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DEGREE OF BEST INVERSE APPROXIMATION BY POLYNOMIALS

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1. Introduction

Let π_n be the space of all algebraic polynomials with degree not exceeding n, and let $\|\cdot\|_p$ denote the L_p norm on the interval $[-1, 1]$. The purpose of this paper is the study of the speed of convergence of the error of *best inverse approximation* in L_p from π_n defined by

$$D_{n,p}(f) = \inf \{\|1 - fP_n\|_p \colon P_n \in \pi_n\}.$$

This problem finds its origin in the method of least-squares inverses which was introduced by E. A. Robinson (cf. [11] and [12, pages 153–175]), in 1963 in connection with deconvolution of 2-length wavelets and inverse filtering in geophysical studies. Since then, this method has also been adopted and generalized in recursive digital filter design (cf. [13]). The validity of these procedures and the mathematical theory have been discussed in [3], where the present problem was proposed.

To avoid trivial cases, we will always consider those functions f which are not identically zero, but with non-empty zero-sets in $[-1, 1]$ and $1 \le p < \infty$. The main result of this paper is the following.

THEOREM 1. *Let $f \not\equiv 0$ be a real analytic function on $[-1, 1]$ and $1 \le p < \infty$. There exist positive constants C_1, C_2 depending only on f with the following properties:*

(a) *if $f(x) = 0$ for some $x \in (-1, 1)$, then*

$$C_1 n^{-1/p} \le D_{n,p}(f) \le C_2 n^{-1/p}, \quad n = 1, 2, \ldots; \tag{1.1}$$

(b) *if $f(x) \neq 0$ for all $x \in (-1, 1)$ but $f(-1)f(1) = 0$, then*

$$C_1 n^{-2/p} \le D_{n,p}(f) \le C_2 n^{-2/p}, \quad n = 1, 2, \ldots. \tag{1.2}$$

It will be seen that the analyticity condition can be weakened. However, since our method of proof depends very heavily on the zero structure of an

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analytic function, we do not state the strongest possible result. In the next section, preparatory results on interpolating polynomials will be established. These results will be applied in Section 3 to prove our main theorem. Final remarks and open questions will be included in the final section.

2. Preliminary results

Throughout this paper, all L_p norms are taken over $[-1, 1]$ unless otherwise stated. We have the following result.

LEMMA 1. *There exists a positive constant C_3 independent of p, $1 \le p \le \infty$, with the following properties. Let $P_n \in \pi_n$. If $P_n(0) = 1$ then*

$$\|P_n\|_p \ge C_3 n^{-1/p}, \tag{2.1}$$

and if $P_n(1) = 1$ then

$$\|P_n\|_p \ge C_3 n^{-2/p}, \tag{2.2}$$

for all $n = 1, 2, \ldots$.

If $1 \le p < \infty$ and $P_n \in \pi_n$, we have

$$\frac{1}{2}\int_{-\pi}^{\pi} |P_n(\cos\theta)\sin\theta|^p \, d\theta = \int_{-1}^{1} (1-x^2)^{(p-1)/2} |P_n(x)|^p \, dx \le \|P_n\|_p^p.$$

Hence, by a result of Nikol'skii on trigonometric polynomials (cf. [10] and [14, p. 229]), we obtain

$$\begin{aligned} \|P_n\|_p &\ge 2^{-1-1/p}(n+1)^{-1/p} \sup |P_n(\cos\theta)\sin\theta| \\ &\ge 2^{-1-1/p}(n+1)^{-1/p} |P_n(0)|, \end{aligned} \tag{2.3}$$

so that (2.1) follows. On the other hand (cf. [14, page 236]),

$$\|P_n\|_\infty \le (p+1)^{1/p} n^{2/p} \|P_n\|_p.$$

Hence, (2.2) follows.

From Lemma 1 or directly from (2.3) there follows:

COROLLARY 1. *Let $0 < \varepsilon < 1$. There exists a positive constant C_ε such that*

$$\sup\{|P_n(x)| : |x| \le 1 - \varepsilon\} \le C_\varepsilon n^{1/p} \|P_n\|_p \tag{2.4}$$

for every polynomial $P_n \in \pi_n$ and $1 \le p \le \infty$.

