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A DEFECT RELATION FOR LINEAR SYSTEMS ON COMPACT COMPLEX MANIFOLDS[1]

BY

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Introduction

In 1979 Shiffman ([7]) conjectured that if $f: \mathbf{C}^m \to \mathbf{P}_n$ is a non-constant meromorphic map and if $D_1, \ldots, D_q$ are distinct hypersurfaces of degree d in $\mathbf{P}_n$ such that no point is contained in the support of $n+1$ distinct D_j and $f(\mathbf{C}^m) \nsubseteq \operatorname{supp} D_j$ for all j, then

$$\sum_{j=1}^{q} \delta_f(D_j) \leq 2n, \tag{1}$$

where δ_f denotes the Nevanlinna defect. To support his conjecture Shiffman proved (1) for a class of meromorphic maps of finite order.

To extend the class that satisfies (1) we use the method of associate maps which was introduced in 1941 by Ahlfors [1], generalized and developed by Weyl [11], Stoll [8], Cowen-Griffiths [4] and Wong [12]. Namely, (1) holds either if $f(\mathbf{C}^m)$ is contained in a line of $\mathbf{P}_n$ or is a projection of a "special exponential map", i.e., an exponential map satisfying (6.1) (see Section 6). More in general we introduce an auxiliary defect τ_f, which we express explicitly and for all meromorphic maps $f: \mathbf{C}^m \to \mathbf{P}_n$ we prove

$$\sum_{j=1}^{q} \delta_f(D_j) \leq n(1 + \tau_f). \tag{2}$$

Therefore in order to prove (1) for all meromorphic maps it would be sufficient to prove $\tau_f \leq 1$.

To add generality we prove (2) for meromorphic maps $f: \mathbf{C}^m \to X$, where X is a compact complex n-dimensional manifold and for $D_1, \ldots, D_q \in |L|$, where L is a spanned line bundle.

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1. Nevanlinna theory

Define

$$\tau(z) = |z|^2 = \sum_{j=1}^{m} |z_j|^2 \quad \text{for any } z = (z_1, \ldots, z_m) \in \mathbf{C}^m.$$

If $r > 0$, we set

$$\mathbf{C}^m[r] = \{z \in \mathbf{C}^m \mid |z| < r\}, \quad \mathbf{C}^m\langle r \rangle = \partial \mathbf{C}^m[r].$$

Define

$$v = dd^c \tau \quad \text{on } \mathbf{C}^m$$

and

$$\sigma = d^c \log \tau \wedge (dd^c \log \tau)^{m-1} \quad \text{on } \mathbf{C}^m - \{0\}$$

where

$$d^c = \left(\frac{i}{4\pi}\right)(\bar{\partial} - \partial).$$

Let ν be a divisor on $\mathbf{C}^m$. For all $0 < r_0 < r$ the *valence function* is defined by

$$N_\nu(r, r_0) = \int_{r_0}^{r} n_\nu(t) t^{-1}\, dt$$

where, with $S_t = \mathbf{C}^m[t] \cap \operatorname{supp} \nu$,

$$n_\nu(t) = \begin{cases} t^{2-2m} \displaystyle\int_{S_t} \nu v^{m-1} & \text{if } m > 1 \\ \displaystyle\sum_{z \in S_t} \nu(z) & \text{if } m = 1 \end{cases}$$

is the *counting function* of ν.

Let L be a non-negative line bundle on the compact complex manifold X, with a hermitian metric κ. Let $f: \mathbf{C}^m \to X$ be a meromorphic map. For $r > r_0 > 0$, define the *characteristic function*

$$T_f(r, r_0, L, \kappa) = \int_{r_0}^{r} \left(\int_{\mathbf{C}^m[t]} f^*(c(L, \kappa)) \wedge v^{m-1} \right) t^{1-2m}\, dt$$

where $c(L, \kappa)$ is the Chern form of L for κ.

Let $s \in \Gamma(X, L)$ be a global section on L and let $D = D[s] \in |L|$ be the divisor associated to s. Define the *valence function* of f for D

$$N_f(r, r_0, D) = N_{\nu_f^D}(r, r_0)$$

for $r > r_0 > 0$ and where $\nu_f^D = f^*(D)$ is the pull-back divisor. If $f(\mathbf{C}^m) \nsubseteq$ supp D and $r > 0$, then

$$m_f(r, D) = \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|s|_{\mathfrak{l}}}{|s \circ f|_\kappa} \sigma$$

is the *compensation function* of f for D, where $\mathfrak{l}$ is a metric on $\Gamma(X, L)$ such that $|s|_{\mathfrak{l}}|s \circ f|_\kappa^{-1} \leq 1$. Such a metric exists since X is compact.

The *First Main Theorem* asserts that

$$T_f(r, r_0, L, \kappa) = N_f(r, r_0, D) + m_f(r, D) - m_f(r_0, D) \tag{1.1}$$

if $r > r_0 > 0$ and $f(\mathbf{C}^m) \nsubseteq$ supp D.

The *defect* of f for $D \in |L|$ is defined by

$$\delta_f(D) = \liminf_{r\to\infty} \frac{m_f(r, D)}{T_f(r, r_0, L, \kappa)}.$$

(1.1) implies $0 \leq \delta_f(D) \leq 1$.

Now assume $X = \mathbf{P}_n$. If f: $\mathbf{C}^m \to \mathbf{P}_n$ is a meromorphic map then we recall that $\mathfrak{u}$: $\mathbf{C}^m \to \mathbf{C}^{n+1}$ is a *representation* for f if $\mathbf{P} \circ \mathfrak{u} = f$ on $\mathbf{C}^m - \mathfrak{u}^{-1}(0) \neq \emptyset$ and the representation $\mathfrak{u}$ is said to be *reduced* if dim $\mathfrak{u}^{-1}(0) \leq m - 2$.

