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ON SUBSPACES OF SPACES WITH AN UNCONDITIONAL BASIS AND SPACES OF OPERATORS[1]

BY

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Abstract

A reflexive Banach space E has an unconditional finite dimensional expansion of the identity iff E has the approximation property and E is a subspace of a space with an unconditional basis. More results are given in the non-reflexive case. The results are applied to show that the non-complementation of $C(E, F)$ in $L(E, F)$ is equivalent to $C(E, F) \neq L(E, F)$ in certain cases such as: E is reflexive, E or F has the b.a.p. and F is a subspace of a space with an unconditional basis.

1. Introduction and preliminaries

Throughout this paper, "operator" means a bounded linear map, a "space" is a Banach space and "subspace" means a closed linear subspace, X, Y, E, F and G will always denote Banach spaces. E' is the dual of E. $L(E, F)$ denotes the space of all operators from E to F normed by the usual sup norm and $C(E, F)$ is the subspace of $L(E, F)$ of the compact operators.

A separable Banach space E has the bounded approximation property (b.a.p.) if and only if there is a sequence $\{A_n\}$ of finite rank operators in E such that $x = \sum A_n x$ for all $x \in E$. If $\sum A_n x$ converges unconditionally for all $x \in E$ then $\{A_n\}$ is called an unconditional finite dimensional expansion of the identity (u.f.d.e.i.) of E. In this case we say that E has the unconditional approximation property (suggested by H. P. Rosenthal). If, in addition, for all n, A_n is a projection and $A_n A_m = 0$ when $n \neq m$, then $\{A_n\}$ is called an unconditional finite dimensional decomposition (u.f.d.d.) of (the identity of) E. Pełczyński and Wojtaszczyk [13] proved that E has an u.f.d.e.i. iff E is complemented in a space with an u.f.d.d. A space with an u.f.d.d. is a subspace of a space with an unconditional basis (see Lindenstrauss and Tzafriri [10]). Hence a space with an u.f.d.e.i. is a subspace of a space with an unconditional basis. We shall show that in certain cases, the converse is also true. In particular we prove that for a reflexive Banach space E having the approximation property, E has an u.f.d.e.i.

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iff E is a subspace of a space with an unconditional basis. We mention an example of a subspace of l_1 having a basis but lacking the unconditional a.p. (i.e. having no u.f.d.e.i.). In the nonreflexive case we prove that if E has a shrinking unconditional basis, M is a subspace of E and M' has the approximation property (a.p.) (and hence the b.a.p.) then M has the unconditional a.p. Finally if N is a quotient of a space with a shrinking unconditional basis and N' has the a.p. then N and N' have the unconditional a.p.

In the last section we study some equivalences of the non-complementation of $C(E, F)$ in $L(E, F)$. It is still an open question whether $C(E, F)$ can be nontrivially complemented in $L(E, F)$. Among the results proven, if $L(E, F) \neq C(E, F)$, E is reflexive, F is a subspace of a space with an unconditional basis and E or F has the b.a.p. then $C(E, F)$ is uncomplemented in $L(E, F)$.

Let $\{x_n\}$ be a sequence in E. If

$$\varepsilon_1(\{x_n\}) = \sup \{\textstyle\sum |f(x_n)| : f \in E', \|f\| \leq 1\}$$

is finite then we say that the series $\sum x_n$ is weakly unconditionally Cauchy. It is well known and easy to see that $\sum x_n$ is weakly unconditional Cauchy iff there exists $K > 0$ such that for every n and scalars $\lambda_1, \ldots, \lambda_n$,

