the proof of Proposition 3.2.

To construct an example where (2) of Proposition 3.2 is satisfied, one can begin with a valuation ring V as in [14, p. 1139], where (0) cannot be expressed as a countable intersection of nonzero principal ideals. Let K denote the quotient field of V and construct a rank one discrete valuation ring of the form $W = K + M$, where M is the maximal ideal of W (for example, take $W = K[[X]]$). Then $R = V + M$ is a valuation ring and M is a prime $(*)$-ideal of R satisfying Condition (2) of Proposition 3.2 [16, Theorem A, p. 560].

One can also in this way readily obtain an example of a valuation ring R in which every prime $(*)$-ideal is finitely generated, but in which there exist

$(*)$-ideals that are not finitely generated: for V and K as above, construct a rank one valuation ring $W = K + M$ such that the maximal ideal M of W is not principal. Using Proposition 3.2, we see that (0) and the maximal ideal of R are the only prime ideals of R that are $(*)$-ideals (in the construction in [14], if P is a nonzero prime of V, then the set of prime ideals of V containing P is countable; moreover, the maximal ideal of V is principal), and each of these ideals is finitely generated. On the other hand, if x is a nonzero element of M, then xW, regarded as an ideal of R, is a nonfinitely generated $(*)$-ideal; this is true because W, regarded as an R-module, is a nonfinitely generated $(*)$-module, which in turn can be seen from the fact that $W/M \simeq K$ is a nonfinitely generated $(*)$-module over $R/M \cong V$. That K is not finitely generated as a V-module is clear. Moreover, since V is a valuation ring with quotient field K, the V-submodules of K are linearly ordered; K is not countably generated as a V-module by the construction of V, and hence K is a $(*)$-module over V by Proposition 3.1. It can be shown that each ideal of V is countably generated, and hence V provides an example of a ring in which $(*)$- and $(**)$-ideals are finitely generated, but over which there exists a $(*)$-module that is not finitely generated.

4. $(**)$-modules over a Noetherian ring are finitely generated

We have already observed (Corollary 1.3) that a $(*)$-module over a Noetherian ring is finitely generated. The main result of this section, Theorem 4.2, asserts that the analogous statement for $(**)$-modules is also valid. This theorem was used in observing that question (Q1) of Section 2 has an affirmative answer if the answer to question (Q2) is affirmative.

Results of Section 3 show that $(**)$-modules (or $(*)$-modules) need not be finitely generated, and hence, after proving Theorem 4.2, we consider in more detail the class $\mathscr{F}$ of rings R such that each $(**)$-module over R is finitely generated; our main results along these lines are contained in Theorems 4.7 and 4.10. Actually, there are three other classes of rings that naturally arise at this point: the class $\mathscr{F}_1$ of rings over which each $(*)$-module is finitely generated and the classes $\mathscr{F}_2, \mathscr{F}_3$ of rings over which each $(**)$-ideal, or $(*)$-ideal, is finitely generated. The inclusion relations $\mathscr{F} \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3$ and $\mathscr{F} \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_3$ are obvious, and we observe later in the section that for N countable, the ring $R_1 = k^N$ of Example 2.4 is in $\mathscr{F}_3 - \mathscr{F}_2$. We do not deal in depth with each of these classes, however, and most of the considerations of the section concern the class $\mathscr{F}$. Theorem 4.2 will follow fairly easily from our first result of the section, Proposition 4.1.