Let $L = H$ be the hyperplane section bundle on $\mathbf{P}_n$ with the metric κ induced by the standard metric on $\mathbf{C}^{n+1}$. Let

$$T_f(r, r_0) = T_f(r, r_0, H, \kappa).$$

If $D = D[\alpha] \in |H|$ is a hyperplane in $\mathbf{P}_n$ and $\mathfrak{u}$: $\mathbf{C}^m \to \mathbf{C}^{n+1}$ is a reduced representation of f then Jensen's formula states that

$$N_f(r, r_0, D) = \int_{\mathbf{C}^m\langle r\rangle} \log |\alpha \circ \mathfrak{u}| \sigma = \int_{\mathbf{C}^m\langle r_0\rangle} \log |\alpha \circ \mathfrak{u}| \sigma. \tag{1.2}$$

(1.1) and (1.2) imply

$$T_f(r, r_0) = \int_{\mathbf{C}^m\langle r\rangle} \log |\mathfrak{u}| \sigma - \int_{\mathbf{C}^m\langle r_0\rangle} \log |\mathfrak{u}| \sigma. \tag{1.3}$$

2. Associated maps

Let B be a holomorphic $(m-1, 0)$ form on $\mathbf{C}^m$. We shall define a differential operator D_B as follows. Let $\mathfrak{u}$: $\mathbf{C}^m \to \mathbf{C}^{n+1}$ be a holomorphic map. Then

$$\mathfrak{u}' = D_B \mathfrak{u}: \mathbf{C}^m \to \mathbf{C}^{n+1}$$

is a holomorphic map defined by

$$d\mathfrak{u} \wedge B = D_B \mathfrak{u}\, dz_1 \wedge \cdots \wedge dz_m.$$

The differential operator D_B can be repeated so we can define

$$\mathscr{u}^{(p)} = D_B^p \mathscr{u} = D_B(D_B^{p-1}\mathscr{u}).$$

Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map and $\mathscr{u}\colon \mathbf{C}^m \to \mathbf{C}^{n+1}$ a representation of f. Take $p = 0, \ldots, n$. Then

$$\mathscr{u}_p = \mathscr{u}_{p,B} = \mathscr{u} \wedge \mathscr{u}' \wedge \cdots \wedge \mathscr{u}^{(p)}$$

is the pth *associated representation.* Obviously

$$\mathscr{u}_p\colon \mathbf{C}^m \to \tilde{G}_{n,p} \subseteq \bigwedge_{p+1} \mathbf{C}^{n+1}$$

is a holomorphic map. (Here $\tilde{G}_{n,p}$ is the Grassmannian cone of $(p + 1)$-planes in $\mathbf{C}^{n+1}$.)

We say that f is *general of order* p for B if and only if $\mathscr{u}_p \not\equiv 0$. Also f is *general* for B if and only if f is general of order n for B, in which case f is general of order p for all $p = 0, \ldots, n$. Then

$$f_p = \mathbf{P} \circ \mathscr{u}_p\colon \mathbf{C}^m \to G_{n,p} = \mathbf{P}(\tilde{G}_{n,p})$$

is a well defined meromorphic map with $\mathscr{u}_p$ as a representation. The meromorphic map f_p is called the pth *associate map* of f for B.

Let Ω_p be the Fubini-Kaehler form on $\mathbf{P}(\bigwedge_{p+1} \mathbf{C}^{n+1})$ respectively on $G_{n,p}$ for $p = 0, \ldots, n$. Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map general for B. Define the pth volume form of f for B by

$$H_p = i_{m-1} m f_p^*(\Omega_p) \wedge B \wedge \bar{B} \quad \text{on } \mathbf{C}^m - I_{f_p}$$

where $i_{m-1} = (i/2\pi)^{m-2}(m-1)!\,(-1)^{(m-1)(m-2)/2}$ and I_{f_p} is the indeterminacy set of f_p. Let h_p be such that $H_p = h_p \upsilon^m$; then we have

$$\text{(2.1)} \qquad h_0 = \frac{|\mathscr{u}_1|^2}{|\mathscr{u}|^4} \quad \text{and } h_p = \frac{|\mathscr{u}_{p-1}|^2 |\mathscr{u}_{p+1}|^2}{|\mathscr{u}_p|^4} \text{ for } 0 < p < n.$$

Define

$$\text{(2.2)} \qquad S_p(r) = \frac{1}{2} \int_{\mathbf{C}^m\langle r\rangle} \log |h_p| \sigma.$$

Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a non-degenerate meromorphic map (i.e., $f(\mathbf{C}^m)$ is not contained in any hyperplane in $\mathbf{P}_n$). Then (see [9] Theorem 7.1) there exists a holomorphic $(m - 1, 0)$ form on $\mathbf{C}^m$ whose coefficients are polynomials of degree at most $n - 1$ and such that f is general for B and

$$\text{(2.3)} \qquad i_{m-1} B \wedge \bar{B} \leq (1 + r^{2n-2})\upsilon^{m-1} \quad \text{on } \mathbf{C}^m[r].$$

