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A NON-REMOVABLE SET FOR ANALYTIC FUNCTIONS SATISFYING A ZYGMUND CONDITION

BY

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1. Introduction

A complex-valued function f defined on the complex plane $\mathbf{C}$ satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, if there exists a constant $C(f)$ such that

$$|f(z+h) - f(z)| \leq C(f)|h|^{\alpha}$$

for all complex z and h. This condition is obviously stronger than

$$|f(z+h) + f(z-h) - 2f(z)| \leq C(f)|h|^{\alpha}$$

which does not necessarily imply the continuity of f. When $\alpha = 1$, the latter condition is usually called the Zygmund condition. We shall denote the classes of bounded continuous functions which satisfy the above conditions respectively by Lip_α and Λ_α. If $0 < \alpha < 1$, it is well known (see [5, Chap. V, Section 4]) that Lip_α and Λ_α are identical but $\mathrm{Lip}_1 \subsetneqq \Lambda_1$.

We shall call a compact subset E of $\mathbf{C}$, a removable set for analytic functions of class Lip_α, resp. Λ_α, provided that every function in Lip_α, resp. Λ_α, which is analytic in $\mathbf{C} \setminus E$ has analytic extension to the entire plane. Dolženko [1] proved that E is removable for analytic functions of class Lip_α, $0 < \alpha < 1$, if and only if E has $(1+\alpha)$-dimensional measure zero. In [6] we showed that this result is also true for the case $\alpha = 1$. Thus the removable sets for analytic functions of class Λ_1 must also have zero $dx\,dy$-measure.

In this paper we shall construct a compact set E of zero $dx\,dy$-measure and a probability Borel measure μ, supported on E, such that its Cauchy transform

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}$$

belongs to Λ_1. Since $\hat{\mu}(z) = -1/z + \cdots$ at ∞, $\hat{\mu}$ cannot be entire.

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Throughout this paper we will denote the Lebesgue measure in the plane by m, and for convenience, we denote by C certain absolute constants, not necessarily the same in different occurrences.

2. Construction of E and μ

We start with the unit square

$$Q = \{z = x_1 + ix_2 : 0 \le x_1 \le 1 \text{ and } 0 \le x_2 \le 1\}$$

of the complex plane. For $n = 1, 2, 3, \dots$ let $\mathcal{G}_n$ be the grid of closed octadic squares of size 8^{-n} which are contained in Q. The members of $\mathcal{G}_n$ will be denoted by $Q_j^{(n)}$ where $j = 1, 2, \dots, (64)^n$.

We divide the squares of each grid into two types, called red and green squares, as follows.

The red squares of $\mathcal{G}_1$ will consist of 32 squares, where 28 of them are those squares that intersect the boundary ∂Q. The remaining 4 squares are chosen arbitrarily in the interior of Q.

Now suppose $n \ge 2$. For each $Q_j^{(n-1)} \in \mathcal{G}_{n-1}$, we choose 32 squares of $\mathcal{G}_n$ which are contained in $Q_j^{(n-1)}$, in such a way that 28 of them intersect $\partial Q_j^{(n-1)}$. As above, the other 4 squares are arbitrarily chosen in the interior of $Q_j^{(n-1)}$. The red squares of $\mathcal{G}_n$ are those chosen in this way, and the rest are green; red squares are labeled $R_j^{(n)}$, green squares $G_j^{(n)}$.

