ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 4, Winter 1992

# NATURALLY REDUCTIVE RIEMANNIAN $S$-MANIFOLDS

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## 1. Introduction

Riemannian $k$-symmetric spaces and, more generally, Riemannian regular $s$-manifolds have been studied by several authors and the general theory is now well established. All such manifolds are homogeneous and the associated canonical connection [5] is determined by the symmetry tensor field $S$ of type $(1, 1)$ which is derived from $s$. Riemannian locally regular $s$-manifolds can then be defined either in terms of $s$ or, more usefully for our purpose, in terms of the invariance of certain tensor fields under the action of $S$ as a field of tangent space endomorphisms. Thus, as well as the first order condition that $\nabla S$ should be $S$-invariant, where $\nabla$ is the Riemannian connection, one requires the second order conditions that $\nabla^2 S$ and the curvature tensor field $R$ are $S$-invariant and then the third order condition that $\nabla R$ is $S$-invariant.

For a regular $s$-manifold, the homogeneous Riemannian structure can be shown to be naturally reductive [5] if and only if $S$ satisfies the additional first order condition $(\nabla_{(I-S)^{-1}X} S)S^{-1}X = 0$ for all vector fields $X$. In turn, this condition can be applied to define the notion of naturally reductive for locally regular $s$-manifolds and one might then ask whether such a first order condition can be used to simplify the higher order conditions given above. The simplest example is afforded by a Riemannian locally symmetric space which can be defined either by local 2-symmetries or by the single condition $\nabla R = 0$. In this case $S = -I$ so the above tensor conditions are trivial except for $\nabla R$ being $S$-invariant, that is $\nabla R = 0$. Moreover, this condition reduces to

$$(\nabla_X R)(X, JX, X, JX) = 0$$

for the Hermitian case. A less trivial case arises with locally 3-symmetric spaces. These are almost Hermitian with almost complex structure $J$ satisfying

$$S = -\frac{1}{2}I + \frac{\sqrt{3}}{2}J$$

Received December 7, 1990
1991 Mathematics Subject Classification. Primary 53B20, 53C25, 53C30, 53C35, 53C40.

and the naturally reductive condition reduces to the nearly Kähler property $(\nabla_X J)X = 0$ for all vector fields $X$. Then, as shown by Gray [3], [4], a nearly Kähler manifold is locally 3-symmetric if and only if

$$(\nabla_X R)(X, JX, X, JX) = 0$$

for all vector fields $X$, the latter condition being equivalent to the $S$-invariance of $\nabla R$. In particular, no second order conditions are required. An analogous result is proved in [10] for naturally reductive locally 4-symmetric spaces where again the second order conditions are redundant. We remark that in this case $S$ determines an $f$-structure [1], [8], [14] since $-1$ may be an eigenvalue. The purpose of the present paper is to show that a similar simplified characterisation holds for all Riemannian naturally reductive locally regular $s$-manifolds.

For general notational purposes we will normally use [5]. We write $\mathscr{T}_q^p$ for the algebra of smooth tensor fields with contravariant and covariant orders $p$ and $q$ respectively; in particular, we write $\mathscr{T}_0^p = \mathscr{T}^p$ and $\mathscr{T}_p^0 = \mathscr{T}_p$. Tensor fields $A$, $B \in \mathscr{T}_1^1$ will often be considered as linear endomorphisms and then composed in the usual way to give $AB \in \mathscr{T}_1^1$. Also, for the curvature tensor field $R$ we use the same symbol to denote its covariant form given by

$$R(X, Y, Z, W) = g(R(Z, W)Y, X)$$

for a Riemannian metric $g$. Finally, for later use, we define $\nabla^2 S$ by the relation

$$(\nabla^2 S)(X, Y, Z) = (\nabla^2_{XY} S)Z = \nabla_X((\nabla_Y S)Z) - (\nabla_{\nabla_X Y} S)Z - (\nabla_Y S)\nabla_X Z,$$