PROPOSITION 4.1. *Let M be a module over the ring R such that M is not finitely generated and let $\{m_i\}_{i=0}^{\infty}$ be a subset of M such that $m_{i+1} \notin Rm_1 + \cdots + Rm_i$ for each positive integer i. Then there exists a strictly ascending sequence*

$M_1 < M_2 < \cdots$ *of submodules of* M *such that*

(1) $m_{i+1} \in M_{i+1} - M_i$ *for each* i,

(2) *if* $N = \bigcup_1^\infty M_i$, *then* M/N *is a torsion module*,

and

(3) *for each* m *in* $M - N$, *the ideal* $N\colon (m) = \{x \in R \mid xm \in N\}$ *is not finitely generated.*

Proof. Let $A_k = Rm_1 + \cdots + Rm_k$ for each positive integer k. Inductively, we choose a collection $\{H_k\}_{k=0}^\infty$ of submodules of M as follows. Let $H_0 = (0)$. Having chosen $H_0, \ldots, H_{k-1}$, we let $\mathscr{S}_k = \{H \mid H$ is a submodule of M, $H_{k-1} \subseteq H$ and $H + A_k < H + A_{k+1} < H + A_{k+2} < \cdots\}$. The set $\mathscr{S}_k$ is partially ordered with respect to inclusion and is nonempty since $H_{k-1} \in \mathscr{S}_k$. Let $\mathscr{T} = \{N_\lambda\}_{\lambda \in \Lambda}$ be a chain of elements of $\mathscr{S}_k$ and set $N = \bigcup_{\lambda \in \Lambda} N_\lambda$. If there exists an integer $i \geq 0$ such that $N + A_{k+i} = N + A_{k+i+1}$, then $m_{k+i+1} = n + b_{k+i}$ for some $n \in N$ and $b_{k+i} \in A_{k+i}$. But $n \in N_\lambda$ for some $\lambda \in \Lambda$ so $m_{k+i+1} \in N_\lambda + A_{k+i}$. It follows that $N_\lambda + A_{k+i} = N_\lambda + A_{k+i+1}$, contrary to the fact that $N_\lambda \in \mathscr{S}_k$. We conclude that $N \in \mathscr{S}_k$. This shows that $\mathscr{S}_k$ is inductive, so by Zorn's Lemma $\mathscr{S}_k$ contains a maximal element H_k. By choice of H_k we have $H_k + A_k < H_k + A_{k+1} \subseteq H_{k+1} + A_{k+1}$ so it follows that $H_1 + A_1 < H_2 + A_2 < \cdots$ is a countably infinite strictly ascending sequence of submodules of M. Let $M_k = H_k + A_k$ for each k; we show that Conditions (1), (2), and (3) are satisfied for the sequence $\{M_k\}_1^\infty$.

The strict inclusion $H_k + A_k < H_k + A_{k+1}$ implies that m_{k+1} is in $M_{k+1} - M_k$ for each k. If $m \in M - N$, then in particular, $m \notin H_1$, so by choice of H_1, we have $H_1 + Rm + A_i = H_1 + Rm + A_{i+1}$ for some positive integer i; this implies that we can write m_{i+1} as $h_1 + rm + c_i$ for some $h_1 \in H_1$, $r \in R$, and $c_i \in A_i$. Then

$$rm = m_{i+1} - h_1 - c_i \in H_1 + A_{i+1} \subseteq M_{i+1},$$

and rm is nonzero (hence $r \neq 0$) because m_{i+1} is not in M_i, while $h_1 + c_i$ is in M_i. This proves that M/N is a torsion module. To prove (3), we use a proof by contradiction. Thus, assume that $m \in M - N$ and that $N\colon (m) = (b_1, \ldots, b_t)$ is finitely generated. There exists a smallest positive integer k such that $b_1 m, \ldots, b_t m \in M_k$. Since $m \notin H_k$, we have

$$H_k + Rm + A_{k+i} = H_k + Rm + A_{k+i+1}$$

for some integer $i \geq 0$, so we can write $m_{k+i+1} = h_k + rm + c_{k+i}$ for some $h_k \in H_k$, $r \in R$, and $c_{k+i} \in A_{k+i}$. Then

$$rm \in H_k + A_{k+i+1} \subseteq M_{k+i+1},$$

so $r \in N\colon (m)$. If $r = s_1 b_1 + \cdots + s_t b_t$, where $s_1, \ldots, s_t \in R$, then

$$\begin{aligned} m_{k+i+1} &= h_k + (s_1 b_1 + \cdots + s_t b_t)m + c_{k+i} \\ &= s_1 b_1 m + \cdots + s_t b_t m + h_k + c_{k+i} \in M_k + H_k + A_{k+i} \subseteq M_{k+i}. \end{aligned}$$