3. General position

Let X be a projective variety of dimension n_0 in $\mathbf{P}_n$. For $p = 1, \ldots, n$, set

$$\tilde{X}_p = \{(x, y) \in X \times G_{n,p} \mid x \in E(y)\}$$

where $E(y) \subseteq \mathbf{P}_n$ denotes the p-plane associated by $y \in G_{n,p}$. We know that the projection $\pi_p\colon \tilde{X}_p \to G_{n,p}$ is proper and holomorphic. Therefore $X_p = \pi_p(\tilde{X}_p)$ is a compact analytic subset of $G_{n,p}$. For any $D = D[\alpha] \in |H|$ hyperplane in $\mathbf{P}_n$ we define

$$u_p(D)\colon X_p \to \mathbf{R}[0, 1]$$

by

$$u_p(D)(x) = \frac{|\mathfrak{x} \llcorner \alpha|^2}{|\mathfrak{x}|^2 |\alpha|^2} \quad \text{for } p = 0, \ldots, n$$

and for $x = \mathbf{P}(\mathfrak{x}) \in X_p$. Here $\mathfrak{x} \llcorner \alpha$ is such that

$$(\mathfrak{x} \llcorner \alpha, \beta) = (\mathfrak{x}, \alpha \wedge \beta) \quad \text{for every } \beta \in \textstyle\bigwedge_p (\mathbf{C}^{n-1})^*.$$

If $D_j = D[\alpha_j]$ are hyperplanes in $\mathbf{P}_n$, $j = 1, \ldots, q$ define

$$c_p = c_p(D_1, \ldots, D_q)\colon X_p \to \mathbf{Z} \quad \text{for } p = 0, \ldots, n$$

by

$$c_p(x) = \#\{j \in \mathbf{N}[1, q] \mid u_p(D_j)(x) = 0\}.$$

DEFINITION 3.1. Let $k_0, k_1 \in \mathbf{N}$ such that $n_0 \le k_1$. We say that $D_1, \ldots, D_q$ are in *general position of order* (k_0, k_1) *with respect to* X if $c_0(x) \le k_1$ for every $x \in X$ and if $\alpha_{j_0}, \ldots, \alpha_{j_{k_1}}$ span a linear subspace of dimension at least $k_0 + 1$ in $(\mathbf{C}^{n+1})^*$ for every choice of $1 \le j_0 < \cdots < j_{k_1} \le q$.

We observe that if $D_1, \ldots, D_q$ are in general position of order (k_0, k_1) with respect to X then

$$n_0 \le k_0 \le \operatorname{Min}(k, n) \tag{3.1}$$

and for any $t \le k_1$ and $1 \le j_0 < \cdots < j_t \le q$ then

$$\dim \bigcap_{h=0}^{t} D_{j_h} \le (n - k_0) + (k_1 - t). \tag{3.2}$$

Now proceeding as in [3] for the proof of Lemma 3.2, for any $x \in X$ we have

$$c_p(x) \le \lambda(k_0, k_1, p) \quad \text{for } p = 0, \ldots, n \tag{3.3}$$

where $\lambda(k_0, k_1, p)$ is an abbreviation for $\operatorname{Min}(k_1, n - k_0 + k_1 - t)$.

Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map not contained in any hyperplane in

$\mathbf{P}_n$. Consider a holomorphic $(m-1, 0)$ form B. Assume f is general for B and $f(\mathbf{C}^m) \subseteq X$, then $f_p(\mathbf{C}^m) \subseteq X_p$ for $p = 0, \ldots, n$. So the map $\phi_p(D) = u_p(D) \circ f_p$ is well defined for every hyperplane D. Set

$$m_p(r, D) = -\int_{\mathbf{C}^m\langle r\rangle} \log \phi_p(D)\sigma.$$

Then we have

$$m_0(r, D) = m_f(r, D), \tag{3.4}$$

$$m_n(r, D) = 0 \quad (\text{since } f_n \text{ is constant}). \tag{3.5}$$

From (3.4) and (3.5) we get

$$\sum_{p=0}^{n-1} (m_p(r, D) - m_{p+1}(r, D)) = m_f(r, D). \tag{3.6}$$

Let $D_1, \ldots, D_q$ be distinct hyperplanes in $\mathbf{P}_n$ in general position of order (k_0, k_1) w.r.t. X. Set

$$Y_p = \bigcup_{j=1}^{q} (u_p(D_j)^{-1}(0)) \leq X_p \quad \text{for } p = 0, \ldots, n.$$

Then, similarly as in [3] for Proposition 4.1 we have

$$\sum_{j=1}^{q} \log\left(\frac{\phi_{p+1}(D_j)}{\phi_p(D_j)^{1-\beta}}\right) \leq \lambda(k_0, k_1, p) \log\left(\sum_{j=1}^{q} \frac{\phi_{p+1}(D_j)}{\phi_p(D_j)^{1-\beta}}\right) + O(1) \tag{3.7}$$

on $\mathbf{C}^m - f_p^{-1}(Y_p)$, where $0 < \beta < 1$.

We note that, by (2.1) and (2.2),

$$-\sum_{p=0}^{n-1} \lambda(k_0, k_1, p) S_p(r)$$

$$= k_1 \int_{\mathbf{C}^m\langle r\rangle} \log |\mathscr{u}| \sigma + (k_1 - k_0) \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{u}_{n-1}|}{|\mathscr{u}_n|} \sigma$$

$$+ \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{u}_{n-k_0}|}{|\mathscr{u}_n|} \sigma.$$

Set $$Q_k(r, f) = \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{u}_{n-k}|}{|\mathscr{u}_n|} \sigma \quad \text{and} \quad Q(r, f) = Q(r, f).$$

Therefore by (1.3) we have

$$-\sum_{p=0}^{n-1} \lambda(k_0, k_1, p) S_p(r) \tag{3.8}$$

$$= k_1 T_f(r, r_0) + (k_1 - k_0) Q(r, f) + Q_{k_0}(r, f) + O(1).$$

4. Defect relation

Before stating the theorem we fix some notations. Let g and h be real valued functions on $\mathbf{R}(r_0, \infty)$. We write $g(r) \lesssim h(r)$ if a subset E of $\mathbf{R}(r_0, \infty)$ with finite Lebesgue measure exists such that $g(r) \leq h(r)$ for all $r \in \mathbf{R}(r_0, \infty) - E$.