and then have the general formula

$$(\nabla^2_{XY} S)Z - (\nabla^2_{YX} S)Z = R(X, Y)SZ - SR(X, Y)Z.$$

## 2. Preliminaries and statement of theorem

We first recall some basic properties of Riemannian regular $s$-manifolds, most details of which can be found in [6]. Let $(M, g)$ be a *smooth*, *connected*, *finite-dimensional Riemannian* manifold and let $s = \{s_x \colon x \in M\}$ be a family of isometries of $(M, g)$ such that each $x \in M$ is an isolated fixed point of the corresponding map $s_x$. We call $s_x$ a *symmetry* at $x$ and say $(M, g)$ is a *Riemannian regular s-manifold* with respect to the given *s-structure* if

$$s_x \circ s_y = s_{s_x(y)} \circ s_x \quad \text{for all } x,\ y \in M.$$

Then $M$ becomes a homogeneous space with respect to the group $G$

generated by $s$. A tensor field $S \in \mathscr{T}_1^1$ is defined by the condition that, for each $x \in M$, $S_x$ is the differential of $s_x$ evaluated at $x$. We call $S$ the *symmetry tensor field* on $M$. It follows from the definition of $s$ that $I - S$ is non-singular at each point of $M$ and $S$ is invariant under the action of each $s_x$.

A tensor field $T \in \mathscr{T}_q^p$ is said to be *S-invariant* if, for all $\omega_1, \dots, \omega_p \in \mathscr{T}_1$ and $X_1, \dots, X_q \in \mathscr{T}^1$,

$$T(\omega_1 S, \dots, \omega_p S, X_1, \dots, X_q) = T(\omega_1, \dots, \omega_p, SX_1, \dots, SX_q)$$

where $(\omega S)X = \omega(SX)$ for $\omega \in \mathscr{T}_1$ and $X \in \mathscr{T}^1$. In particular $P \in \mathscr{T}_q^1$ and $Q \in \mathscr{T}_q$ are $S$-invariant if and only if, for all $X_1, \dots, X_q \in \mathscr{T}^1$,

$$SP(X_1, \dots, X_q) = P(SX_1, \dots, SX_q)$$

and

$$Q(X_1, \dots, X_q) = Q(SX_1, \dots, SX_q).$$

Thus it can be seen that the tensor fields $g$, $R$, $\nabla R$, $\nabla S$, $\nabla S^{-1}$ and $\nabla^2 S$ are $S$-invariant. If we regard $S$ as a field of endomorphisms on $M$ then the $S$-invariance of $g$ is equivalent to $S$ being *orthogonal* at each point of $M$. Also we note that if any tensor field $T$ is $S$-invariant then $T$ is $S^k$-invariant for any $k \in Z$.

Because of its regular $s$-structure, we may consider a Riemannian regular $s$-manifold $(M, g)$ as a *reductive* homogeneous space with respect to a group of isometries preserving $S$ and we write the corresponding *canonical connection* [5] as $\tilde{\nabla}$. Then as shown in [2], $\nabla$ and $\tilde{\nabla}$ are related by

$$(2.1) \qquad \nabla_X Y - \tilde{\nabla}_X Y = (\nabla_{(I-S)^{-1}X} S) S^{-1} Y \quad \text{for all } X, Y \in \mathscr{T}^1.$$

We note from [5] that the homogeneous space $(M, g)$ is *naturally reductive* with $\nabla$ as the natural torsion free connection if and only if $\nabla$ and $\tilde{\nabla}$ have the same geodesics, that is, if and only if [10]

$$(2.2) \qquad (\nabla_{(I-S)^{-1}X} S) S^{-1} X = 0 \quad \text{for all } X \in \mathscr{T}^1.$$

Next, we consider local analogues. Thus a *Riemannian locally regular s-manifold* is a Riemannian manifold $(M, g)$ together with a family $s = \{s_x : x \in M\}$ of local isometries such that each $x \in M$ is an isolated fixed point of $s_x$ and the symmetry tensor field $S$, defined as above, is smooth and locally invariant by each $s_x$. For convenience of notation, we now make the following definition.

DEFINITION. A *Riemannian S-manifold*, denoted by $(M, g, S)$, is a Riemannian manifold $(M, g)$ together with a tensor field $S \in \mathscr{T}_1^{\,1}$ such that $g$ and $\nabla S$ are $S$-invariant and $I - S$ is non-singular. We call any such $S$ a *symmetry tensor field* on $(M, g)$ and say $(M, g, S)$ is *naturally reductive* if (2.2) is satisfied.