This contradicts the fact that $M_{k+i} = H_{k+i} + A_{k+i}$ is properly contained in $H_{k+i} + A_{k+i+1}$. Therefore N: (m) is not finitely generated, and the proof of Proposition 4.1 is complete.

THEOREM 4.2. *A $(**)$-module over a Noetherian ring is finitely generated.*

Proof. Assume that M is a module over a Noetherian ring and that M is not finitely generated. Choose a sequence $\{m_i\}_1^\infty$ of elements of M such that $m_{k+1} \notin Rm_1 + \cdots + Rm_k$ for each k, and choose a strictly ascending sequence $\{M_i\}_1^\infty$ of submodules of M satisfying conditions (1)–(3) of Proposition 4.1. Because R is Noetherian, (3) implies that $M = N = \bigcup_{i=1}^\infty M_i$. Therefore M is finitely generated.

One of the first test cases of Theorem 4.2 that we considered was that of abelian groups. Paul Hill pointed out to us that in that case Theorem 4.2 follows from a result of Kulikov [30], [31, p. 175], [13, Corollary 18.4] to the effect that each abelian group G is the union of an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of subgroups, where each G_i is a direct sum of cyclic groups.

In connection with Theorem 4.2, Paul Eakin asked if a nonfinitely generated module over a Noetherian ring will always have as a homomorphic image a module that is countably generated but not finitely generated. Ed Enochs, using results of Matlis on injective envelopes of simple modules over Noetherian rings, answered Eakin's question in the affirmative. This yields, of course, another proof that $(**)$-modules over a Noetherian ring are finitely generated.

We proceed to show that certain classes of non-Noetherian rings also have the property described in Theorem 4.2.

LEMMA 4.3. *Assume that A is an ideal of the ring R such that $A = \bigcup_{i=1}^\infty A_i$, where $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending sequence of nilpotent ideals of R. If R/A is in the class $\mathscr{F}$, then so is R.*

Proof. Let M be a $(**)$-module over R. Since R/A is in $\mathscr{F}$, it follows that M/AM is finitely generated, so there exists a finitely generated submodule B of M such that $M = B + AM$. The equality $A = \bigcup_1^\infty A_i$ implies that $AM = \bigcup_1^\infty A_iM$, and hence $M = \bigcup_1^\infty (B + A_iM)$. Since M is a $(**)$-module, it follows that $M = B + A_iM$ for some i. Then for each positive integer k, $M = B + A_i^kM$, and since A_i is nilpotent, it follows that $M = B$.

COROLLARY 4.4. *Assume that R is a 0-dimensional quasi-local ring with maximal ideal M.*

(1) *If M is nilpotent, then $R \in \mathscr{F}$.*
(2) *If R is a chained ring, then $R \in \mathscr{F}$.*

Proof. (1) is obvious from Lemma 4.3, and (2) follows from the same result since $M = \bigcup_{k=1}^\infty M_k$, where M_k is the ideal of R generated by $\{x \in R \mid x^k = 0\}$; that each M_k is nilpotent (in fact, $M_k^k = (0)$) depends upon the assumption that R is a chained ring.

The converse of Lemma 4.3 is valid with no restrictions on the ideal—that is, the homomorphic image of an element of $\mathscr{F}$ is in $\mathscr{F}$.