We set

$$\tau_k(f) = \limsup_{r \to \infty} \frac{Q_k(r, f)}{T_f(r, r_0)}$$

and $\tau_f = \tau_1(f)$.

THEOREM 4.1. (SECOND MAIN THEOREM AND DEFECT RELATION). *Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be a non-degenerate, transcendental, meromorphic map. Let $D_1, \ldots, D_q$ be hyperplanes in $\mathbf{P}_n$ in general position of order (k_0, k_1) with respect to a projective variety X of dimension n_0 in $\mathbf{P}_n$. Then*

$$\sum_{j=1}^{q} m_f(r, D_j) \lesssim k_1 T_f(r, r_0) + (k_1 - k_0)Q(r, f) + Q_{k_0}(r, f) + O(\log r T_f(r, r_0)), \tag{4.1}$$

and

$$\sum_{j=1}^{q} \delta_f(D_j) \leq k_1 + (k_1 - k_0)\tau_f + \tau_{k_0}(f). \tag{4.2}$$

Proof. Since the proof is rather long and since it is similar to the proof of the Second Main Theorem in [4] for $m = 1$ and in [9] or [12] for the general case, we shall give here only a sketch of it.

By (3.6) we have

$$\sum_{j=1}^{q} m_f(r, D_j) = \sum_{j=1}^{q} \sum_{p=0}^{n} (m_p(r, D_j) - m_{p+1}(r, D_j)). \tag{4.3}$$

Let $\gamma_p = \mathrm{Max}_{1 \leq j \leq q}\, m_p(r_0, D_j)$ and

$$\beta(r) = \frac{1}{q(T_{f_p}(r, r_0) + \gamma_p)}.$$

Since $T_{f_p}(r, r_0) \to \infty$ for $r \to \infty$ then there exists $r' > r_0$ such that $0 < \beta(r) < 1$ for all $r > r'$. Using (3.7) and proceeding similarly as in [9] for Lemma 11.4 we get

$$\lambda(k_0, k_1, p)S_p(r) + \sum_{j=1}^{q} (m_p(r, D_j) - m_{p+1}(r, D_j)) + O(1) \leq \frac{\lambda(k_0, k_1, p)}{2} \int_{\mathbf{C}^m\langle r\rangle} \log\left(\sum_{j=1}^{q} \frac{\phi_{p+1}(D_j)}{\phi_p(D_j)^{1-\beta(r)}}\, h_p\right)\sigma. \tag{4.4}$$

Using Ahlfors Estimates (see [9] Theorem 10.3) and proceeding as in [9] for lemma 11.5 we get

$$\int_{\mathbf{C}^m\langle r\rangle} \log\left(\sum_{j=1}^{q} \frac{\phi_{p+1}(D_j)}{\phi_p(D_j)} h_p\right)\sigma \le O(\log rT_f(r, r_0)). \tag{4.5}$$

Then (4.4) and (4.5) yield

$$\lambda(k_0, k_1, p)S_p(r) + \sum_{j=1}^{q} (m_p(r, D_j) - m_{p+1}(r, D_j)) \le O(\log rT_f(r, r_0)). \tag{4.6}$$

Therefore by (3.8), (4.3) and (4.6) we get (4.1). Since f is transcendental, by the definition of τ_f, $\tau_k(f)$ and (4.1) we get

$$\begin{aligned}\sum_{j=1}^{q} \delta_f(D_j) &= \sum_{j=1}^{q} \liminf_{r\to\infty} \frac{m_f(r, D_j)}{T_f(r, r_0)} \\ &\le \liminf_{r\to\infty}\left(\sum_{j=1}^{q} \frac{m_f(r, D_j)}{T_f(r, r_0)}\right) \\ &\le k_1 + (k_1 - k_0)\tau_f + \tau_{k_0}(f), \qquad \text{Q.E.D.}\end{aligned}$$

We observe that

$$\begin{aligned}Q_n(r, f) &= T_f(r, r_0) - N_\theta(r, r_0) \\ &= O(1) \\ &\le T_f(r, r_0) + O(1)\end{aligned} \tag{4.7}$$

where θ is the Wronskian divisor of f. More generally,

$$Q_k(r, f) = \sum_{s=n-k+1}^{n} \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{U}_{s-1}|}{|\mathscr{U}_s|}\,\sigma$$

and since (see [9] Proposition 10.6)

$$\begin{aligned}S_p(r) &= \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{U}_{p-1}|}{|\mathscr{U}_p|}\,\sigma - \int_{\mathbf{C}^m\langle r\rangle} \frac{|\mathscr{U}_p|}{|\mathscr{U}_{p+1}|}\,\sigma \\ &\mathrel{\underset{\bullet}{\le}} O(\log rT_f(r, r_0))\end{aligned}$$

we have

$$Q_k(r, f) \mathrel{\underset{\bullet}{\le}} kQ(r, f) + O(\log rT_f(r, r_0)). \tag{4.8}$$

Therefore (4.7) and (4.8) imply the following result.