Riemannian $S$-manifolds and (locally) regular $s$-manifolds are related as follows.

THEOREM 2.1 [2]. *Let $(M, g)$ be a Riemannian locally regular s-manifold with symmetry tensor field $S$. Then $(M, g, S)$ is a Riemannian S-manifold for which $\nabla^2 S$, $R$ and $\nabla R$ are S-invariant. Conversely, any $(M, g, S)$ for which $\nabla^2 S$, $R$ and $\nabla R$ are S-invariant is a locally regular s-manifold with symmetry tensor field $S$. Moreover, any complete simply connected Riemannian locally regular s-manifold is a Riemannian regular s-manifold.*

As shown below, on any $(M, g, S)$ distributions $\mathscr{D}_0$ and $\mathscr{D}_i$, $i = 1, \ldots, r$ are determined by the eigenspaces of $S$ where $\mathscr{D}_0$ corresponds to the $-1$ eigenspace of $S$. Since $S$ is orthogonal an almost complex structure $J$ is determined on $M$ when $-1$ is not an eigenvalue of $S$; moreover, $(M, g)$ is then almost Hermitian. If $-1$ is an eigenvalue of $S$ then $J$ is defined similarly but with $JX_0 = 0$ for any $X_0 \in \mathscr{D}_0$. As remarked earlier, in this latter case $J$ may be regarded as an $f$-structure on $(M, g)$ although this notation is not used here. Our purpose is to prove the following theorem.

THEOREM 2.2. *Let $(M, g, S)$ be a naturally reductive Riemannian S-manifold with associated eigenspace distributions $\mathscr{D}_0$ and $\mathscr{D}_i$, $i = 1, \ldots, r$ as above. Then $(M, g, S)$ is a locally regular s-manifold with associated symmetry tensor field $S$ if and only if*

$$(\nabla_{V_0} R)(X_0, Y_0, Z_0, W_0) = 0 \quad \text{for all } X_0, Y_0, Z_0, W_0, V_0 \in \mathscr{D}_0$$

*and*

$$(\nabla_{X_i} R)(X_i, JX_i, X_i, JX_i) = 0 \quad \text{for each } X_i \in \mathscr{D}_i,\ i = 1, \ldots, r.$$

From now on we consider an arbitrary naturally reductive Riemannian $S$-manifold $(M, g, S)$. The proof of the theorem depends largely on a case by case study of the curvature tensor field restricted to eigenspace distributions of $S$. In the remainder of this section we describe these distributions and prove some lemmas for later use.

For any $Q \in \mathscr{T}_q$ we define $\overline{Q} \in \mathscr{T}_q$ by

$$\overline{Q}(X_1, \ldots, X_q) = Q(SX_1, \ldots, SX_q) - Q(X_1, \ldots, X_q)$$

for all $X_1, \ldots, X_q \in \mathscr{T}^1$. Thus, $Q$ is $S$-invariant if and only if $\overline{Q} = 0$. Then we prove:

LEMMA 2.3. *Let $Q \in \mathscr{T}_q$ and suppose $X_1, \ldots, X_q \in \mathscr{T}^1$ such that for all positive integers $m$,*

$$\overline{Q}(S^m X_1, \ldots, S^m X_q) = \overline{Q}(X_1, \ldots, X_q).$$

*Then $\overline{Q}(X_1, \ldots, X_q) = 0$.*

*Proof.* For each $m \geq 1$,

$$\begin{aligned} Q(S^{m+1}X_1, \ldots, S^{m+1}X_q) &- Q(S^m X_1, \ldots, S^m X_q) \\ &= Q(SX_1, \ldots, SX_q) - Q(X_1, \ldots, X_q) \end{aligned}$$

and by adding the first $n$ of these equations we have

$$\begin{aligned} Q(S^{n+1}X_1, \ldots, S^{n+1}X_q) &- Q(SX_1, \ldots, SX_q) \\ &= nQ(SX_1, \ldots, SX_q) - nQ(X_1, \ldots, X_q). \end{aligned}$$

Since $g$ is $S$-invariant, $S$ is orthogonal at each point so the left hand side of the above equation is pointwise bounded as $n \to \infty$ and the lemma follows immediately.