Either (1) or (2) of Corollary 4.4 is sufficient to show that the class $\mathscr{F}$ contains rings that are not Noetherian. For example, if K is a field, then $K[\{X_i\}_{i=1}^{\infty}]/(\{X_iX_j\})$ is a non-Noetherian ring satisfying the hypotheses of Corollary 4.4 and (1), while V/B, for V a rank one nondiscrete valuation ring and B an ideal of V that is not prime, is a non-Noetherian ring satisfying the hypotheses of Corollary 4.4 and (2). In Theorem 4.10 we extend (2) of Corollary 4.4 to the case of a chained ring of arbitrary finite dimension.

In [17], Gilmer considers two types of rings, which he calls W-rings and W^*-rings. We next show that each W^*-ring is in $\mathscr{F}$; the definitions are as follows. The ring R is a *W-ring* if each ideal of R is uniquely expressible as a finite irredundant intersection of primary ideals belonging to distinct prime ideals, and a *W^*-ring* is a W-ring in which each primary ideal contains a power of its radical.[11] The terms *W-domain* and *W^*-domain* mean an integral domain that is a W-ring or W^*-ring, respectively. Each W-ring is a finite direct sum of one-dimensional W-domains and zero-dimensional quasi-local rings, and conversely; moreover, the sum is a W^*-ring if and only if each of the summands is a W^*-ring. An integral domain D is a W-domain if and only if $\dim D \leq 1$ and each nonzero element of D belongs to only finitely many maximal ideals of D; to obtain a characterization of W^*-domains one merely tacks on the extra condition in the definition. For references to these results, see [35] and [17]. In the next result we use the notation Ann (x) to denote the *annihilator* of the element x of an R-module M—that is, Ann $(x) = \{r \in R \mid rx = 0\}$.

PROPOSITION 4.5. *Assume that N is a torsion module over a one-dimensional W-domain D. Let $\{P_\alpha\}_{\alpha \in A}$ be the set of maximal ideals of D, and for each $\alpha \in A$, let N_α be the submodule of N consisting of 0, together with all nonzero x such that P_α is the radical of the annihilator of x. Then N is the direct sum of its family $\{N_\alpha\}_{\alpha \in A}$ of submodules.*

Proof. It is routine to prove that each N_α is a submodule of N. If y is a nonzero element of N, then Ann (y) is a finite intersection $\bigcap_{i=1}^{k} Q_i$ of primary ideals, where $P_{\alpha_i} = \operatorname{rad} Q_i$. If $k = 1$, then clearly $y \in \sum_{\alpha \in A} N_\alpha$, and if $k > 1$, then we note that $A_1 + \cdots + A_k = D$, where $A_j = \bigcap_{i \neq j} Q_i$, since $A_1 + \cdots + A_k$ is contained in no maximal ideal of D. Thus if $1 = a_1 + \cdots + a_k$, where

[11] Krull in [29, p. 1] called a ring R a *Laskersch Ring* if each ideal of R is a finite intersection of primary ideals of R (the designation was for E. Lasker, who in [33] proved that a polynomial ring in finitely many indeterminates over a field has this property); for the same concept, Evans in [11, p. 507] and Gilmer in [15, Exercise 5, p. 455] use the term *Laskerian ring*, while Heinzer and Ohm use *Lasker ring* in [20, p. 74]. The distinguishing feature of W-rings among such rings is the uniqueness of the primary decomposition (assuming irredundance and distinct belonging prime ideals). A ring in which each ideal is a finite intersection of primary ideals and in which each primary ideal contains a power of its radical is called *strongly Laskerian* in [15, Exercise 5, p. 455].