COROLLARY 4.2. *With the same notations as in Theorem* 4.1 *we have*

$$\sum_{j=1}^{q} m_f(r, D_j) \,\dot{\leq}\, \begin{cases} k_1(T_f(r, r_0) + Q(r, f)) \\ \qquad + O(\log rT_f(r, r_0)) \\ 2k_1 T_f(r, r_0) + O(\log rT_f(r, r_0)) \\ & \text{if } n = 1 \\ (k_1 + 1)T_f(r, r_0) + (k_1 - n)Q(r, f) \\ \qquad + O(\log rT_f(r, r_0)) \\ & \text{if } k_0 = n \\ (n + 1)T_f(r, r_0) + O(\log rT_f(r, r_0)) \\ & \text{if } k_1 = k_0 = n \end{cases} \tag{4.9}$$

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \begin{cases} k_1(1 + \tau_f) \\ 2k_1 & \text{if } n = 1 \\ k_1 + 1 + (k_1 - n)\tau_f & \text{if } k_0 = n \\ n + 1 & \text{if } k_1 = k_0 = n. \end{cases} \tag{4.10}$$

Remark 4.3. If $k_1 = k_0 = n$ and $X = \mathbf{P}_n$ then "general position of order (k_0, k_1) with respect to $\mathbf{P}_n$" is the same as "general position". Therefore (4.10) for $k_0 = k_1 = n$ is the classical result.

5. An application

First we fix some notations and recall some known results. Let Y be a compact, complex, n-dimensional manifold. Let L be a line bundle over Y. Set $N + 1 = \dim_{\mathbf{C}} \Gamma(Y, L)$. Let $\psi\colon Y \to \mathbf{P}_N$ be the *dual classification map*. Then L is *spanned* if and only if ψ is a holomorphic map. In addition, if L is spanned, we have that $\psi(Y)$ is a projective variety in $\mathbf{P}_N$. If H is the hyperplane section bundle over $\mathbf{P}_N$ then $\psi^*(H) = L$ and $\psi^*\colon \Gamma(\mathbf{P}_N, H) \to \Gamma(Y, L)$ is an isomorphism.

DEFINITION 5.1. Let $D_1, \ldots, D_q$ be divisors of L. We say that $D_1, \ldots, D_q$ are in *general position* if no point of Y is contained in $n + 1$ distinct D_j.

We shall need later the following general assumptions.

(A1) Let Y be a compact, complex n-dimensional manifold and L a line bundle over Y with hermitian metric $\psi^*(\kappa)$ the pull-back of the metric in the hyperplane section bundle H over $\mathbf{P}_N$. Set $n_0 = \dim \psi(Y)$.

(A2) Let $f: \mathbf{C}^m \to Y$ be a meromorphic map. Set

$$h = \psi \circ f: \mathbf{C}^m \to \mathbf{P}_N.$$

Assume h is not constant.

(A3) Let $\mathbf{P}_s \subseteq \mathbf{P}_N$ be a subspace of minimal dimension such that $h(\mathbf{C}^m) \subseteq \mathbf{P}_s$. Define $\tilde{h}: \mathbf{C}^m \to \mathbf{P}_s$ by $h(z) = \tilde{h}(z)$ for every $z \in \mathbf{C}^m$. If $\iota: \mathbf{P}_s \subsetneq \mathbf{P}_N$ is the inclusion then $h = \iota \circ \tilde{h}$. We have $\tilde{h}$ non-degenerate.

(A4) Let B be a holomorphic $(m-1, 0)$ form on $\mathbf{C}^m$. Assume $\tilde{h}$ is general for B and $i_{m-1} B \wedge \bar{B} \leq (1 + r^{2s-2})v^{m-1}$ on $\mathbf{C}^m[r]$.

(A5) Let $D_1, \ldots, D_q$ be distinct divisors of L in general position such that $f(\mathbf{C}^m) \nsubseteq \operatorname{supp} D_j$ for $j = 1, \ldots, q$. Assume $q \geq n + 1$.

DEFINITION 5.2. Assume (A1)–(A4). Then we define

$$Q(r, f) = Q(r, \tilde{h}), \tag{5.1}$$

$$\tau_f = \tau_{\tilde{h}}. \tag{5.2}$$

THEOREM 5.3. *Assume* (A1)–(A5). *Abbreviate* $T_f(r, r_0, L, \psi^*(\kappa))$ by $T_f(r, r_0)$. *Then*

$$\sum_{j=1}^{q} m_f(r, D_j) \leq \begin{cases} n(T_f(r, r_0) + Q(r, f)) + O(\log rT_f(r, r_0)) \\ 2nT_f(r, r_0) + O(\log rT_f(r, r_0)) \quad \text{if } s = 1 \end{cases} \tag{5.3}$$

and

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \begin{cases} n(1 + \tau_f) \\ 2n \quad \text{if } s = 1. \end{cases} \tag{5.4}$$

Proof. Let $\tilde{D}_1, \ldots, \tilde{D}_q$ be hyperplanes in $\mathbf{P}_N$ such that $\psi^*(\tilde{D}_j) = D_j$. Then

$$\begin{aligned} T_f(r, r_0) &= T_h(r, r_0), \\ N_f(r, r_0, D_j) &= N_h(r, r_0, \tilde{D}_j), \\ m_f(r, D_j) &= m_h(r, \tilde{D}_j). \end{aligned} \tag{5.5}$$

Moreover $\tilde{D}_1, \ldots, \tilde{D}_q$ are in general position of order (n_0, n) with respect to $\psi(Y)$. Let $P_j = \iota^*(\tilde{D}_j)$ be hyperplanes in $\mathbf{P}_s$. Then we have that $P_1, \ldots, P_q$ are in general position of order (n_0', n) with respect to $X = \psi(Y) \cap \mathbf{P}_s$ where $n_0' = \operatorname{Max}(\dim X, n_0 - N + s)$. Since $h = \iota \circ \tilde{h}$ we have

$$\begin{aligned} T_h(r, r_0) &= T_h(r, r_0, H, \kappa) = T_{\tilde{h}}(r, r_0, \iota^* H, \iota^* \kappa) = T_{\tilde{h}}(r, r_0), \\ N_h(r, r_0, \tilde{D}_j) &= N_{\tilde{h}}(r, r_0, P_j), \\ m_h(r, \tilde{D}_j) &= m_{\tilde{h}}(r, P_j) + O(1). \end{aligned} \tag{5.6}$$