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leq K \sup |\lambda_i| .$$

For any $x' \in E'$ and $y \in F$ denote by $x' \otimes y$ the operator mapping $x \to x'(x)y$. If $\{e_n\}$ is an unconditional basis of E with an unconditional constant K and if $\{e_n'\}$ is the sequence of coefficient functionals and if $A \in L(F, E)$, $B \in L(E, G)$ then

$$\left\| \sum_{i=1}^{n} \lambda_i A^* e_i' \otimes B e_i \right\| \leq K \|A\| \; \|B\| \sup |\lambda_i|$$

for all n and $\lambda_1, \ldots, \lambda_n$. Hence $\varepsilon_1(\{A^* e_i' \otimes Be\}) < \infty$ (as observed by Lust [11]). Finally if $\{A_n\}$ is a sequence in $L(E, F)$ such that $\sum A_n x$ converges unconditionally for every x in E then $\|\sum \lambda_i A_i x\| \leq \varepsilon_1(\{A_n x\})$ for every finite sequence $\{\lambda_i\}$ with $|\lambda_i| \leq 1$. By the uniform boundedness principle $\|\sum \lambda_i A_i\|$ is uniformly bounded for such sequences—i.e. $\sum A_i$ is weakly unconditionally Cauchy.

2. The main tool

LEMMA 1. *Let $\{T_n\}$ and $\{B_n\}$ be sequences in $C(X, Y)$ and let $T \in L(X, Y)$. Assume that $T_n x$ tends to Tx in norm (respectively, weakly) for every $x \in X$ and that*

$$\phi\left(T_n^* f - \sum_{i=1}^{n} B_i^* f \right) \to 0 \quad \textit{for} \quad f \in Y', \phi \in X'' \tag{2.1}$$

Then there is a sequence $\{A_n\}$ *of finite linear combinations of the* T_i*'s so that* $\sum_{i=1}^{\infty} A_i x = Tx$ *(respectively, in the weak topology) for all* $x \in X$ *and*

(i) $\sum A_n x$ *converges unconditionally for all* $x \in X$ *if* $\sum B_n x$ *does, and*
(ii) $\sum A_n$ *is weakly unconditionally Cauchy in* $C(X, Y)$ *if* $\sum B_n$ *is.*

Proof. Put $V_n = T_n - \sum_{i=1}^{n} B_i$, then $\phi(V_n^* f) \to 0$ for every $f \in Y'$ and every $\phi \in X''$. By Kalton [8, Corollary 3], $\{V_n\}$ is weakly null. By a well known theorem of Mazur, there is a strongly null sequence $\{U_n\}$ of convex combinations of the V_n's of the form

$$U_n = \sum_{i \in I_n} \lambda_i V_i \tag{2.2}$$

where

$$I_n = \{i\colon p_{n-1} < i \le p_n\}, \quad 0 = p_0 < p_1 < p_2 < \cdots, \tag{2.3}$$

and

$$\lambda_i \ge 0 \text{ for all } i, \qquad \sum_{i \in I_n} \lambda_i = 1 \text{ for all } n, \tag{2.4}$$

and such that $\|U_n\| < 2^{-n}$ for all n.

Now let us put $I_0 = \emptyset$ and define $A_n = \sum_{i \in I_n} \lambda_i T_i - \sum_{i \in I_{n-1}} \lambda_i T_i$ for all n. It is clear that $\sum_{i=1}^{n} A_i x = \sum_{i \in I_n} \lambda_i T_i x \to Tx$ for all $x \in X$, in the strong (respectively, weak) topology of Y. Suppose now that $\sum A_n x$ converges unconditionally. For every natural i put

$$\alpha_i = \sum_{i \le j \in I_n} \lambda_j; \qquad \beta_i = \sum_{i > j \in I_n} \lambda_j \tag{2.5}$$

when n is determined by $i \in I_n$.

Straightforward calculation gives

$$A_n = \sum_{i \in I_n} \alpha_i B_i + \sum_{i \in I_{n-1}} \beta_i B_i + U_n - U_{n-1} \tag{2.6}$$

and

$$\sum_n A_n x = \sum_n \left(\sum_{i \in I_n} \alpha_i B_i x \right) + \sum_n \left(\sum_{i \in I_{n-1}} \beta_i B_i x \right) + \sum_n (U_n - U_{n-1}) x. \tag{2.7}$$

Since $0 \le \alpha_i$, $\beta_i \le 1$, the first two series on the right side of (2.7) converge unconditionally (the I_n's are pairwise disjoint). And $\sum (U_n - U_{n-1})x$ is (even) absolutely convergent. Hence $\sum A_n x$ converges unconditionally. From (2.6) (ii) is derived in the same fashion. ∎

Remark. The idea of the construction of the A_n's is taken from Pełczyński. See [14, p. 446]. The present form was suggested by the referee in a slightly different form.