We also have the following, which is in the opposite direction to Lemma 1:

LEMMA 2. *There exists a positive constant C_4 and polynomials $P_n, Q_n \in \pi_n$, $n = 1, 2, \ldots$ such that $P_n(0) = 1$, $Q_n(1) = 1$ and $\|P_n\|_p \le C_4 n^{-1/p}$, $\|Q_n\|_p \le C_4 n^{-2/p}$, for $1 \le p < \infty$.*

The main tool in the proof of this lemma is a quadrature formula technique for finding extremal polynomials. This useful technique was possibly first used by Freud [7] in proving a result of Erdös and Turán [6]. More recently, many interesting applications have been given by R. A. DeVore, for example, in his lecture notes [5]. Another application occurs in [1]. Let $L_n \in \pi_n$ be the Legendre polynomials. We define the polynomials P_n as follows: $P_1 \equiv 1$ and $P_n = P_{4m-2}$ for $4m - 2 \le n \le 4m + 1$, $m = 1, 2, \ldots$, where

$$P_{4m-2}(x) = (L_{2m-1}(x)/xL'_{2m-1}(0))^2.$$

Clearly, $P_n(0) = 1$ for all n. Using the quadrature formula technique (cf. [5], pp. 180–181) we can conclude that

$$\|P_n\|_1 \le Cn^{-1} \tag{2.5}$$

and

$$\sup_{1/2 \le |x| \le 1} |P_n(x)| \le Cn^{-1} \tag{2.6}$$

for all $n = 1, 2, \ldots$, where C is an absolute constant. From Corollary 1 and (2.5), we conclude that $\|P_n\|_\infty$ is a bounded sequence in n. If $1 < p < \infty$, then

$$\|P_n\|_p^p \le \|P_n\|_\infty^{p-1}\|P_n\|_1 \le C\|P_n\|_\infty^{p-1}n^{-1},$$

by (2.5). That is, we have proved the inequality for P_n in the lemma.

We will now construct $\{Q_n\}$. Each Q_n will be non-negative and monotone increasing on $[-1, 1]$. We choose $Q_{2m} = Q_{2m-1}$ for $m = 1, 2, \ldots$ where

$$Q_{2m-1}(x) = c_m \int_{-1}^{x} \left(\frac{L_m(t)}{t - t_{m,m}}\right)^2 dt$$

and c_m is a normalizing constant chosen so that $Q_{2m-1}(1) = 1$. Here,

$$-1 < t_{1,m} < t_{2,m} < \cdots < t_{m,m} < 1$$

are the zeros of the mth Legendre polynomial L_m in increasing order. Hence,

$$Q_{2m-1}(1) = \int_{-1}^{1} \lambda_m(t)\, dt,$$

and integration by parts gives

$$\|Q_{2m-1}\|_1 = \int_{-1}^{1} (1 - t)\lambda_m(t)\, dt,$$

where

$$\lambda_m(t) = c_m[L_m(t)/(t - t_{m,m})]^2.$$

Now, let $A_i(m)$ be the weight associated with $t_{i,m}$ in the Gauss-Legendre quadrature formula with nodes at the zeros of L_m. Using this quadrature formula, and the fact that $\lambda_m(t_{i,m}) = 0$ for $i = 1, \ldots, m-1$, we find

$$\begin{aligned}\|Q_{2m-1}\|_1 &= \sum_{i=1}^{m} A_i(m)(1 - t_{i,m})\lambda_m(t_{i,m}) \\ &= A_m(m)(1 - t_{m,m})\lambda_m(t_{m,m}) \\ &= (1 - t_{m,m}) \int_{-1}^{1} \lambda_m(t)\, dt \\ &= (1 - t_{m,m})Q_{2m-1}(1) \\ &= 1 - t_{m,m}.\end{aligned}$$

The result follows since a theorem of Bruns (cf. [5, p. 20]) shows that

$$C'm^{-2} \leq 1 - t_{m,m} \leq C''m^{-2}.$$

We will also need the following lemma on Hermite interpolation by polynomials. A related lemma appears in [2].

LEMMA 3. *Let* $-1 \leq x_1 < x_2 < \cdots < x_r \leq 1$ *and let* k *be a non-negative integer. Then there exists a positive constant* C_6 *such that for any integer* $n \geq r(k+1) - 1$ *and real numbers* $c_{ij}(n)$, $1 \leq i \leq r$ *and* $0 \leq j \leq k$, *there exists a polynomial* $P_n \in \pi_n$ *with*

$$P_n^{(j)}(x_i) = c_{ij}(n), \quad 1 \leq i \leq r,\ 0 \leq j \leq k, \tag{2.7}$$

$$\|P_n\|_p \leq C_6 n^{-1/p} \max\{|c_{ij}(n)| : 1 \leq i \leq r,\ 0 \leq j \leq k\}, \tag{2.8}$$

for all p, $1 \leq p \leq \infty$.