Now we define a sequence $\{\varphi_n\}$ $(n = 1, 2, \dots)$ of Rademacher functions as follows:

$$\varphi_n(z) = \begin{cases} 1 & \text{if } z \in \operatorname{int} G_j^{(n)} \text{ for some } j, \\ -1 & \text{if } z \in \operatorname{int} R_j^{(n)} \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then, by induction, we derive a sequence of functions $\{f_n\}$ as follows. We set

$$f_1(z) = \begin{cases} 1 + \varphi_1(z) & \text{if } z \in Q, \\ 0 & \text{otherwise.} \end{cases} \tag{2a}$$

If f_n has been defined, then

$$f_{n+1}(z) = \begin{cases} f_n(z) + \varphi_{n+1}(z) & \text{if } f_n(z) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2b}$$

It is clear that each f_n assumes only nonnegative integral values, $f_n \le n + 1$, and f_n is constant on the interior of any octadic squares of size 8^{-n}. Furthermore, since

$$\iint_{Q_j^{(n)}} \varphi_{n+1}\, dm = 0,$$

we obtain

$$\iint f_n \, dm = 1, \quad n = 1, 2, 3, \ldots \tag{3}$$

and

$$\iint_{Q_j^{(n)}} f_{n+k} \, dm = \iint_{Q_j^{(n)}} f_n \, dm \tag{4}$$

for all $k = 1, 2, 3, \ldots$. Therefore the sequence $\{f_n\}$ converges to a unique probability Borel measure in the weak-star topology.

THEOREM 1. *Let μ be the limit of the sequence $\{f_n\}$ in the weak-star topology and E be the support of μ. Then $m(E) = 0$.*

Proof. For $n = 1, 2, 3, \ldots$ define

$$S_n(z) = 1 + \varphi_1(z) + \varphi_2(z) + \cdots + \varphi_n(z), \quad z \in Q.$$

it is well known that $S_n = 0$ infinitely often almost everywhere (see [2, Chap. XIV]). The support E omits the interior of any square Q_n such that $S_n = 0$. Hence $m(E) = 0$.

3. Density property of μ

For each square I let $\delta(I) = \mu(I)/m(I)$. If I is an octadic square of size 8^{-n}, it is easy to see that this ratio is equal to the common value of f_n in $\operatorname{int} I$. Thus,

$$|\delta(I) - \delta(I')| \leq 2 \tag{5}$$

for any two adjacent octadic square of the same size.

Property (5) is essential in proving the following theorem.

THEOREM 2. *Let*

$$S = \left\{ z = a + \rho e^{i\varphi} \colon \varphi_0 \leq \varphi \leq \varphi_0 + \theta, 0 \leq \rho \leq r \right\}$$

and

$$S' = \left\{ z = a + \rho e^{i\varphi} \colon \varphi_0 + \theta \leq \varphi \leq \varphi_0 + 2\theta, 0 \leq \rho \leq r \right\}$$

be two adjacent sectors of the same center and size. Then there exists an absolute *constant C such that*

$$|\mu(S) - \mu(S')| \leq Cm(S). \tag{6}$$

Proof. The proof we give here is based on a method used by Kahane in [4].

First we assume that the angle θ is not too small, say $\theta \geq \pi/4$, so that the length of the arc on ∂S is comparable with radius r. Under this assumption we see that

$$\tfrac{1}{8}\pi r^2 \leq m(S) \leq \tfrac{1}{2}\pi r^2. \tag{7}$$

An octadic square contained in int S will be called maximal if any expanded octadic square crosses the boundary ∂S. Let p be the smallest integer such that there exists an octadic square of size 8^{-p} contained in int S. For each $j = p, p+1, p+2, \dots$ let $\{\omega_k^j\}$, $k = 1, 2, \dots, n_j$, be the collection of those maximal squares of size 8^{-j} contained in int S. Then

$$\operatorname{int} S = \bigcup_{j=p}^{\infty}\left(\bigcup_{k=1}^{n_j} \omega_k^j\right)$$

and with a little computation we can show that

$$m\left(\bigcup_{k=1}^{n_j} \omega_k^j\right) \leq Cr8^{-j}. \tag{8}$$

Now let ω^* be an octadic square of size 8^{-p+1} which intersects S. There are at most N such squares, where N is independent of S. Since each ω_k^j is contained in some ω^*, it follows from (5) that