Let E have an unconditional basis $\{e_i\}$ with coefficient functionals $\{e_i'\}$. Let M be a subspace of E. Denote by $j\colon M \to E$ the inclusion, put $g_i = e_i'|_M = j^* e_i'$ and put $S_n = \sum_{i=1}^n g_i \otimes e_i$. If $U_n = \sum_{i=1}^n e_i' \otimes e_i$ are the partial sum projections defined by $\{e_n\}$ in E then $S_n = U_n|_M = U_n \circ j \in C(M, E)$.

PROPOSITION 1. *Let E and M be as above and let F be a Banach space. Suppose that $\{T_n\}$ is a sequence of finite rank operators in $L(F, M)$ pointwise convergent to some $T \in L(F, M)$ (i.e. $\|T_n x - Tx\| \to 0$ for all $x \in F$) and assume*

$$\phi(T_n^* j^* f - T^* S_n^* f) \to 0 \quad \text{for } f \in E',\ \phi \in F'' \tag{2.8}$$

Then there exists a sequence $\{A_n\}$ of finite rank operators in $L(F, M)$ such that $\sum A_n x$ converges unconditionally to Tx for all $x \in F$.

The sequence $\{A_n\}$ has each of the forms

$$A_n = \sum_{i \in I_n} \lambda_i T_i - \sum_{i \in I_{n-1}} \lambda_i T_i, \tag{2.9}$$

$$A_n = \sum_{i \in I_n} \alpha_i T^* g_i \otimes e_i + \sum_{i \in I_{n-1}} \beta_i T^* g_i \otimes e_i + R_n \tag{2.10}$$

where the λ_i, I_n, α_i, β_i satisfy the conditions (2.3), (2.4) *and* (2.5) *and $R_n \in L(F, E)$, $\|R_n\| \leq 2^{2-n}$.*

Proof. We use Lemma 1 where $F = X$, $E = Y$, and $B_n = T^* j^* e_i' \otimes e_i$. Since

$$S_n T = \sum_{i=1}^n T^* j^* e_i' \otimes e_i,$$

$T^* S_n^* = \sum_{i=1}^n B_i^*$ and (2.1) is satisfied. It is clear that $\sum B_n x$ converges unconditionally for every $x \in F = X$. (2.9) and (2.10) are simple consequences of the proof of Lemma 1. ▮

The following is a special case of Proposition 1.

PROPOSITION 2. *Let E have an unconditional basis, let M be a subspace of E and let $S_n\colon M \to E$ be the partial sum projections of the basis restricted to M. Suppose $\{T_n\}$ is a sequence of finite rank operators in $L(M, M)$, pointwise convergent to the identity such that*

$$\phi(T_n^* j^* f - S_n^* f) \to 0 \quad \text{for } f \in E',\ \phi \in M'' \tag{2.11}$$

then M has an u.f.d.e.i.

Let E have an unconditional basis $\{e_i\}$ with $\{e_i'\}$ the coefficient functionals. Put $U_n = \sum_{i=1}^n e_i' \otimes e_i$ the basis projections. Let N be a quotient of E with $q\colon E \to N$ the quotient map. Then for any space F, $L(N, F)$ may be isometrically embedded in $L(E, F)$ by $\alpha\colon L(N, F) \to L(E, F)$; $\alpha(T) = T \circ q$. Clearly, $\alpha(C(N, F)) \subset C(E, F)$.

PROPOSITION 3. *Let E and N be as above. Suppose $\{T_n\}$ is a sequence of finite rank operators in $L(N, F)$ pointwise convergent to some $T \in L(N, F)$ and assume*

$$\phi(U_n^* q^* T^* - q^* T_n^*) f \to 0 \quad \text{for } f \in F',\ \phi \in E''. \tag{2.12}$$

Then there is a sequence $\{A_n\}$ in $L(N, F)$ of finite rank operators such that $\sum A_n x$ converges unconditionally to Tx for all $x \in N$.