We remark that this lemma will be applied usually to the case when $Q$ and $\overline{Q}$ are replaced by $R$ and $\overline{R}$. Next, define the connection $\tilde{\nabla}$ on $M$ as in (2.1). Then for all $X, Y \in \mathscr{T}^1$,

$$(\tilde{\nabla}_X S)Y = (\nabla_X S)Y - (\nabla_{(I-S)^{-1}X} S)Y + S(\nabla_{(I-S)^{-1}X} S)S^{-1}Y = 0$$

since $\nabla S$ is $S$-invariant. Thus $S$ is parallel on $M$ with respect to $\tilde{\nabla}$ and it follows that the eigenvalues of $S_x$, $x \in M$, and their multiplicities are independent of $x$. Since $S$ is orthogonal its distinct non-real eigenvalues have the form

$$e^{\pm i\theta_1} = c_1 \pm is_1, \ldots, e^{i\theta_r} = c_r \pm is_r,$$

where $0 < \theta_1, \ldots, \theta_r < \pi$ and, for brevity of notation, we write $\cos\theta_j, \sin\theta_j$ as $c_j, s_j$ for $j = 1, \ldots, r$. Also, $-1$ is the only possible real eigenvalue since $S - I$ is non-singular. Then smooth disjoint $S$-invariant distributions $\mathscr{D}_0, \mathscr{D}_1, \ldots, \mathscr{D}_r$ are defined on $M$ by

$$\mathscr{D}_0 = \ker(S + I),$$

and

$$\mathscr{D}_j = \ker\left(S^2 - 2c_j S + I\right) \quad \text{for } j \in [r]$$

where, from now on, we write $\{1, 2, \ldots, r\}$ as $[r]$. Clearly any $X \in \mathscr{T}^1$ has a unique decomposition into a sum of *distribution vector fields*, that is $X = X_0 + X_1 + \cdots + X_r$ where $X_0 \in \mathscr{D}_0$ and $X_j \in \mathscr{D}_j$ for $j \in [r]$. We remark that the possibilities of $\mathscr{D}_0$ or all $\mathscr{D}_j$, $j \in [r]$ being vacuous are not excluded. To avoid needless repetition, we indicate distribution vector fields by their appropriate suffixes such as $X_0$ or $X_j$ often without reference to the corresponding $\mathscr{D}_0$ or $\mathscr{D}_j$. Next define $S_0$, $S_j \in \mathscr{T}_1^1$ by $S_0 X = SX_0$, $S_j X = SX_j$ for $j \in [r]$. Then define $I_j, J_j$ by $S_j = c_j I_j + s_j J_j$ where $I_j X = X_j$. Clearly

$$J_j^2 X_j = -X_j \quad \text{and} \quad I_j X_0 = I_j X_i = J_j X_0 = J_j X_i = 0$$

for $i$, $j \in [r]$, $i \neq j$. Now write $J = J_1 + \cdots + J_r$ and note that $J^3 + J = 0$ and $g(JX_i, JX_j) = g(X_i, X_j)$ for $i$, $j \in [r]$. Since the eigenvalues of $S$ are constant, it follows that each $I_j$ and $J_j$ is a polynomial in $S$ and $S^{-1}$ with constant coefficients. Hence each $I_j$, $J_j$, $\nabla I_j$ and $\nabla J_j$ is smooth and $S$-invariant. The same properties hold for $S_0$ and $\nabla S_0$. We also define $I_0 = -S_0$ since then $I_0 X_0 = X_0$ for all $X_0$.

LEMMA 2.4. (i) *Let $i$, $j \in [r]$ and let $T \in \mathscr{T}_2$. If*

$$T(SX_i, SY_j) = T(X_i, Y_j) \quad \textit{for all } X_i, Y_j$$

*then*

(a) $T(X_i, Y_j) = 0$ *if* $i \neq j$,

(b) $T(JX_i, Y_j) + T(X_i, JY_j) = 0$ *if* $i = j$.

*Similarly, if*

$$T(SX_i, SY_j) = -T(X_i, Y_j) \quad \textit{for all } X_i, Y_j$$

*then*

(c) $T(X_i, Y_j) = 0$ *if* $c_i + c_j \neq 0$

(d) $T(JX_i, Y_j) - T(X_i, JY_j) = 0$ *if* $c_i + c_j = 0$.