Hence by (5.5) and (5.6) we get

$$
\begin{aligned}
T_f(r, r_0) &= T_{\tilde{h}}(r, r_0), \\
N_f(r, r_0, D_j) &= N_{\tilde{h}}(r, r_0, P_j), \qquad (5.7)\\
m_f(r, D_j) &= m_{\tilde{h}}(r, P_j) + O(1).
\end{aligned}
$$

Applying (4.9) and (4.10) to the map $\tilde{h}$ and hyperplanes $P_1, \ldots, P_q$ and using (5.7) we obtain (5.3) and (5.4), Q.E.D.

6. Exponential maps

In the previous sections we found that if f: $\mathbf{C}^m \to \mathbf{P}_n$ is a meromorphic map such that $f(\mathbf{C}^m) \subseteq \mathbf{P}_1$, then

$$Q(r, f) \leq T_f(r, r_0) + O(1).$$

Our aim, in this section, is to extend this result to a wider class of meromorphic maps.

Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be a non-constant meromorphic map with

$$\mathscr{u} = (f_0, \ldots, f_n)$$

as reduced representation. We say that f is an *exponential map* if $f_j = \psi_j \exp \phi_j$, where ψ_j and ϕ_j are holomorphic functions on $\mathbf{C}^m$ for $j = 0, \ldots, n$, and there exists a holomorphic function u on $\mathbf{C}^m$ such that if $h_j = \psi_j u^{-1}$ then

$$T_{h_j}(r, r_0) = o(T_f(r, r_0)).$$

We also say that the holomorphic function u satisfying the above condition is *admissible* for f.

We note that if f is an exponential map then f is transcendental (see Mori [5]).

Let $\mathscr{u} = (\psi_0 \exp \phi_0, \ldots, \psi_n \exp \phi_n)$ be the reduced representation of the exponential map f. Then we set

$$R(\mathscr{u}) = (\exp \phi_0, \ldots, \exp \phi_n)$$

and

$$I(\mathscr{u}) = (\exp(-\phi_0), \ldots, \exp(-\phi_n)).$$

Then $R(f) = \mathbf{P} \circ R(\mathscr{u})$ and $I(f) = \mathbf{P} \circ I(\mathscr{u})$ are exponential maps. We say that f is a special exponential map (S.E.M.) if

$$T_{I(f)}(r, r_0) = T_{R(f)}(r, r_0) + o(T_f(r, r_0)). \qquad (6.1)$$

Definition 6.1. Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. We say that $f \in \mathscr{R}$ (or $f \in \mathscr{R}_S$) when the following are satisfied.

(i) There exist an exponential map (or an S.E.M.) g: $\mathbf{C}^m \to \mathbf{P}_N$ and a linear map λ: $\mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ such that $f = \mathbf{P}(\lambda) \circ g$.

(ii) If $\mathscr{u}$ and $\mathscr{g}$ are reduced representations of f and g respectively and if u is a holomorphic function on $\mathbf{C}^m$ such that $u\mathscr{u} = \lambda \circ \mathscr{g}$ then u is admissible for g.

(iii) g is non-degenerate.

(iv) $\lambda(\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0) \neq 0$ for $j = 0, \ldots, N$.

Let $f \in \mathscr{R}$ then (g, λ) defined above satisfying (i)–(iv) is called a *decomposition* of f.

Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. We also must define $Q(r, f)$ when f is degenerate.

Let $\mathbf{P}_k \subseteq \mathbf{P}_n$ be the subspace of minimal dimension such that $f(\mathbf{C}^m) \subseteq \mathbf{P}_k$. Then $\tilde{f}$: $\mathbf{C}^m \to \mathbf{P}_k$ defined by $\tilde{f}(z) = f(z)$ for every $z \in \mathbf{C}^m$ is non-degenerate. Let B be a holomorphic $(m-1, 0)$ form on $\mathbf{C}^m$ whose coefficients are polynomials of degree at most $k-1$ and therefore satisfying (2.3). Assume $\tilde{f}$ is general for B. Then we define

$$Q(r, f) = Q(r, \tilde{f}).$$

If $f \in \mathscr{R}$ with (g, λ) as decomposition then $\tilde{f} \in \mathscr{R}$ with $(g, \tilde{\lambda})$ as decomposition, where $\tilde{\lambda}$: $\mathbf{C}^{N+1} \to \mathbf{C}^{k+1}$ is defined by $\tilde{\lambda}(z) = \lambda(z)$ for every $z \in \mathbf{C}^{N+1}$.

PROPOSITION 6.2. *For every $f \in \mathscr{R}_S$ we have*

$$Q(r, f) \mathrel{\underset{\cdot}{\leq}} T_f(r, r_0) + o(T_f(r, r_0)). \tag{6.2}$$

As a direct consequence of Proposition 6.2 we have the following result.

THEOREM 6.3. *Assume that* (A1)–(A5) *holds. Then if $h(\mathbf{C}^m) \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_N$ or if $h \in \mathscr{R}_S$ we have*

$$\sum_{j=1}^{q} m_f(r, D_j) \mathrel{\underset{\cdot}{\leq}} 2nT_f(r, r_0) + o(T_f(r, r_0)), \tag{6.3}$$

$$\sum_{j=1}^{q} \delta_f(D_j) \leq 2n. \tag{6.4}$$

Before proving Proposition 6.2 we want to show that $\mathscr{R}_S$ is not empty. In fact it extends the class of meromorphic maps for which Shiffman [7] proved (6.4).

