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AUTOMATIC CONTINUITY IS EQUIVALENT TO UNIQUENESS OF INVARIANT MEANS

BY

JOSEPH ROSENBLATT[1]

0. Let G be a group and suppose that (X, B, m) is a probability space on which G acts as a group of invertible measure-preserving transformations. Given a function $f\colon X \to R$ and $g \in G$, the translation ${}_g f$ is defined by ${}_g f(x) = f(g^{-1}x)$ for all $x \in X$. This defines a representation of G as a group of isometries of the Lebesgue spaces $L_p(X)$, $1 \le p \le \infty$. This representation will be called the *regular representation of* G. A *G-invariant mean* (for this representation of G as measure-preserving transformation of (X, B, m)) is a positive linear functional ϕ on $L_\infty(X)$ such that $\phi(1) = 1$ and $\phi({}_g f) = \phi(f)$ for all $g \in G$ and $f \in L_\infty(X)$. The integral with respect to m, denoted $\int dm$, is one such G-invariant mean. We say that the action of G has a *unique G-invariant mean* if and only if $\int dm$ is the only G-invariant mean. Because there will not be a unique G-invariant mean when the group action by G is not ergodic, we may assume when discussing invariant means that G acts ergodically on (X, B, m); i.e., if $A \in B$ and $m(gA \,\Delta\, A) = 0$ for all $g \in G$, then either $m(A) = 0$ or $m(A) = 1$. The question of uniqueness of G-invariant means is of considerable interest and has been successfully settled in many cases in recent years. See Drinfeld [4], Margulis [7], Rosenblatt [12], Sullivan [15].

A *G-invariant linear form* on $L_p(X)$ is a linear functional λ on $L_p(X)$ such that $\lambda({}_g f) = \lambda(f)$ for all $g \in G$ and $f \in L_p(X)$. We say that the representation of G on (X, B, m) has L_p *automatic continuity* if any G-invariant linear form on $L_p(X)$ is continuous in the L_p-norm topology. If the σ-algebra of invariant sets is infinite, then generally L_p automatic continuity fails to hold, so we may assume that G acts ergodically on (X, B, m) when discussing automatic continuity. Notice also that if the action by G is ergodic, then any continuous G-invariant linear form on $L_p(X)$, $1 \le p < \infty$, must be a constant multiple of $\int dm$. This is not necessarily the case on $L_\infty(X)$ because of the possible non-uniqueness of G-invariant means. Whether or not the representation of G has automatic continuity can hinge on the algebraic and/or topological properties of G, as well as the function space

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under consideration. See Bourgain [1], Meisters [8], Meisters and Schmidt [9], Rosenblatt [11], Saeki [13], Willis [16], and Woodward [17].

The main results below will show that uniqueness of G-invariant means and automatic continuity of the representation are equivalent for countable groups.

0.1 THEOREM. *Let G be a countable group which acts as an ergodic group of measure-preserving transformations of a probability space (X, B, m). Then there is a unique G-invariant mean on $L_\infty(X)$ if and only if any G-invariant linear form on $L_2(X)$ is continuous.*

Further discussion of this theorem, details of the proof, and related results are all included in the next section.

1. To see that automatic continuity is equivalent to a condition about the representation of G which is related to uniqueness of G-invariant means, we will need a couple of basic results about a linear operator S and its dual operator S^*.

1.1 PROPOSITION. *Let X and Y be Banach spaces and let S be a continuous linear operator from X to Y. Let S^* be the adjoint operator from Y^* to X^*.*
(a) *S is onto if and only if there exists $\delta > 0$ such that for all $y^* \in Y^*$, $\|S^*y^*\| \geq \delta\|y^*\|$.*
(b) *S^* is onto if and only if there exists $\delta > 0$ such that for all $x \in X$, $\|Sx\| \geq \delta\|x\|$.*

Proof. (a) If S is onto then there exists a constant K such that for all $y \in Y$, $\|y\| \leq 1$, there exists $x \in X$, $\|x\| \leq K$, such that $Sx = y$. Hence,

$$\begin{aligned}\|S^*y^*\| &= \sup_{\|x\|\leq 1} |S^*y^*(x)| = \sup_{\|x\|\leq 1} |y^*(Sx)| \\ &= \frac{1}{K} \sup_{\|x\|\leq K} |y^*(Sx)| \geq \frac{1}{K} \sup_{\|y\|\leq 1} |y^*(y)| \geq \frac{1}{K}\|y^*\|.\end{aligned}$$

Conversely, if S^* satisfies the inequality condition, then S^* is one-to-one and, as a simple argument with Cauchy sequences shows, S^* has closed range. So S is onto by Lemma VI.6.3 in Dunford and Schwartz [5].