(ii) *Let $i \in [r]$ and suppose given $P \in \mathscr{T}_m$ and $X_{1i}, X_{2i}, \ldots, X_{mi} \in D_i$ such that*

$$P(JX_{1i}, X_{2i}, \ldots, X_{mi}) + P(X_{1i}, JX_{2i}, \ldots, X_{mi}) + \cdots$$
$$+ P(X_{1i}, X_{2i}, \ldots, JX_{mi}) = 0.$$

*Then $\bar{P}(X_{1i}, X_{2i}, \ldots, X_{mi}) = 0$.*

*Proof.* (i) We have

$$T(SX_i, SY_j) = \pm T(X_i, Y_j) = T(S^{-1}X_i, S^{-1}Y_j)$$

which implies

$$(c_i c_j \mp 1)T(X_i, Y_j) + s_i s_j T(JX_i, JY_j) = 0$$

and

$$c_i s_j T(X_i, JX_j) + c_j s_i T(JX_i, Y_j) = 0.$$

Then (i) follows easily.

(ii) The assumption on $P$ implies that the function

$$\mathbf{R} \to \mathbf{R};\ t \mapsto P(e^{tJ}X_{1i}, e^{tJ}X_{2i}, \dots, e^{tJ}X_{mi})$$

is constant. Hence,

$$P(e^{tJ}X_{1i}, e^{tJ}X_{2i}, \dots, e^{tJ}X_{mi}) = P(X_{1i}, X_{2i}, \dots, X_{mi})$$

and (ii) follows by choosing $t = \theta_i$ in this equation.

Next, we consider $\nabla S$ which, by assumption, is $S$-invariant and hence $S^{-1}$-invariant. Then for $i$, $j$, $k \in [r]$ and for all $X_j, Y_k$,

$$(2.3)\qquad (c_i I_i + s_i J_i)(\nabla_{Y_k} S)X_j = I_i(\nabla_{(c_k I + s_k J)Y_k} S)(c_j I + s_j J)X_j,$$

$$(2.4)\qquad (c_i I_i - s_i J_i)(\nabla_{Y_k} S)X_j = I_i(\nabla_{(c_k I - s_k J)Y_k} S)(c_j I - s_j J)X_j.$$

Addition gives

$$(2.5)\qquad (c_i - c_j c_k)I_i(\nabla_{Y_k} S)X_j = s_j s_k I_i(\nabla_{JY_k} S)JX_j,$$

hence

$$(c_i - c_j c_k)I_i(\nabla_{JY_k} S)JX_j = s_j s_k I_i(\nabla_{Y_k} S)Xj$$

from which

$$(2.6)\qquad \left((c_i - c_j c_k)^2 - s_j^2 s_k^2\right)I_i(\nabla_{Y_k} S)X_j = 0.$$

Now

$$(c_i - c_j c_k)^2 - s_j^2 s_k^2 = c_i^2 + c_j^2 + c_k^2 - 2c_i c_j c_k - 1$$

so is symmetric in $i, j, k$. If $(c_i - c_j c_k)^2 - s_j^2 s_k^2 = 0$ then $\cos\theta_k = \cos(\theta_i \pm \theta_j)$ for any permutation of $i, j, k$. In this case we define $\alpha_{ijk} = 1$ if $\theta_i + \theta_j + \theta_k = 2\pi$ or $\theta_k = \theta_i + \theta_j$ and $\alpha_{ijk} = -1$ if $\theta_j = \theta_k + \theta_i$ or $\theta_i = \theta_k + \theta_j$; these are the only possible relations between $\theta_i, \theta_j, \theta_k$. Then $\cos\theta_k = \cos(\theta_i + \alpha_{ijk}\theta_j)$ for any permutation of $i, j, k$, where we recall that $0 < \theta_l < \pi$ for all $l \in [r]$. From its definition, $\alpha_{ijk}$ can be seen to satisfy

(i) $\alpha_{ijk} = \alpha_{jik}$,
(ii) $\alpha_{ijk}\alpha_{kij}\alpha_{jki} = 1$,
(iii) $\alpha_{ijk} = s_j(c_j - c_i c_k)/s_i(c_i - c_j c_k)$.