The proof is very much the same as in Proposition 1 and thus omitted.

3. Unconditional finite dimensional expansion of the identity

THEOREM 1. *Let E have an unconditional basis and let M be a reflexive subspace of E. Then M has an u.f.d.e.i. if and only if M has the a.p.*

Proof. The "only if" part is trivial. For the "if" part the result follows easily from Proposition 2. ▌

COROLLARY 1. *Let G be reflexive then G has an u.f.d.e.i. if and only if G has the a.p. and G is a subspace of a space with an unconditional basis.*

COROLLARY 2. *Let E have the unconditional a.p. and let F be a reflexive subspace of E. Then F has the unconditional a.p. if and only if F has the a.p.*

Examples (1) Let E be a subspace of L_p, $1 < p < \infty$. If E has the a.p. then by Theorem 1, E has the unconditional a.p.

(2) Let E be the space defined by Lindenstrauss [9] as the subspace of l_1 spanned by the sequence $\{x^n\}$ in l_1 where $x^n = (x_i^n)$ is given by $x_n^n = 1$, $x_{2n+1}^n = x_{2n+2}^n = -\frac{1}{2}$ and $x_i^n = 0$ for i other than n, $2n+1$, $2n+2$. Then E has a basis but E has no u.f.d.e.i. (see [13]).

Example 2 shows that Theorem 1 cannot be simply generalized by deleting "reflexive". Hence, more conditions must be added in the non-reflexive case.

In the proof of the next theorem we need the following fact essentially contained in Johnson, Rosenthal and Zippin [6].

PROPOSITION 4. *Let E' be separable and suppose E' has the a.p. Then there is a sequence of finite rank operators T_n: $E \to E$ such that $T_n x \to x$ and $T_n^* f \to f$ for all $x \in E, f \in E'$.*

Proof. E' is separable and has the a.p. By Grothendieck [5], E' has the b.a.p. Hence there is a bounded sequence of finite rank operators R_n: $E' \to E'$ so that $R_n f \to f$ for $f \in E'$. By [6, Corollary 3.2] we may assume $R_n = A_n^*$ for suitable A_n: $E \to E$. It follows that $A_n x \to x$ weakly for every x in E. Let $\{x_i\}$ be a dense sequence in E. Using induction and Mazur's theorem define sequences $0 = p_0 < p_1 < \cdots$ of integers and $\lambda_1, \lambda_2, \ldots$ of non-negative reals such that when $I_n = \{j: p_{n-1} < j \le p_n\}$, $\sum_{j \in I_n} \lambda_j = 1$ and $\|\sum_{j \in I_n} \lambda_j A_j x_i\| \le 1/n$ for all n and all $1 \le i \le n$. Now put $T_n = \sum_{j \in I_n} \lambda_j A_j$. Since $\|A_n\|$ is bounded, $\|T_n\|$ is bounded and the $T_n x_i \to x_i$ for all i implies $T_n x \to x$ for all $x \in E$. $T_n^* f \to f$ for all $f \in E'$ follows from the construction. ▌

THEOREM 2. *Let E have a shrinking unconditional basis and let M be a subspace of E such that M' has the a.p. Then M has an u.f.d.e.i.*

Proof. The assumptions of Prop. 2 are satisfied using Prop. 4. ∎

If $\{e_i\}$ is a boundedly complete basis of E, then E is isometric to the dual of $[e_i']$—the subspace of E' spanned by the coefficient functionals $\{e_i'\}$. The following theorem was proved by P. Saphar.