$a_i \in A_i$, then $y = a_1 y + \cdots + a_n y$; moreover, $Q_j a_j y \subseteq \text{Ann}(y) \cdot y = (0)$ so that $a_i y \in N_{\alpha_i}$. It follows that $N = \sum N_\alpha$ in either case. To prove that the sum $\sum N_\alpha$ is direct, it suffices to prove that $N_{\alpha_0} \cap (N_{\alpha_1} + \cdots + N_{\alpha_t}) = (0)$ for each finite subset $\{\alpha_i\}_{i=0}^t$ of A. Thus, if $n_{\alpha_0} = \sum_{i=1}^t n_{\alpha_i}$, where $n_{\alpha_i} \in N_{\alpha_i}$, then $\text{Ann}(n_{\alpha_0})$ and $\text{Ann}(n_{\alpha_1}) \cap \cdots \cap \text{Ann}(n_{\alpha_t})$ annihilate n_{α_0}, and these two ideals generate D. Hence $n_{\alpha_0} = 0$ and N is the direct sum of the family $\{N_\alpha\}$, as asserted.

LEMMA 4.6. *Assume that M is a $(**)$-module over the domain J and that F is a free submodule of M such that $dM \subseteq F$ for some nonzero element d of J. Then F is finitely generated.*

Proof. If F were not finitely generated, then F could be expressed as $G \oplus Jx_1 \oplus Jx_2 \oplus \cdots$, where G is a free submodule of F and $\{x_i\}_{i=1}^\infty$ is a countably infinite free subset of F. Let

$$F_i = G \oplus Jx_1 \oplus \cdots \oplus Jx_i$$

for each i, and let $M_i = \{x \in M \mid dx \in F_i\}$. Each M_i is a submodule of M, $M_1 \subseteq M_2 \subseteq \cdots$, and $M = \bigcup_1^\infty M_i$ (since $F = \bigcup_1^\infty F_i$). Since M is a $(**)$-module, $M = M_k$ for some k. Therefore $F = M \cap F = M_k \cap F$, but $M_k \cap F = F_k$ since $\{x_i\}_1^\infty$ is free. This contradicts the fact that F_k is properly contained in F, and hence F is finitely generated, as asserted.

THEOREM 4.7. *A $(**)$-module over a W^*-ring is finitely generated.*

Proof. Let R be a W^*-ring, and express R as the direct sum of a family $\{R_j\}_{j=1}^m \cup \{D_i\}_{i=1}^n$ of zero-dimensional quasi-local W^*-rings R_j and one-dimensional W^*-domains D_i. It is straightforward to show that a finite direct sum of rings S_k is in the class $\mathscr{F}$ if and only if each S_k is in $\mathscr{F}$; hence we consider the cases where $R = R_j$ or $R = D_i$.

That each R_j is in $\mathscr{F}$ follows from (1) of Corollary 4.4; we prove directly that each $D = D_i$ is in $\mathscr{F}$. Let M be a $(**)$-module over D. There exists a free submodule F of M such that $N = M/F$ is a torsion D-module; also, N is a $(**)$-module. Let $\{P_\alpha\}_{\alpha \in A}$ be the set of maximal ideals of D, and let N_α be defined as in Proposition 4.5. Since N is the direct sum of the family $\{N_\alpha\}_{\alpha \in A}$ and since N is a $(**)$-module, only finitely many elements α of A are such that $N_\alpha \neq (0)$; let $\{\alpha_i\}_{i=1}^t$ be the subset of A consisting of such elements α. Each N_{α_i} is a $(**)$-module over D. For a fixed i and for a positive integer j, we let

$$T_{ij} = \{y \in N_{\alpha_i} \mid P_{\alpha_i}^j \text{ is contained in Ann}(y)\};$$

each T_{ij} is a submodule of N_{α_i}, $T_{i1} \subseteq T_{i2} \subseteq \cdots$, and since each primary ideal of D contains a power of its radical, $N_{\alpha_i} = \bigcup_{j=1}^\infty T_{ij}$. We conclude that for each i, there exists a positive integer k_i such that $N_{\alpha_i} = T_{ik_i}$—that is, $P_{\alpha_i}^{k_i}$ annihilates N_{α_i}. Therefore $\bigcap_{i=1}^t P_{\alpha_i}^{k_i}$ annihilates $N = N_{\alpha_1} + \cdots + N_{\alpha_t}$. Choose a nonzero element d in $\bigcap_1^t P_{\alpha_i}^{k_i}$. Then $dM \subseteq F$, and Lemma 4.6 implies that F is finitely generated. Now M/dM is a $(**)$-module over the zero-dimensional

W^*-ring D/dD, and hence by a case of Theorem 4.7 already proved, M/dM is finitely generated. Because F and $M/F \simeq (M/dM)/(F/dM)$ are finitely generated, we conclude that M is finitely generated. This completes the proof of Theorem 4.7.