(b) If S^* is onto, then by (a), there exists $\delta > 0$ such that for all $x^{**} \in X^{**}$, $\|S^{**}x^{**}\| \geq \delta\|x^{**}\|$. The evaluation mappings $e_X\colon X \to X^{**}$ and $e_Y\colon Y \to Y^{**}$ are isometries and $S^{**}e_X(x) = e_Y(Sx)$ for all $x \in X$. Hence, S satisfies the inequality condition of (b). Conversely, if S satisfies the inequality condition, then S is one-to-one and has closed range. Thus, Theorem VI.6.2 in Dunford and Schwartz [5] proves that S^* is onto. ∎

Using this proposition, we can determine the dual condition to automatic continuity. Let

$$L_p^0(X) = \left\{ f \in L_p(X) \colon \int f\,dm = 0 \right\}.$$

1.2 Definition. Let G be a group which acts as a group of measure-preserving transformations of a probability space (X, B, m). The action of G is said to have the L_p *mean-zero weak containment property* if for all $g_1, \dots, g_n \in G$, and $\varepsilon > 0$, there exists $f \in L_p^0(X)$ such that $\|f\|_p = 1$ and $\|_{g_i}f - f\|_p < \varepsilon$ for all $i = 1, \dots, n$.

Remark. In other words, the action of G has the L_p mean-zero weak containment property if and only if the regular representation of G restricted to $L_p^0(X)$ weakly contains the identity representation.

1.3 Proposition (Schmidt [14]). *The L_p mean-zero weak containment property for $1 \le p < \infty$ is equivalent to the L_2 mean-zero weak containment property.*

Proof. If the action is not ergodic, then the mean-zero weak containment holds on all $L_p(X)$, $1 \le p < \infty$. So we can assume that the action is ergodic and then the arguments in Schmidt [14] apply. ∎

Thus, if G has Kazhdan's property T as a discrete group, and G acts ergodically on (X, B, m), then the action of G does not have the L_p mean-zero weak containment property for any p, $1 \le p < \infty$, because it does not have the L_2 mean-zero weak containment property. However, if G is amenable as a discrete group, and (X, B, m) is non-atomic, then the representation of G does have the mean-zero weak containment property on all $L_p(X)$, $1 \le p < \infty$. Groups, like the free group on two generators, which are neither amenable nor have Kazhdan's property T fall somewhere between these extremes and can have different representations as measure-preserving transformations which may or may not have the mean-zero weak containment property. In fact, we have the following theorem.

1.4 Theorem. *The discrete group G has Kazhdan's property T if and only if for all representations of G as an ergodic group of measure-preserving transformations, the L_2 mean-zero weak containment property fails to hold. The discrete group G is amenable if and only if for all representations as an* *(ergodic) group of measure-preserving transformations of a non-atomic probability space, the L_2 mean-zero weak containment property holds.*

Proof. The first part is in Rosenblatt [10] and Connes and Weiss [3]. See also Schmidt [14]. The second part is essentially in Rosenblatt [12], see in particular the proof of Theorem 3.8. ∎

In Rosenblatt [10], [11] and Willis [16], the failure of the mean-zero weak containment property is shown to imply automatic continuity results. In fact, the following theorem holds. The idea for this proof comes from Willis [16].

1.5 THEOREM. *Let G be a countable group which acts as an ergodic group of measure-preserving transformations of a probability space (X, B, m). If $1 \le p \le \infty$, and if q is the index conjugate to p, then the representation of G has L_p automatic continuity if and only if the L_q mean-zero weak containment fails to hold.*

Proof. First, there is L_p automatic continuity if and only if $L_p^0(X)$ is this linear span of the functions of the form ${}_gf - f$ for $g \in G$ and $f \in L_p^0(X)$. To see this, note that when $L_p^0(X)$ is the linear span, then any G-invariant linear form λ must be zero on $L_p^0(X)$. By linearity of λ, it follows that λ is a scalar multiple of the integral with respect to m and is automatically continuous. Conversely, when $1 \le p < \infty$, this linear span is L_p-norm dense in $L_p^0(X)$ because the action of G is ergodic. So automatic continuity fails unless the linear span equals $L_p^0(X)$. The same argument applies when $p = \infty$, if there is a unique G-invariant mean on $L_\infty(X)$. But if there is not a unique G-invariant mean on $L_\infty(X)$, then the linear space Y which is the L_∞-norm closed linear span of functions of the form ${}_gf - f$ for $g \in G$ and $f \in L_\infty(X)$ has infinite codimension in $L_\infty(X)$ by the main theorem in Chou [2]. In this case there is a discontinuous linear functional on $L_\infty(X)/Y$, and therefore there is a discontinuous G-invariant linear form on $L_\infty(X)$.