$$|\delta(\omega_k^j) - \delta(\omega^*)| \leq 2N + j - p + 1$$

and that

$$\begin{aligned}
|\mu(S) - m(S)\delta(\omega^*)| &= \left|\sum_{j=p}^{\infty}\sum_{k=1}^{n_j} m(\omega_k^j)(\delta(\omega_k^j) - \delta(\omega^*))\right| \\
&\leq \sum_{j=p}^{\infty}\sum_{k=1}^{n_j} m(\omega_k^j)(2N + j - p + 1) \\
&\leq 2Nm(S) + Cr\sum_{j=p}^{\infty} 8^{-j}(j - p + 1)
\end{aligned}$$

by (8). Therefore we obtain

$$\begin{aligned}|\mu(S) - m(S)\delta(\omega^*)| &\le 2Nm(S) + Cr8^{-p}\sum_{j=1}^{\infty} j8^{-j}\\ &\le 2Nm(S) + Cr^2\\ &\le Cm(S)\end{aligned}$$

by (7).

If we choose ω^* which intersects both S and S', then

$$|\mu(S) - \mu(S')| \le |\mu(S) - m(s)\delta(\omega^*)| + |\mu(S') - m(S')\delta(\omega^*)| \le Cm(S).$$

This proves the theorem in case $\theta \ge \pi/4$.

We now consider the case when $\theta \le \pi/4$. We use circles centered at a to divide S into truncated sectors S_j, $j = 1, 2, 3, \ldots$ which have the property that the lengths of all sides of S_j are comparable with $r\theta$. Note that the perimeter of each S_j is $O(r\theta)$. Let S_j' be the corresponding sectors in S'. Then by an argument as above we can show that

$$|\mu(S_j) - \mu(S_j')| \le Cm(S_j)$$

for each $j = 1, 2, 3, \ldots$ It follows that

$$\begin{aligned}|\mu(S) - \mu(S')| &\le \sum_{j=1}^{\infty} |\mu(S_j) - \mu(S_j')|\\ &\le C\sum_{j=1}^{\infty} m(S_j)\\ &\le Cm(S).\end{aligned}$$

Remark. In view of (5) and the technique used in proving Theorem 2, we note that there exists an absolute constant C such that

$$|\mu(I) - \mu(I')| \le Cm(I) \tag{9}$$

for any two adjacent squares of the same size. Then it follows from (9) that

$$\mu(I) \le Ct^2 \log\frac{1}{t} \tag{10}$$

for every closed square of size $t \le t_0$. Then $\hat{\mu}$ is defined and continuous on the entire complex plane with modulus of continuity ω satisfying

$$\omega(\delta) = O\left(\delta \log\frac{1}{\delta}\right) \tag{11}$$

for small $\delta > 0$. See [3, chapter 3, Section 4].

4. The Zygmund condition

In this section we shall show that $\hat{\mu}$ satisfies a Zygmund condition. For this purpose, we define

$$\Phi(z, y) = (P_y * \hat{\mu})(z)$$

for all z and $y > 0$, where

$$P_y(z) = \frac{y}{\left(x_1^2 + x_2^2 + y^2\right)^{3/2}}, \quad z = x_1 + ix_2,$$

is the Poisson Kernel, modulo a constant, of the upper half space. It is well known (see [5, Chapter V, Section 4]) that $\hat{\mu}$ satisfies a Zygmund condition if and only if there exists a constant A such that

$$\left|\frac{\partial^2 \Phi}{\partial y^2}(z, y)\right| \leq \frac{A}{y}$$

for all z and $y > 0$. Since Φ is harmonic, this condition is equivalent to

$$\left|\frac{\partial^2 \Phi}{\partial z\, \partial \bar{z}}(z, y)\right| \leq \frac{A}{y}.$$

Thus, in view of the equality

$$\frac{\partial \Phi}{\partial \bar{z}} = -\pi(P_y * \mu),$$

it is sufficient to prove the conditions

(12a) $$\left|\frac{\partial P_y}{\partial x_1} * \mu(z, y)\right| \leq \frac{A}{y},$$

(12b) $$\left|\frac{\partial P_y}{\partial x_2} * \mu(z, y)\right| \leq \frac{A}{y}.$$

We shall give a proof of (12a). The proof of (12b) is technically the same.