Our final considerations of this section show that each finite-dimensional chained ring is in F. In Lemma 4.9, we use the term *pseudo-radical of* R to mean the intersection of the set of nonzero prime ideals of the ring R; see [18, p. 275], [15, Exercise 3, p. 58].

LEMMA 4.8. *Let A be a finitely generated ideal of the ring R. If M is a $(**)$-module over R, then AM is a $(**)$-module over R.*

Proof. Let $C_1 \subset C_2 \subset \cdots$ be a sequence of submodules of AM such that $AM = \bigcup_1^\infty C_i$. If we set $B_i = \{m \in M \mid Am \subset C_i\}$, then $B_1 \subseteq B_2 \subseteq \cdots$ is a chain of submodules of M. Suppose that $A = (a_1, \ldots, a_k)$ and let $m \in M$. Then for each i, $1 \leq i \leq k$, there exists C_{λ_i} such that $a_i m \in C_{\lambda_i}$. If $t = \max\{\lambda_1, \ldots, \lambda_k\}$, then $Am \subseteq C_t$—that is, $m \in B_t$. Thus $M = \bigcup_1^\infty B_i$. Since M is a $(**)$-module, $M = B_n$ for some n. Therefore, $AM = AB_n \subseteq C_n$, so $AM = C_n$.

LEMMA 4.9. *Let A be an ideal contained in the pseudo-radical of R, and assume that A is countably generated. If R/A is in $\mathscr{F}$, then so is R.*

Proof. If R is not an integral domain, then each element of A is nilpotent and the result follows from Lemma 4.3. Thus, we assume that R is an integral domain. By the case in which R is not a domain, it then follows that if A_1 is a nonzero ideal of R contained in A, then either $A_1 = A$ or A_1 is not prime, and in either case, R/A_1 is in $\mathscr{F}$.

Assume that R is not in $\mathscr{F}$ and let M be a $(**)$-module that is not finitely generated. Choose a nonzero element y in A. Then M/yM is a $(**)$-module over R/yR, and hence is finitely generated as a (R/yR)-module and as an R-module. Thus, there exists a finitely generated submodule B of M such that $M = B + yM$. It is also the case that $M = B + y^n M$ for each positive integer n. Since M is not finitely generated, it follows that $y^n M$ is contained in no finitely generated submodule of M. Thus, we choose $\{m_i\}_1^\infty \subset M$ such that

$$y^{i+1} m_{i+1} \notin Rym_1 + \cdots + Ry^i m_i \quad \text{for each } i.$$

By Proposition 4.1, there exists an infinite strictly ascending sequence $M_1 < M_2 < \cdots$ of submodules of M such that

$$y^{i+1} m_{i+1} \in M_{i+1} - M_i \quad \text{for each } i$$

and such that M/N is a torsion module, where $N = \bigcup_1^\infty M_i$. Since M is a $(**)$-module, $M \neq N$—that is, $M/N \neq (0)$. For each positive integer i, set $V_i = \{m \in M \mid y^i m \in N\}$. It is clear that $V_1 \subseteq V_2 \subseteq \cdots$ is a chain of submodules of M and we note that $M = \bigcup_1^\infty V_i$. For if $m \in M$, then $B =$