Next, by subtracting (2.4) from (2.3) we have

$$s_i J_i(\nabla_{Y_k} S) X_j = s_j c_k I_i(\nabla_{Y_k} S) J X_j + s_k c_j I_i(\nabla_{JY_k} S) X_j \tag{2.7}$$

and from (2.5) and (2.7),

$$s_i(c_i - c_j c_k) J_i(\nabla_{Y_k} S) X_j + s_j(c_j - c_k c_i) I_i(\nabla_{Y_k} S) J X_j = 0. \tag{2.8}$$

Then from (2.6) and (2.8) we obtain:

LEMMA 2.5. *For any $i, j, k \in [r]$ either $I_i(\nabla_{Y_k} S) X_j = 0$ for all $X_j, Y_k$ or $\cos\theta_k = \cos(\theta_i + \alpha_{ijk}\theta_j)$. Moreover, if $\cos\theta_k = \cos(\theta_i + \alpha_{ijk}\theta_j)$ then*

$$J_i(\nabla_{Y_k} S) X_j + \alpha_{ijk} I_i(\nabla_{Y_k} S) J X_j = 0$$

*for all $X_j, Y_k$.*

Next, we write (2.2) as

$$(\nabla_X S)(I - S^{-1}) X = 0 \quad \text{for all } X \in \mathscr{T}^1. \tag{2.9}$$

Then for all $i, j, k \in [r]$ and for all $X_j, Y_k$

$$I_i(\nabla_{Y_k} S)\big((1 - c_j) I + s_j J\big) X_j + I_i(\nabla_{X_j} S)\big((1 - c_k) I + s_k J\big) Y_k = 0. \tag{2.10}$$

Also, the relation $g(SY, SY) = g(Y, Y)$ implies

$$g(\nabla_X S) Y, SY\big) = 0 \quad \text{for all } X, Y \in \mathscr{T}^1. \tag{2.11}$$

Hence, for all $X_i, Y_j, Z_k$,

$$g\big((\nabla_{Z_k} S) Y_j, S X_i\big) + g\big((\nabla_{Z_k} S) X_i, S Y_j\big) = 0. \tag{2.12}$$

Then as a consequence of (2.10) and (2.12) we have:

LEMMA 2.6. *Let $i, j, k \in [r]$ and suppose $g(\nabla_{Z_k} S)Y_j, X_i) = 0$ for all $X_i, Y_j, Z_k$. Then this equation holds for any permutation of $X_i, Y_j, Z_k$.*

The remaining lemmas in this section provide information on the $S$-invariance of the curvature tensor field $R$. We use, throughout, the associated tensor field $\bar{R}$ defined above.

LEMMA 2.7. *Let $i, j, k, l \in [r]$. Then the following relations hold.*

(i) *If $I_i(\nabla_{Y_k} S)X_j = I_i(\nabla_{Z_l} S)X_j = 0$ for all $X_j, Y_k, Z_l$ then, for all $W_i, X_j, Y_k, Z_l$,*

$$\bar{R}(W_i, X_j, Y_k, Z_l) = 0 \quad \text{if } i \neq j$$

*and*

$$\bar{R}(JW_i, X_j, Y_k, Z_l) + \bar{R}(W_i, JX_j, Y_k, Z_l) = 0 \quad \text{if } i = j.$$

(ii) *$\bar{R}(X_j, X_i, X_i, JX_i) = 0$ for all $X_i, X_j$.*

(iii) *If $i \neq j$ and $I_j(\nabla_{X_j} S)X_i = 0$ for all $X_i, X_j$ then*

$$\bar{R}(Y_j, X_i, X_i, X_j) = \bar{R}(Y_j, JX_i, X_i, X_j) = 0 \quad \text{for all } X_i, X_j, Y_j.$$

*Proof.* (i) Since $I_i(\nabla_{Y_k} S)X_j = 0$ for all $X_j, Y_k$ then

$$I_i\left(\nabla^2_{Z_l Y_k} S\right)X_j + \left(\nabla_{Z_l} I_i\right)\left(\nabla_{Y_k} S\right)X_j + I_i\left(\nabla_{(\nabla_{Z_l} I_k)Y_k} S\right)X_j + I_i\left(\nabla_{Y_k} S\right)\left(\nabla_{Z_l} I_j\right)X_j = 0$$

from which it follows that

$$SI_i\left(\nabla^2_{Z_l Y_k} S\right)X_j = I_i\left(\nabla^2_{SZ_l SY_k} S\right)SX_j.$$