THEOREM 3 (Saphar). *Let E have an unconditional basis $\{e_i\}$ and let M be a subspace of E. Put $N = E/M$.*

(a) *If $\{e_i\}$ is shrinking and N' has the a.p. then there is an u.f.d.e.i. $\{A_n\}$ of N such that $\{A_n^*\}$ is an u.f.d.e.i. of N'.*

(b) *If $\{e_i\}$ is boundedly complete and M is closed in $\sigma(E, [e_i'])$ then M has an u.f.d.e.i. if and only if M has the a.p.*

Proof. (a) By Prop. 4 there exists a sequence $\{T_n\}$ in $L(N, N)$ of finite rank operators such that $T_n x \to x$ and $T_n^* f \to f$ for all $x \in N, f \in N'$. Now by Proposition 3 there is an u.f.d.e.i. $\{A_n\}$ of N. As in Proposition 1 the A_n have the form (2.9) and thus $\sum_{i=1}^n A_i^* = \sum_{i \in I_n} \lambda_i T_i^*$ and $\{A_n^*\}$ is an u.f.d.e.i. of N'.

(b) Follows from (a) by duality. ∎

Example. Let E be a subspace of l_1 closed in $\sigma(l_1, c_0)$ and having the a.p. By Theorem 3, E has an u.f.d.e.i.

4. Applications to spaces of operators

Let E and F be infinite dimensional Banach spaces. Consider the following properties:

(a) $L(E, F) = C(E, F)$;
(b) $L(E, F)$ contains no isomorphic copy of l_∞;
(c) $C(E, F)$ contains no isomorphic copy of c_0;
(d) $C(E, F)$ is complemented in $L(E, F)$.

Kalton [7] proved that (a), (b), (c) and (d) are equivalent when E has an u.f.d.e.i. His proof does not depend on this particular property of E for the implications (a) $\Rightarrow$ (b) and (the trivial) (a) $\Rightarrow$ (d). Using a theorem of Nissenzweig [12] and Josefson [7] it is easy to extend Kalton's proof to give (b) $\Rightarrow$ (c). Tong and Wilken [17] proved (d) $\Rightarrow$ (a) when F has an unconditional basis.

The results of the preceding section together with the mentioned results yield some cases when (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). We shall give some more results.

The following lemma is basically due to Kalton.

LEMMA 2. *Let E be weakly compactly generated and suppose there exists a sequence $\{T_n\}$ in $C(E, F)$ such that $\sum T_n x$ converges unconditionally to Tx for every x in E where $T \in L(E, F)$ is non-compact. Then $C(E, F)$ is uncomplemented in $L(E, F)$.*

Proof. (1) First we prove the special case where E is separable. Since T is non-compact, $\sum T_n$ diverges in $C(E, F)$. Thus, there is a sequence $0 = p_0 < p_1 < \cdots$ of integers such that $B_k = \sum_{p_{k-1}+1}^{p_k} T_n$ satisfies $\inf \|B_k\| > 0$. Clearly $\sum B_n x$ also converges unconditionally for all $x \in E$. Using standard methods of the uniform boundedness principle, it is easy to see that there is some $K > 0$ such that $\sum |f(B_n x)| \leq K\|f\| \, \|x\|$ for every $x \in E$ and $f \in F'$ (see also introduction). The map $\phi\colon l_\infty \to L(E, F)$ defined by $\phi(\xi)x = \sum \xi_n T_n x$ is well defined, linear and bounded and $\phi(c_0) \subset C(E, F)$. Now proceed as in the proof of (iv)$\Rightarrow$(v) in Theorem 6 of Kalton [7].

(2) Now let E be any WCG space. T is not compact. Thus, there is a separable subspace E_0 of E such that the restriction of T to E_0, $T|_{E_0}$ is non-compact. E is WCG and E_0 is separable. By Amir and Lindenstrauss [1] there is a separable complemented subspace E_1 of E containing E_0. Let P be a projection of E on E_1. If there was a projection Q of $L(E, F)$ on $C(E, F)$ then $Q_1\colon L(E_1, F) \to C(E_1, F)$ defined by $Q_1(S) = Q(SP)|_{E_1}$ would be a projection of $L(E_1, F)$ on $C(E_1, F)$, in contradiction with the first part of the proof (since $\sum T_n|_{E_1} x$ converges unconditionally to $T|_{E_1}$ for all $x \in E_1$). Hence no such Q exists. ∎

THEOREM 4. *Let E and F be infinite dimensional and suppose one of the following cases occurs:*

(1) *E is reflexive, F is a subspace of some Banach space G with an unconditional basis, and E or F has the b.a.p.*

(2) *E is weakly compactly generated, F is a subspace of some G having a shrinking unconditional basis and E' or F' has the b.a.p.*

(3) *E is a quotient of some G with a shrinking unconditional basis and either E' has the b.a.p. or F' is separable and has the b.a.p.*

Then (a), (b), (c) and (d) are equivalent.