PROPOSITION 6.4. *Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be an exponential map with*

$$\mathscr{u} = (\psi_0 \exp \phi_0, \ldots, \psi_n \exp \phi_n)$$

as reduced representation. If one of the following conditions is satisfied then f is an S.E.M.

1. *There exists an isometry* $\alpha\colon \mathbf{C}^m \to \mathbf{C}^n$ *such that* $-\phi_j = \phi_j \circ \alpha$ *for* $j = 0, \ldots, n$.

2. *There exist a holomorphic function* ϕ *on* $\mathbf{C}^m$ *and real numbers* $\lambda_0, \ldots, \lambda_n$ *such that* $\phi_j = \lambda_j \phi$ *for* $j = 0, \ldots, n$.

Proof. If f satisfies condition 1 then since σ is invariant by isometry we get $T_{R(f)}(r, r_0) = T_{I(f)}(r, r_0)$ and therefore (6.1).

Suppose now that f satisfies condition 2. Let

$$(j_0, \ldots, j_n)$$

be a permutation of $(0, 1, \ldots, n)$ such that $\lambda_{j_0} \le \cdots \le \lambda_{j_n}$. Let $\alpha_k = \lambda_{j_k} - \lambda_{j_0}$ and $a = \lambda_{j_0}$. Then there exist constants $c_1 > c_0 > 0$ such that

$$\text{(6.5)} \qquad c_0 |e^{2a\phi}|(1 + |e^{\phi}|^2)^{\alpha_n} \le \sum_{j=0}^{n} |e^{\phi}|^{2\lambda_j} \le c_1 |e^{2a\phi}|(1 + |e^{\phi}|^2)^{\alpha_n}.$$

Therefore if $h = \mathbf{P}(1, e^{\Phi})\colon \mathbf{C}^m \to \mathbf{P}_1$ then, by (1.3),

$$T_{R(f)}(r, r_0) = \alpha_n T_h(r, r_0) + O(1)$$

and

$$T_{I(f)}(r, r_0) = \alpha_n T_h(r, r_0) + O(1).$$

Hence (6.1) is satisfied, Q.E.D.

Remark 6.5. (a) Condition 1 in Proposition 6.4 is clearly satisfied when all ϕ_j are homogeneous polynomials of the same degree or in general when $\phi_j = \sum_{k=0}^{\infty} P_{jk}$ where P_{jk} are homogeneous polynomials of degree $2^h(2k + 1)$ for a fixed $h \in \mathbf{Z}[0, \infty)$. For example when $h = 0$ then ϕ_j are odd functions.

(b) Processed as in [2] for the proof of Proposition 6.1 it is possible to prove that all the meromorphic maps considered in [7] by Shiffman are in $\mathscr{R}$. Moreover if (g, λ) is a decomposition of a meromorphic map in [7] then

$$g = \mathbf{P}(\psi_0 \exp P_0, \ldots, \psi_N \exp P_N)$$

where all P_j are homogeneous polynomials of the same degree. Therefore by (a) we have that $\mathscr{R}_S$ extends the Shiffman class.

Proof of Proposition 6.2. Let (g, λ) be a decomposition of f. Then from (6.1) and

$$\text{(6.6)} \qquad T_g(r, r_0) \,\|\!\le T_f(r, r_0) + o(T_f(r, r_0)),$$

$$T_{R(g)}(r, r_0) \lesssim T_g(r, r_0) + o(T_f(r, r_0)), \tag{6.7}$$

$$Q(r, g) \lesssim T_{I(g)}(r, r_0) + o(T_f(r, r_0)), \tag{6.8}$$

$$Q(r, f) \lesssim Q(r, g) + o(T_f(r, r_0)), \tag{6.9}$$

we will get (6.2). Therefore we will prove (6.6)–(6.9). First we note that (6.6) is a direct consequence of Proposition 4.3 in [2]. Let $\mathscr{g} = (\psi_0 \exp \phi_0, \dots, \psi_N \exp \phi_N)$ and $\mathscr{u}$ be reduced representations of g and f respectively and u a holomorphic function such that $u\mathscr{u} = \lambda \circ \mathscr{g}$. Then if $h_j = \psi_i/u$ we have by assumption $T_{h_j}(r, r_0) = o(T_g(r, r_0))$. Since

$$|R(\mathscr{g})| \leq \left(\sum_{j=1}^{N} |h_j|^{-2} \right)^{1/2} |u|^{-1} |\mathscr{g}|$$

and

$$\int_{\mathbf{C}^m\langle r\rangle} \log \left(\sum_{j=0}^{N} |h_j|^2 \right)^{1/2} \sigma \leq \sum_{j=0}^{N} T_{h_j}(r, r_0) + O(1) \leq o(T_g(r, r_0)),$$

and we have (6.6), we get (6.7).