$\{x \in R \mid xm \in N\} \neq (0)$ since M/N is a torsion module. But then since y is in the pseudo-radical of R, $y^n \in B$ for some n—that is, $y^n m \in N$ and $m \in V_n$ as we wished to prove. Since M is a $(**)$-module, it follows that $M = V_n$ for some n—that is, $y^n M \subseteq N$. Therefore $y^n M = y^n M \cap N = \bigcup_1^\infty (y^n M \cap M_i)$. By Lemma 4.8, $y^n M$ is a $(**)$-module, so $y^n M = y^n M \cap M_t$ for some integer t; in other words, $y^n M \subseteq M_t$. If we set $s = \max\{n, t\}$, then $y^{s+1} m_{s+1} \in y^s M \subseteq y^n M \subseteq M_t \subseteq M_s$, contradicting the fact that $y^{s+1} m_{s+1} \notin M_s$. It follows that M is finitely generated and that R is in $\mathscr{F}$.

THEOREM 4.10. *Assume that R is a finite-dimensional chained ring. Then R is in $\mathscr{F}$.*

Proof. We use induction on the dimension of R, the result being true for R of dimension 0 by (2) of Corollary 4.4. If the result is true for chained rings of dimension less than k, where k is positive, and if R is a k-dimensional chained ring, then let $P_0 < P_1 < \cdots < P_k$ be the chain of proper prime ideals of R. Choose $y \in P_1 - P_0$. The ideal yR/P_0 of the valuation ring R/P_0 is contained in the pseudo-radical of R/P_0 and is countably generated. Since $(R/P_0)/(yR/P_0) \simeq R/yR$ is in $\mathscr{F}$ by the induction hypothesis, Lemma 4.9 implies that R/P_0 is in $\mathscr{F}$. As in Corollary 4.4, we have $P_0 = \bigcup_{k=1}^\infty M_k$, where M_k is the ideal of R generated by $\{x \in R \mid x^k = 0\}$, and $M_k^k = (0)$. Hence R is in $\mathscr{F}$ by Lemma 4.3.

Recall that we have observed in Section 3 that there exist valuation rings with $(**)$-ideals that are not finitely generated. In those examples the valuation rings have uncountably many prime ideals. It is not known to us if valuation rings with countably many prime ideals necessarily belong to $\mathscr{F}$.

We return briefly to our comments in the introduction to this section concerning the classes $\mathscr{F}$, $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$. As mentioned at the end of Section 3, there exists a valuation ring V such that $(*)$- and $(**)$-ideals of V are finitely generated, but such that the quotient field K of V is a nonfinitely generated $(*)$-module. This example shows that the inclusions $\mathscr{F} \subseteq \mathscr{F}_2$ and $\mathscr{F}_1 \subseteq \mathscr{F}_3$ are proper. If R_1 is the ring of Example 2.4, then an argument concerning supports of functions shows that $(*)$-ideals of R are finitely generated if and only if the set N is countable. Hence, for N countably infinite, R_1 is in the class $\mathscr{F}_3$ but not in the class $\mathscr{F}_2$. Thus the inclusion $\mathscr{F}_2 \subseteq \mathscr{F}_3$ is proper. We have not been able to determine whether the inclusion $\mathscr{F} \subseteq \mathscr{F}_1$ is proper—that is, whether there exists a ring R such that $(*)$-modules over R are finitely generated, but such that there exists a $(**)$-module over R that is not finitely generated. If R_1 is the ring of Example 2.4, with k the field of real numbers and N the set of positive integers, then Roger Wiegand pointed out to us that the existence of P-points in $\beta N - N$, where βN is the Stone-Čech compactification of N, implies that R_1 is not in the class $\mathscr{F}_1$. Assuming the continuum hypothesis, it is known that $\beta N - N$ contains P-points [W. Rudin, Homogeneity problems in the theory of Čech compactifications, Duke Math. J., 23 (1956), 409–419] or

[L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, 1960, p. 100], and thus R_1 is not in $\mathscr{F}_1$, modulo the continuum hypothesis.

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