Proof. As mentioned before $(a)\Rightarrow(b)\Rightarrow(c)$ and $(a)\Rightarrow(d)$ always hold. We have only to prove that $(c)\Rightarrow(a)$ and $(d)\Rightarrow(a)$. Suppose (a) fails, then there is a non-compact T_0 in $L(E, F)$. We will show that this implies the existence of a series $\sum A_n$ in $C(E, F)$ and a non-compact T in $L(E, F)$ such that $\sum A_n x$ converges unconditionally to Tx for every x in E. This means (see the introduction) that $\sum A_n$ is divergent and weakly unconditionally Cauchy. By Bessaga and Pełczyński [2], (c) fails and by Lemma 2, (d) fails. So all that is left to do is prove the existence of the series $\sum A_n$ for each of the three cases of the theorem.

Case 1. As in the proof of Lemma 2 (second part) there is a separable subspace E_1 of E and a projection P in E, with $P(E) = E_1$, such that $T_0|_{E_1}$ is not compact. Put $T = T_0 P$, then T is not compact. Since either E_1 or F is a separable space having the b.a.p. (and E_1 is complemented in E) it is easy to see that there is a sequence $\{R_n\}$ of finite rank operators from E to F so that $\|R_n x - Tx\| \to 0$ for all x in E. Let $\{e_i\}$ and $\{e_i'\}$ be the unconditional basis of G

and its sequence of coefficient functionals and $D_i = e_i' \otimes e_i$. Let j: $F \to G$ be the inclusion and $C_i = D_i j$. Then, since E is reflexive,

$$\phi\left(R_n^* j^* f - \sum_{i=1}^{n} T^* C_i^* f\right) = f\left(jR_n \phi - \sum_{i=1}^{n} C_i T\phi\right)$$

tends to 0 for all f in G' and ϕ in $E'' = E$. The assumptions of Lemma 1 are now satisfied with $X = E$, $Y = G$, $T_n = jR_n$ and $B_i = C_i T$. Since $\sum B_i x$ is unconditionally convergent for every x in E', there is (by Lemma 1) a series $\sum A_n$ as claimed.

Case 2. Put $T = T_0$. F' is separable and $T^*(F')$ is separable. Since either E' or F' has the b.a.p. there is a bounded sequence H_n: $F' \to E'$ of finite rank operators such that $H_n f \to T^* f$ for all f in F'. Let $\{f_i\}$ be a dense sequence in F'. For each n there is by [6, Lemma 3.1] a weak* continuous operator K_n: $F' \to E'$ such that range $K_n \subset$ range H_n, $K_n f_i = H_n f_i$ for $i = 1, 2, \ldots, n$ and $\|K_n\| < 2\|H_n\|$. By the weak* continuity of K_n, $K_n = R_n^*$ for some R_n: $E \to F$. Clearly $R_n^* f \to T^* f$ for all f in F' and this implies

$$R_n x \overset{\omega}{\to} Tx \quad \text{for all } x \text{ in } E.$$

Now define D_i and C_i as in Case 1, with $\{e_i\}$ shrinking. This yields $\sum C_i^* f = j^*(\sum D_i^* f) = j^*(f) = f|_F$ for all f in G'. Hence

$$\phi\left(R_n^* j^* f - \sum_{i=1}^{n} T^* C_i^* f\right) \to 0$$

is true again as in Case 1 and the rest is alike.