Now enumerate G as a sequence $(g_n\colon n \ge 1)$. Let D_N be the subspace of functions of the form

$$\sum_{i=1}^{N} {}_{g_i}f_i - f_i \quad \text{where } f_1, \dots, f_N \in L_p^0(X).$$

As in Rosenblatt [10, Theorem 15], a Baire category argument shows that there is L_p automatic continuity if and only if for some N, $D_N = L_p^0(X)$. Fix N and define the linear operator

$$S_N\colon \bigoplus_{i=1}^{N} L_p^0(X) \to L_p^0(X)$$

by

$$S_N(f_1,\dots,f_N) = \sum_{i=1}^{N} {}_{g_i}f_i - f_i \quad \text{for } f_1,\dots,f_N \in L_p^0(X).$$

Then L_p automatic continuity is equivalent to S_N being onto for some N. Assume first that $1 \le p < \infty$. By Proposition 1.1a), S_N is onto if and only if S_N^* satisfies a lower norm estimate. But the dual operator S_N^* is

$$S_N^*\colon L_q^0(X) \to \bigoplus_{i=1}^{N} L_q^0(X)$$

given by

$$S_N^* f = \left({}_{g_1^{-1}}f - f,\dots,{}_{g_N^{-1}}f - f\right).$$

Hence, there is L_p automatic continuity if and only if there exists N and $\delta > 0$ such that for all $f \in L_q^0(X)$, $\|f\|_q = 1$, for some $i = 1,\dots,N$, $\|{}_{g_i^{-1}}f - f\|_q \ge \delta$. This last condition is exactly the failure of L_q mean-zero weak containment. When $p = \infty$, we use Proposition 1.1b) instead because now S_N is the dual of the operator

$$T_N\colon L_1^0(X) \to \bigoplus_{i=1}^{N} L_1^0(X)$$

given by

$$T_N f = \left({}_{g_1^{-1}}f - f,\dots,{}_{g_N^{-1}}f - f\right).$$

Thus, as above, there is L_∞ automatic continuity if and only if the mean-zero weak containment property fails to hold on $L_1(X)$. ∎

Remarks. (a) One implication in this theorem is false if the group is not countable. For example, if the group is the circle group T, then by Meisters and Schmidt [9], there is automatic continuity for T on $L_2(T, m)$ where m is the usual normalized Lebesgue measure. However, because G is abelian (and hence amenable as a discrete group), the mean-zero weak containment property holds for all $L_p(T)$, $1 \le p < \infty$. The argument above does show though that if the mean-zero weak containment property fails to hold on $L_q(X)$, $1 < q \le \infty$, then there is L_p automatic continuity for the conjugate index p. Indeed, in this case there would be a finitely-generated subgroup H of G such that there is automatic continuity on $L_p(X)$ given only invariance by H.

(b) It is not clear if, in the case that there is more than one G-invariant mean on $L_\infty(X)$, the linear span of functions of the form ${}_g f - f$ for $g \in G$ and $f \in L_\infty^0(X)$ is a closed subspace.

(c) In Theorem 1.10, it is shown that automatic continuity for countable groups G always fails for $L_1(X)$ when (X, B, m) is non-atomic. Thus, algebraic properties of G play a role in Theorem 1.5 only when $1 < p \leq \infty$.

1.6 COROLLARY. *For a countable group G which acts as an ergodic group of measure-preserving transformations of a probability space (X, B, m), there is automatic continuity on some $L_p(X)$, $1 < p \leq \infty$, if and only if there is automatic continuity on all $L_p(X)$, $1 < p \leq \infty$.*

Remark. Part of the above can be made quite explicit. Suppose that $g_1, \dots, g_n \in G$ and $\delta > 0$ are such that for all $f \in L_2^0(X)$, $\|f\|_2 = 1$, there exists some g_i with $\|{}_{g_i} f - f\|_2 \geq \delta$. Let

$$\mu \colon L_p^0(X) \to L_p^0(X)$$

be the operator given by

$$\mu f = \frac{1}{n+1}\left(f + \sum_{i=1}^{n} {}_{g_i} f\right).$$

Then $\|\mu\|_p < 1$ for all p, $1 < p < \infty$. See Rosenblatt [10], [11]. Therefore, given $f \in L_p^0(X)$, the series $F = f + \sum_{k=1}^{\infty} \mu^k f$ converges in L_p-norm and defines a function F which satisfies

$$f = F - \mu F = \sum_{i=1}^{n} \left(\frac{F}{n+1}\right) - {}_{g_i}\left(\frac{F}{n+1}\right).$$

This shows explicitly why any linear form on $L_p(X)$ which is invariant under $g_1, \dots, g_n$ must be a constant times $\int dm$.