A corresponding property holds for $I_i(\nabla^2_{Y_k Z_l} S)X_j$ so, from the relation

$$g\left(\left(\nabla^2_{Y_k Z_l} S\right)X_j - \left(\nabla^2_{Z_l Y_k} S\right)X_j, SW_i\right) = R(SW_i, SX_j, Y_k, Z_l) - R(W_i, X_j, Y_k, Z_l)$$

we see immediately that, for all $W_i, X_j, Y_k, Z_l$,

$$\bar{R}(SW_i, SX_j, Y_k, Z_l) - \bar{R}(W_i, X_j, Y_k, Z_l) = 0.$$

By fixing $Y_k$ and $Z_l$ we can apply Lemma 2.2 and then (i) follows.

(ii) We see from (2.9) and Lemma 2.5 that

$$I_j\left(\nabla_{X_i} S\right)X_i = I_j\left(\nabla_{JX_i} S\right)X_i = 0 \quad \text{for all } X_i.$$

Then the same proof as above shows that for all $X_i$, $X_j$.

$$\bar{R}(SX_j, SX_i, X_i, JX_i) - \bar{R}(X_j, X_i, X_i, JX_i) = 0. \tag{2.13}$$

We have also

$$I_j\left(\nabla_{X_i} S^{-1}\right)X_i = -S^{-1} I_j(\nabla_{X_i} S) S^{-1} X_i = -S^{-1} I_j(\nabla_{X_i} S)(c_i I - s_i J) X_i = 0$$

and, similarly, $I_j(\nabla_{JX_i} S^{-1})X_i = 0$. Hence, as in the proof of (i),

$$\bar{R}\left(S^{-1}X_j, S^{-1}X_i, X_i, JX_i\right) - \bar{R}(X_j, X_i, X_i, JX_i) = 0 \tag{2.14}$$

and (ii) is an easy consequence of (2.13) and (2.14).

(iii) This follows by the argument used for (ii).

LEMMA 2.8. (i) *Let $i$, $j$, $k$, $l \in [r]$ and suppose for all $X_j, Y_k, Z_l$ and for some non-zero $\alpha \in \mathbf{R}$,*

$$J_i(\nabla_{Y_k} S)X_j + \alpha I_i(\nabla_{Y_k} S)JX_j = 0$$

*and*

$$J_i(\nabla_{Z_l} S)X_j + \alpha I_i(\nabla_{Z_l} S)JX_j = 0.$$

*Then for all $W_i, X_j, Y_k, Z_l$,*

$$\bar{R}(JW_i, X_j, Y_k, Z_l) - \alpha \bar{R}(W_i, JX_j, Y_k, Z_l) = 0$$

*if $i \neq j$ or if $i = j$ and $\alpha = -1$.*

(ii) *Let $i$, $j$, $k \in [r]$ and suppose for all $X_i$, $Z_k$ and for some $\alpha \in \mathbf{R}$,*

$$J_j(\nabla_{Z_k} S)X_i + \alpha I_j(\nabla_{Z_k} S)JX_i = 0.$$

*Then for all $X_i, Y_j, Z_k$,*

$$\bar{R}(JY_j, X_i, X_i, Z_k) - \alpha \bar{R}(Y_j, JX_i, X_i, Z_k) = 0$$

*if $i \neq j$ or if $i = j$ and $\alpha \neq 1$.*

*Proof.* (i) One easily proves, as in Lemma 2.7, that

$$\begin{aligned} &S\left(J_i\left(\nabla^2_{Y_k Z_l} S\right)X_j + \alpha I_i\left(\nabla^2_{Y_k Z_l} S\right)JX_j\right) \\ &\qquad = J_i\left(\nabla^2_{SY_k SZ_l} S\right)SX_j + \alpha I_i\left(\nabla^2_{SY_k SZ_l} S\right)JSX_j \end{aligned}$$

with the corresponding relation also holding when $Y_k, Z_l$ are exchanged. It then follows that

$$\begin{aligned}&\bar{R}(SJW_i, SX_j, Y_k, Z_l) - \alpha\bar{R}(SW_i, SJX_j, Y_k, Z_l)\\&\quad= \bar{R}(JW_i, X_j, Y_k, Z_l) - \alpha\bar{R}(W_i, JX_j, Y_k, Z_l).\end{aligned}$$