We assume without loss of generality that $z = 0$. If (r, θ) are the polar coordinates of (x_1, x_2), then

$$\frac{\partial P_y}{\partial x_1} = \frac{-2x_1 y}{\left(x_1^2 + x_2^2 + y^2\right)^{5/2}} = \frac{-2ry\cos\theta}{\left(r^2 + y^2\right)^{5/2}}.$$

For y fixed, the function $\varphi(r) = ry/(r^2 + y^2)^{5/2}$ is increasing when $0 \le r \le y/2$ and decreasing when $r \ge y/2$. The maximum value of φ is $M = \varphi(y/2) = C/y^3$ and

$$\iint \tilde{\varphi}\left(\sqrt{x_1^2 + x_2^2}\right) dx_1\, dx_2 \le \frac{C}{y}, \tag{13}$$

where

$$\tilde{\varphi}(r) = \begin{cases} \varphi(y/2) & \text{if } 0 \le r \le y/2, \\ \varphi(r) & \text{otherwise.} \end{cases}$$

Let n be a positive integer and consider the points $Mj2^{-n}$, $j = 1, 2, \ldots, 2^n - 1$ in the range of φ. Let a_j, b_j be the inverse images by f of $Mj2^{-n}$ with $b_j > a_j$, and define

$$\alpha_j(r) = \begin{cases} 1 & \text{if } 0 \le r \le a_j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_j(r) = \begin{cases} 1 & \text{if } 0 \le r \le b_j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_n = M2^{-n} \sum_{j=1}^{2^n - 1} (\beta_j - \alpha_j).$$

We can verify easily that

$$0 \le \varphi(r) - \varphi_n(r) \le M/2^n, \quad r \ge 0.$$

Next, let $\theta_k = \cos^{-1}(k/2^n)$, $k = 1, 2, \ldots, 2^n$ and define

$$\gamma_k(\theta) = \begin{cases} 1 & \text{if } \theta_k \le \theta \le \pi/2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_n = \gamma_{2^n} - 2^{-n} \sum_{k=1}^{2^n} \gamma_k.$$

With a little computation we can show that

$$0 \le \cos\theta - \psi_n(\theta) \le 1/2^n, \quad 0 \le \theta \le \pi/2.$$

Now, consider the sectors

$$A_{j,k} = \{(x_1, x_2) \colon 0 \le r \le a_j,\ \theta_k \le \theta \le \pi/2\},$$

$$B_{j,k} = \{(x_1, x_2) \colon 0 \le r \le b_j,\ \theta_k \le \theta \le \pi/2\}.$$

Then, if we consider $\varphi_n\psi_n$ as a function of (x_1, x_2), it follows that

$$\varphi_n\psi_n = M2^{-n}\sum_j (\chi_{B_{j,2^n}} - \chi_{A_{j,2^n}}) + M4^{-n}\sum_{j,k} (\chi_{B_{j,k}} - \chi_{A_{j,k}}).$$

Furthermore, by (13) we obtain

(14)

$$M2^{-n}\sum_j \left[m(B_{j,2^n}) + m(A_{j,2^n})\right] + M4^{-n}\sum_{j,k} \left[m(B_{j,k}) + m(A_{j,k})\right] \leq C/y.$$

We now extend $\varphi_n\psi_n$ to the entire plane, in such a way that the resulting extension is odd in x_1 and even in x_2. Then we apply Theorem 2 to each pair of correspondent sectors whose adjacent side lies on the x_2-axis. It follows from (14) that

$$\left|\int \varphi_n\psi_n \, d\mu\right| \leq C/y.$$

Finally,

$$\left|\frac{\partial P_y}{\partial x_1} * \mu(0, y)\right| = \lim_{n\to\infty}\left|\int \varphi_n\psi_n \, d\mu\right| \leq \frac{C}{y}.$$

and (12a) is proved.

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