Because of the association via dual operators of automatic continuity with the mean-zero weak containment property, there is necessarily an association of automatic continuity with uniqueness of invariant means. This was observed already in Rosenblatt [12], but the argument here completes this connection. The theorem that is needed is this one. See Rosenblatt [12] and Schmidt [14] for the details.

1.7 THEOREM. *Let G be a countable group which acts as a group of measure-preserving transformations of a probability space (X, B, m). Then G has a unique invariant mean on $L_\infty(X)$ if and only if the mean-zero weak containment property fails to hold for some (all) $L_p(X)$, $1 \leq p < \infty$.*

Remark. See Schmidt [14] for the analogue of Theorem 1.4 relating uniqueness of invariant means to either Kazhdan's property T or amenability.

Theorem 1.5 and Theorem 1.7 have the next theorem as a corollary. This proves Theorem 0.1.

1.8 COROLLARY. *If G is a countable group which acts as an ergodic group of measure-preserving transformations of a probability space (X, B, m), then G has a unique invariant mean on $L_\infty(X)$ if and only if there is automatic continuity on $L_p(X)$ for all (some) p, $1 < p \leq \infty$. Also, for any countable group G which acts as measure-preserving transformations of a probability space (X, B, m), there is a unique G-invariant mean on $L_\infty(X)$ if and only if there is a unique G-invariant linear form on $L_\infty(X)$ up to multiplication by a scalar.*

1.9 COROLLARY. *Let G be a countable group which acts as a group of measure-preserving transformations of a probability space (X, B, m). Then the L_∞-norm closure of the linear span of all functions of the form ${}_gf - f$, $g \in G$ and $f \in L_\infty(X)$, is equal to $L_\infty^0(X)$ if and only if there exists $g_1, \dots, g_n \in G$ such that for all $f \in L_\infty^0(X)$, there exists $f_1, \dots, f_n \in L_\infty^0(X)$ such that $f = \sum_{i=1}^n {}_{g_i}f_i - f_i$.*

Example. Here is an example of a two generator group, not having Kazhdan's property T, such that the uniqueness of invariant means and the automatic continuity above hold. Let (X, B, m) be the torus $T \times T$ with the usual Lebesgue measure. Let G be the group generated by the two transformations $g_1(x, y) = (y, x)$ and $g_2(x, y) = (x, xy)$. In Rosenblatt [12], it is shown that there exists a unique $\langle g_1, g_2 \rangle$-invariant mean on $L_\infty(T \times T)$. Hence, there is also automatic continuity on $L_p(T \times T)$ with respect to $\langle g_1, g_2 \rangle$ for all p, $1 < p \leq \infty$. See also Rosenblatt [10] where this example is discussed.

Remark. For the results above to hold, the group G needs to be countable. For example, there is automatic continuity on any $L_p(T)$, $1 < p < \infty$, by Bourgain [1], but there are many invariant means on $L_\infty(T)$. However, generally, if there is more than one invariant mean, then the mean-zero weak containment property does hold. See Rosenblatt [12]. It seems likely also that there can exist a very large amenable group of measure-preserving transformations of $[0, 1]$ in Lebesgue measure such that there is a unique invariant mean. Such a construction will probably need the Continuum Hypothesis or Martin's Axiom in the same way there were used in Yang [18] and Foreman [6] where it is shown, under these axioms respectively, that there exists a locally finite (hence, amenable) group G of permutations of the integers Z such that the bounded functions on Z admits a unique G-invariant mean.

Generally, it is recognized that on L_1 spaces automatic continuity will fail. In the context above, this is the corresponding theorem.

1.10 THEOREM. *Let G be a countable group which acts as an ergodic group of measure-preserving transformations of a probability space* (X, B, m). *Assume* (X, B, m) *is non-atomic. Then there are discontinuous G-invariant linear forms on* $L_1(X)$.

Proof. To show that there are discontinuous G-invariant linear forms on $L_1(X)$, it suffices to show that for any $g_1, \dots, g_n \in G$, the vector space of functions of the form $\sum_{i=1}^n g_i f_i - f_i$ where each $f_i \in L_1(X)$ cannot be $L_1^0(X)$. Suppose on the other hand that this happens. Then there is a constant K such that for all $f \in L_1^0(X)$, $\|f\|_1 \le 1$, there exists $f_1, \dots, f_n \in L_1(X)$ such that $f = \sum_{i=1}^n g_i f_i - f_i$ and $\sum_{i=1}^n \|f_i\|_1 \le K$.