To prove this result, we first construct a sequence of polynomials $h_n \in \pi_n$ satisfying $h_n^{(j)}(0) = \delta_{j,0}$, $0 \leq j \leq k$, and

$$\|h_n\|_{L^\infty[-2,2]} \leq C, \quad \|h_n\|_{L^1[-2,2]} \leq Cn^{-1} \tag{2.9}$$

for some absolute positive constant C and all n. Here and throughout, $\delta_{j,k}$ denotes the usual Kronecker delta symbol. Indeed, by Lemma 2, there exist $k_n \in \pi_n$ with $k_n(0) = 1$ and $\|k_n\|_p \leq C_4 n^{-1/p}$, $1 \leq p \leq \infty$. Set $k_n = 1 - l_n$, and $\tilde{h}_{nk} = 1 - l_n^k$. Since $\|l_n\|_\infty \leq C_4 + 1$ and

$$\tilde{h}_{nk} = (1 - l_n)(1 + l_n + \cdots + l_n^{k-1}) = k_n(1 + l_n + \cdots + l_n^{k-1}),$$

we have

$$\|\tilde{h}_{nk}\|_\infty \leq C_4 \frac{(C_4 + 1)^k - 1}{(C_4 + 1) - 1} \leq D \quad \text{and} \quad \|\tilde{h}_{nk}\|_1 \leq Dn^{-1}$$

for all n. Furthermore, it is clear that $\tilde{h}_{nk}^{(j)}(0) = \delta_{j,0}$, $0 \leq j \leq k$. Defining the other $\tilde{h}_n$ by

$$\tilde{h}_{kn+1} = \tilde{h}_{kn+2} = \cdots = \tilde{h}_{k(n+1)-1} = \tilde{h}_{kn}$$

and $h_n(x) = \tilde{h}_n(x/2)$, we see that $\{h_n\}$ has the desired properties.

Choose $m = r(k+1) - 1$ and the polynomials $t_{ij} \in \pi_m$ so that

$$t_{ij}^{(l)}(x_s) = \delta_{i,s}\,\delta_{j,l}, \quad i, s = 1, \ldots, r \text{ and } j, l = 0, \ldots, k.$$

Then the sequence of polynomials

$$q_n(x) = \sum_{i=1}^{r} \sum_{j=0}^{k} c_{ij} t_{ij}(x) h_{n-m}(x - x_i)$$

will have all the required interpolation properties. Since also

$$|q_n(x)| \leq \max |c_{ij}| \sum_{i=1}^{r} \sum_{j=0}^{k} \|t_{ij}\|_\infty |h_{n-m}(x - x_i)|,$$

and, for $n = m, m+1, \ldots,$

$$\|q_n\|_\infty \leq \max |c_{ij}| C' \quad \text{and} \quad \|q_n\|_1 \leq \max |c_{ij}| C''(n+1)^{-1},$$

it follows that

$$\|q_n\|_p \leq \max |c_{ij}| C'''(n+1)^{-1/p}$$

for some absolute constants C', C'', and C'''.

We also have the following result, the proof of which is analogous to that of Lemma 3 above except that one uses the end-point case rather than the mid-point case of Lemma 2.

LEMMA 4. *Let k be a non-negative integer. Then there exists a positive constant C_7 such that for any $n \geq 2k+1$ and real numbers $c_{1j}(n)$, $c_{2j}(n)$, $0 \leq j \leq k$, there exists a $P_n \in \pi_n$ with*

$$P_n^{(j)}(-1) = c_{1j}(n), \; P_n^{(j)}(1) = c_{2j}(n), \quad 0 \leq j \leq k,$$

$$\|P_n\|_p \leq C_7 n^{-2/p} \max_{0 \leq j \leq k} \{\max(|c_{1j}|, |c_{2j}|\} \quad \textit{for all } p, \; 1 \leq p \leq \infty.$$

3. Proof of the main theorem

We now have all the tools for the proof of Theorem 1. We first prove the upper bound in part (a). Write

$$f(x) = (x - x_1)^{k_1} \cdots (x - x_r)^{k_r} h(x) \equiv g(x)h(x)$$

where $h(x) \neq 0$ on $[-1, 1]$, and $-1 \leq x_1 < \cdots < x_r \leq 1$, with at least one x_i in $(-1, 1)$. Let $K = \max(k_1, \ldots, k_r)$. By [9], (cf. [8] for more complete results), there exists a constant C_8 depending on f and a sequence $\{P_n\}$, $P_n \in \pi_n$, such that

$$\left\| \left(\frac{1}{h}\right)^{(j)} - P_n^{(j)} \right\|_\infty \leq C_8 (n+1)^{-2},$$