Case 3. Put $T = T_0$. Either E' or F' is separable and has the b.a.p. Hence we can construct the sequences $\{R_n\}$ and $\{D_i\}$ as in Case 2. Let q: $G \to E$ be the quotient map and use Lemma 1 with $X = G$, $Y = F$, $T_n = R_n q$ and $B_i = TqD_i$. We have again a shrinking basis so (2.1) is satisfied. By Lemma 1, there is a sequence $\{\tilde{A}_n\}$ of finite linear combinations of the $R_n q$'s so that $\sum \tilde{A}_n x = Tqx$ unconditionally for every x in G. Now define $A_n = \sum \delta_i R_i$ if $\tilde{A}_n = \sum \delta_i R_i q$. $\sum A_n$ is the series which was claimed to exist. ∎

Remark. If $E = F$ and (d) fails, then there is an operator T: $E \to E$ where T is *not* of the form $T = \lambda I + K$, λ scalar and $K \in C(E, E)$ (I = the identity).

THEOREM 5. *Let E and F be reflexive and suppose F or E' is a subspace of a Banach space with an unconditional basis. Then $C(E, F)$ is either reflexive or non-isomorphic to a dual space.*

Remark. The isometric version is known to be true more generally (see [4]).

Proof. Suppose $C(E, F)$ is non-reflexive. Let $\{D_n\}$ be a bounded sequence in $C(E, F)$ which has no weakly convergent subsequence. By [4, Corollary 1.3] there is a subsequence $\{S_k = D_{n_k}\}$ which is weakly Cauchy. In particular $\lim f(S_k x)$ exists for every $f \in F'$ and $x \in E$. Since F is reflexive $\lim f(S_k x) = f(Tx)$ for some $T \in L(E, F)$. T is not compact because otherwise, by Kalton [8, Corollary 3],

$$S_k \overset{\omega}{\to} T$$

in contradiction to the construction of $\{D_n\}$. Now we must look at the two following cases:

Case 1. F is a subspace of a space G with an unconditional basis $\{e_i\}$ and coefficient functionals $\{e_i'\}$. Put j: $F \to G$ the embedding, $T_n = jS_n$, $K_n = e_n' \otimes e_n$ and $B_n = K_n jT$. Then the conditions of Lemma 1 are satisfied with $X = E$, $Y = G$. $\sum K_n$ is weakly unconditionally Cauchy and thus, so is $\sum B_n$. By Lemma 1 there exists a weakly unconditionally Cauchy series $\sum A_n$ such that $\sum A_n x = Tx$ weakly. As in the proofs of Lemma 2 and Th. 4 there is a projection P in E such that $E_1 = P(E)$ is separable and $T|_{E_1}$ is not compact. $C(E_1, F)$ is separable and isomorphic to a complemented subspace of $C(E, F)$. $\sum A_n|_{E_1}$ is a divergent series and weakly unconditionally Cauchy. By Bessaga and Pełczyński [2], c_0 is isomorphic to a (complemented) subspace of $C(E_1, F)$. Thus c_0 is isomorphic to a complemented subspace of $C(E, F)$. By Bessaga and Pełczyński [3], $C(E, F)$ is not isomorphic to a dual space.

Case 2. E' is a subspace of a space with an unconditional basis. By Case 1 $C(F', E')$ is either reflexive or non-isomorphic to a conjugate space. But here $C(E, F)$ and $C(F', E')$ are isometric by $T \leftrightarrow T^*$. ∎

Example. $C(l_p, l_q)$ is reflexive (respectively, not isomorphic to a conjugate space) if $1 < q < p < \infty$ ($1 < p < q < \infty$). See [4]. We conclude with two open problems.

Problem 1. Is Theorem 5 true even when the condition F or E' is a subspace of a space with an unconditional basis is dropped?

Problem 2. Let E be an infinite dimensional subspace of a space with an unconditional basis. Is there always some T: $E \to E$ not of the form $\lambda I + K$, λ scalar, K in $C(E, E)$? (The problem is open for any infinite dimensional Banach space and was raised by Lindenstrauss. See the remark after Theorem 4.)

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