Set $\tilde{\mathscr{g}} = u^{-1}\mathscr{g}$. Then $\tilde{\mathscr{g}}_k = u^{-(k+1)}\mathscr{g}_k$ and

$$\begin{aligned} Q(r, g) &= \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{g}_{N-1}|}{|\mathscr{g}_N|} \sigma \\ &= \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\tilde{\mathscr{g}}_{N-1}|}{|\tilde{\mathscr{g}}_N|} \sigma - \int_{\mathbf{C}^m\langle r\rangle} \log |u| \sigma \\ &\leq \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\tilde{\mathscr{g}}_{N-1}|}{|\tilde{\mathscr{g}}_N|} \sigma. \end{aligned} \tag{6.10}$$

Write $\tilde{\mathscr{g}} = (\tilde{g}_0, \dots, \tilde{g}_N)$ where $\tilde{g}_j = h_j \exp \phi_j$ for $j = 0, \dots, N$. Then

$$\tilde{\mathscr{g}}^{(k)} = (\tilde{g}_0^{(k)}, \dots, \tilde{g}_N^{(k)})$$

and $\tilde{g}_j^{(k)} = d_{kj} \exp \phi_j$ where d_{kj} are meromorphic functions defined recursively by

$$d_{kj} = d'_{k-1,j} + \phi'_j d_{k-1,j} \text{ for } k \in \mathbf{N} \quad \text{and} \quad d_{0j} = h_j.$$

Set $\Phi = (d_{ij})$ for $i, j = 0, \dots, N$, $\Phi_k = (d_{ij})$ for $i = 0, \dots, N-1$ and $j = 0, \dots, k-1, k+1, \dots, N$ and $\tilde{\psi}_k = \det \Phi_k (\det \Phi)^{-1}$. Then it is not difficult to see that

$$|\tilde{\mathscr{g}}_{N-1}|\, |\tilde{\mathscr{g}}_N|^{-1} = \left(\sum_{j=0}^{N} |\tilde{\psi}_k e^{-\phi_k}|^2 \right)^{1/2}.$$

Proceeding as for the proof of (6.7) we get

$$(6.11)\qquad \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\tilde{\mathscr{g}}_{N-1}|}{|\tilde{\mathscr{g}}_N|}\,\sigma \le T_{I(g)}(r, r_0) + \sum_{k=0}^{N} T_{\tilde{\psi}_k}(r, r_0) + O(1).$$

Now a standard technique in Value Distribution Theory and the Lemma of the Logarithmic Derivative (see Vitter [10]) give us

$$(6.12)\qquad T_{\tilde{\psi}_k}(r, r_0) \lesssim o(T_g(r, r_0)).$$

Then (6.10), (6.11), (6.12 and (6.6) imply (6.8).

In order to prove (6.9), without loss of generality we may assume f is non-degenerate. Consider $\varepsilon \in \bigwedge_{N-n} \mathbf{C}^{N+1}$ such that $E(\mathbf{P}(\varepsilon)) = \operatorname{Ker} \lambda$. Then there exist constants $c_1 > c_0 > 0$ such that

$$c_0 |\mathscr{g}_k \wedge \varepsilon| \le |(\lambda \circ \mathscr{g})_k| = |u|^{k+1} |\mathscr{u}_k| \le c_1 |\mathscr{u}_k \wedge \varepsilon|$$

for $k = 0, \ldots, n$. Therefore

$$\int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{u}_{n-1}|}{|\mathscr{u}_n|}\,\sigma \le \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{g}_{n-1} \wedge \varepsilon|}{|\mathscr{g}_n \wedge \varepsilon|}\,\sigma + \int_{\mathbf{C}^m\langle r\rangle} \log |u|\,\sigma + O(1).$$

Since u is admissible for g, $N_u(r, r_0, 0) = o(T_g(r, r_0))$. Hence

$$(6.13)\qquad Q(r, g) \le \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{g}_{n-1} \wedge \varepsilon|}{|\mathscr{g}_n \wedge \varepsilon|}\,\sigma + o(T_f(r, r_0)).$$

Choose an orthonormal base $e_0, \ldots, e_N$ in $\mathbf{C}^{N+1}$ such that

$$\varepsilon = e_{n+1} \wedge \cdots \wedge e_N.$$

Define $\alpha_k \in (\bigwedge_N \mathbf{C}^{N+1})^*$ by

$$\alpha_k(x) = x \wedge e_k \quad \text{for } x \in \bigwedge_N \mathbf{C}^{N+1} \text{ and } k = 0, \ldots, N.$$

Set

$$\mathscr{h}_{(k)} = \mathscr{g}_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_N$$

and

$$h_{(k)} = \mathbf{P}(\mathscr{h}_{(k)}) \colon \mathbf{C}^m \to \mathbf{P}(\bigwedge_N \mathbf{C}^{N+1}) \simeq \mathbf{P}_N.$$

Then by Ahlfors Estimate, and since f is transcendental, we get

$$\int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{h}_{(k)1} \llcorner \alpha_k|}{|\mathscr{h}_{(k)} \llcorner \alpha_k|\,|\mathscr{h}_{(k)}|}\,\sigma \lesssim o(T_f(r, r_0)).$$

Moreover we have (see [8] Hilfsatz 4)

$$|\mathscr{h}_{(k)_1} \llcorner \alpha_k| = |\mathscr{g}_{k-2} \wedge e_k \wedge \cdots \wedge e_N| \, |\mathscr{g}_k \wedge e_{k+1} \wedge \cdots \wedge e_N|.$$

Therefore we have

$$(6.14) \quad \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{g}_{k-2} \wedge e_k \wedge \cdots \wedge e_N|}{|\mathscr{g}_{k-1} \wedge e_k \wedge \cdots \wedge e_N|} \sigma \leq \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{g}_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_N|}{|\mathscr{g}_k \wedge e_{k+1} \wedge \cdots \wedge e_N|} \sigma + o(T_f(r, r_0))$$

for $k = 1, \ldots, N$. Applying (6.14) recursively we have

$$(6.15) \quad \int_{\mathbf{C}^m\langle r\rangle} \log \frac{|\mathscr{g}_{n-1} \wedge \varepsilon|}{|\mathscr{g}_n \wedge \varepsilon|} \sigma \leq Q(r, g) + o(T_f(r, r_0))$$

and by (6.13) we get (6.9), Q.E.D.

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