$n = 0, 1, \ldots, j = 0, 1, \ldots, K$. In particular, this means that $P_n^{(j)}(x)$ is bounded, $j = 0, 1, \ldots, K$. Lemma 3 now guarantees the existence of a sequence of polynomials $\{Q_n\}$, $Q_n \in \pi_n$, so that

$$Q_n^{(j)}(x_i) = -P_n^{(j)}(x_i), \quad i = 1, \ldots, r; j = 0, \ldots, K,$$

and

$$\|Q_n\|_p \leq C_9 n^{-1/p}.$$

Then we have

$$(P_n + Q_n)^{(j)}(x_i) = 0, \quad i = 1, \ldots, r; j = 0, \ldots, K,$$

so that, with $m = r(K+1) - 1$,

$$(P_n + Q_n)(x) = g(x)t_{n-m}(x),$$

where $t_{n-m} \in \pi_{n-m}$ and

$$\left\| \frac{1}{h(x)} - g(x)t_{n-m}(x) \right\|_p \leq C_{10} n^{-1/p}.$$

Therefore, we have

$$\begin{aligned} D_{n,p}(f) \leq \|1 - ft_{n-m}\|_p &= \left\| h\left(\frac{1}{h} - gt_{n-m}\right) \right\|_p \\ &\leq \|h\|_\infty \left\| \frac{1}{h} - gt_{n-m} \right\|_p \leq C_2 n^{-1/p} \end{aligned}$$

which completes the proof of the upper bound in part (a).

We now proceed to the proof of the lower bound in part (a). Let $Q_n^* \in \pi_n$ be chosen such that

$$D_{n,p}(f) = \|1 - fQ_n^*\|_p.$$

We may suppose without loss of generality that one of the interior zeros of f is at zero. Then $f(x) = x^k h(x)$ where $h(x)$ is analytic and $|h(x)| \geq \theta > 0$ on $[-\alpha, \alpha]$ for some $\theta > 0$ and $0 < \alpha < 1$. Now, by what we just proved, we have

$$\|1 - x^k h(x) Q_n^*(x)\|_{L^p[-\alpha,\alpha]} \leq D_{n,p}(f) \leq C_2 n^{-1/p}.$$

This implies that

$$\|x^k Q_n^*(x)\|_{L^p[-\alpha,\alpha]} \leq \frac{1}{\theta} \|x^k h(x) Q_n^*(x)\|_{L^p[-\alpha,\alpha]} \leq C_{11}, \quad n = 0, 1, \ldots,$$

and so, by the inequality between L^p and L^∞ norms for algebraic polynomials (see [14], page 236),

$$\|x^k Q_n^*(x)\|_{L^\infty[-\alpha,\alpha]} \leq C_{12} n^2. \tag{3.1}$$

Since h is analytic, we can find a sequence $\{P_n\}$, $P_n \in \pi_n$, so that

$$\|h - P_n\|_{L\infty[-\alpha,\alpha]} \leq C_{13} n^{-4}. \tag{3.2}$$

Then we have

$$\begin{aligned} D_{n,p}(f) &\geq \|1 - x^k h(x) Q_n^*(x)\|_{L^p[-\alpha,\alpha]} \\ &\geq \|1 - x^k P_n(x) Q_n^*(x)\|_{L^p[-\alpha,\alpha]} - \|x^k Q_n^*(x)(h(x) - P_n(x))\|_{L^p[-\alpha,\alpha]}. \end{aligned}$$

By Lemma 1, (3.1), and (3.2), we find that for all sufficiently large n,

$$\| 1 - x^k P_n(x) Q_n^*(x) \|_{L^p[\alpha,\alpha]} \geq C_{14} n^{-1/p}$$

and

$$\| x^k Q_n^*(x)(h(x) - P_n(x)) \|_{L^p[-\alpha,\alpha]} \leq C_{15} n^{-2},$$

so that

$$D_{n,p}(f) \geq C_1 n^{-1/p}, \quad n = 1, 2, \ldots.$$

This completes the proof of part (a) of the theorem. The proof of part (b) of Theorem 1 follows almost exactly the same lines as that of part (a), where we replace Lemma 3 by Lemma 4.

4. Final remarks

As remarked in Section 1, the analyticity condition in Theorem 1 can be weakened. In fact the theorem holds for any function f which can be written in the form $f = qh$ where q is a polynomial and $h \in C^{k+4}[-1, 1]$ with $h(x) \neq 0$ in $[-1, 1]$ and k is the maximum order of the zeros of q in $[-1, 1]$. Under the analyticity condition we have the following "saturation result".

THEOREM 2. *Let $f \not\equiv 0$ be a real analytic function on $[-1, 1]$ and let $1 \leq p < \infty$. Then*

$$\lim_{n\to\infty} n^{1/p} D_{n,p}(f) = 0 \Leftrightarrow f \text{ is zero free in } (-1, 1), \tag{4.1}$$

and

$$\lim_{n\to\infty} n^{2/p} D_{n,p}(f) = 0 \Leftrightarrow f \text{ is zero free in } [-1, 1].$$

This result follows immediately from Theorem 1 and the obvious fact that if f is real analytic and zero-free in $[-1, 1]$ then $D_{n,p}(f) = O(n^{-k})$ for any $k > 0$ and $1 \leq p \leq \infty$.

There are many questions still to be answered in this area. For instance, the problems of finding $D_{n,p}(|x|^\alpha)$, of simultaneous inverse approximation, and of inverse quasi-rational approximation, are mentioned in [4].

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