Fix M and $N = 2^M$. Let $g_0 = e$, the identity element in G. Choose measurable sets $W_1, \dots, W_N$, a measurable set V of positive measure, and $h_1, \dots, h_N \in G$ with these properties. First, $V \subset W_1 \subset \cdots \subset W_N$. Also

$$W_{k+1} \supset g_i W_k \cup g_i^{-1} W_k \text{ and } W_{k+1} \setminus W_k \supset h_k V$$

for all $i = 1, \dots, n$ and $k = 1, \dots, N-1$.

Since N and n are fixed, this is possible by first just choosing W_1 of small measure and then choosing each

$$W_{k+1} \supset \bigcup_{i=0}^{n} g_i W_k \cup \bigcup_{i=1}^{n} g_i^{-i} W_k$$

with

$$W_{k+1} \Big\backslash \left(\bigcup_{i=0}^{n} g_i W_k \cup \bigcup_{i=1}^{n} g_i^{-1} W_k \right)$$

having small positive measure. If these choices of small measure are made sufficiently small, then the choice of W_k can be carried out for the finite number of values $k = 1, \dots, N$.

By ergodicity of G, there is $h_1 \in G$ and V_1 of positive measure, $W_1 \supset V_1$, such that $W_2 \setminus W_1 \supset h_1 V_1$. Again, by ergodicity of G, there is $h_2 \in G$ and V_2 of positive measure, $V_1 \supset V_2$, such that $W_3 \setminus W_2 \supset h_2 V_2$. Inductively we continue to choose h_k and V_k of positive measure, $V_{k-1} \supset V_k$, such that $W_{k+1} \setminus W_k \supset h_k V_k$. Let $V = V_N$ to complete the construction.

Now define

$$f = \frac{s_M}{m(V)} 1_V + \sum_{k=1}^{M} \frac{-1}{2^k m(V)} 1_{h_{2^k-1}V}$$

where

$$s_M = \sum_{k=1}^{M} \frac{1}{2^k} = 1 - \frac{1}{2^M}.$$

Then $f \in L_1^0(X)$ and

$$\|f\|_1 = 2\left(1 - \frac{1}{2^M}\right)$$

because the construction guarantees that the sets $h_k V$ are pairwise disjoint. Choose $f_1, \ldots, f_n \in L_1(X)$ such that $f = \sum_{i=1}^n {}_{g_i}f_i - f_i$ and $\sum_{i=1}^n \|f_i\|_1 \leq 2K$. We can compute

$$\begin{aligned}
\sum_{k=1}^{N-1} \left| \int_{W_k} f\,dm \right| &= \sum_{s=0}^{M-1} \sum_{i=2^s}^{2^{s+1}-1} \left| \int_{W_i} f\,dm \right| \\
&= \sum_{s=0}^{M-1} 2^s \left| \int_{W_{2^s}} f\,dm \right| \\
&= \sum_{s=0}^{M-1} 2^s \left(s_M - \sum_{k=1}^{s} \frac{1}{2^k} \right) \\
&= \sum_{s=0}^{M-1} 1 - \frac{2^s}{2^M} \\
&= M - 1 + \frac{1}{2^M}.
\end{aligned}$$

But on the other hand, for each i, the sets $g_i^{-1}W_k \,\Delta\, W_k$, $k = 1, \ldots, N$, are pairwise disjoint. Hence, we also have the estimate,

$$\begin{aligned}
\sum_{k=1}^{N-1} \left| \int_{W_k} f\,dm \right| &\leq \sum_{k=1}^{N-1} \sum_{i=1}^{n} \int_{g_i^{-1}W_k \Delta W_k} |f_i|\,dm \\
&\leq \sum_{i=1}^{n} \int |f_i|\,dm \\
&\leq 2K.
\end{aligned}$$

Since M is arbitrary and K is fixed, this is impossible. ■

Remarks. (a) There are many examples of automatic continuity failing in an L_1 space. See Woodward [17], which gives the precedent for the argument above, or Saeki [13]. However, the following question does not seem to be answered: can there be a group G (which is amenable) such that every TILF on $L_1(X)$ is continuous? It is possible that a construction of a very large group using the Continuum Hypothesis or Martin's Axiom could give such an example. This remark should be compared to the remark made after Corollary 1.9.

(b) Theorem 1.10 above shows that in general the L_∞ mean-zero weak containment property does hold. This should be possible to prove directly, but it is not apparent exactly how to do this.

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The Ohio State University
Columbus, Ohio