By fixing $Y_k$ and $Z_l$ and applying Lemma 2.4 (i) to this equation we obtain (i).
(ii) For all $X_i$, $Z_k$, we have

$$J_j(\nabla_{Z_k}S)X_i + \alpha I_j(\nabla_{Z_k}S)JX_i = 0$$

and

$$J_j(\nabla_{X_i}S)X_i + \alpha I_j(\nabla_{X_i}S)JX_i = 0$$

with the same equations holding for $S^{-1}$ in place of $S$. We then obtain

$$\begin{aligned}&\bar{R}(SJY_j, SX_i, X_i, Z_k) - \alpha\bar{R}(SY_j, SJX_i, X_i, Z_k)\\&\quad= \bar{R}(JY_j, X_i, X_i, Z_k) - \alpha\bar{R}(Y_j, JX_i, X_i, Z_k)\end{aligned}$$

and the corresponding equation with $S^{-1}$ in place of $S$. Then (ii) follows as before.

Lemma 2.9. *For all $X, Y \in \mathscr{T}^1$,*

$$R((I-S)X, (I-S)Y, X, Y) - \bar{R}((I-S^{-1})X, (I-S^{-1})Y, X, Y) = 0.$$

*In particular,*

$$\sum_{i,j=1}^{r} s_j(1-c_i)\big(\bar{R}(X_i, JY_j, X, Y) + \bar{R}(JX_j, Y_i, X, Y)\big) = 0$$

*where $X = X_1 + X_2 + \cdots + X_r$ and $Y = Y_1 + Y_2 + \cdots + Y_r$.*

*Proof.* Let $X$, $Y$, $Z \in \mathscr{T}^1$. Then (2.11) implies

$$g((\nabla^2_{XY}S)Z, SZ) + g((\nabla_Y S)Z, (\nabla_X S)Z) = 0$$

Also, from (2.9) we have

$$(\nabla^2_{YX}S)(I-S^{-1})X - (\nabla_X S)(\nabla_Y S^{-1})X = 0.$$

Since

$$g\big((\nabla^2_{XY}S)(I-S^{-1})Y-(\nabla^2_{YX}S)(I-S^{-1})Y,S(I-S^{-1})X\big)$$
$$=R\big(S(I-S^{-1})X,S(I-S^{-1})Y,X,Y\big)$$
$$-R\big((I-S^{-1})X,(I-S^{-1})Y,X,Y\big),$$

the lemma follows easily.

Next, we show that Lemma 2.9 can be extended to higher powers of $S$.

LEMMA 2.10. *For all $X$, $Y\in\mathscr{T}^1$ and all positive integers $m$,*

$$\overline{R}((I-S^m)X,(I-S^m)Y,X,Y)$$
$$-\overline{R}((I-S^{-m})X,(I-S^{-m})Y,X,Y)=0.$$

*In particular,*

$$\sum_{i,j=1}^{r}\sin m\theta_j(1-\cos m\theta_i)\big(\overline{R}(X_i,JY_j,X,Y)+\overline{R}(JX_j,Y_i,X,Y)\big)=0$$

*where $X=X_1+X_2+\cdots+X_r$ and $Y=Y_1+Y_2+\cdots+Y_r$.*

*Proof.* We first show that $(\nabla_XS^m)(I-S^{-m})X=0$ for all $X\in\mathscr{T}^1$ and for any positive integer $m$. Thus, by induction on $m$ we obtain

$$\nabla_XS^m=\sum_{k=0}^{m-1}S^k(\nabla_XS)S^{m-1-k}.$$

Then from (2.9) and the $S$-invariance of $\nabla S$,

$$\begin{aligned}(\nabla_XS^m)(I-S^{-m})X&=\big(\nabla_{(I+S+\cdots+S^{m-1})X}S\big)S^{m-1}(I-S^{-m})X\\&=\big(\nabla_{(I+S+\cdots+S^{m-1})X}S\big)(I-S^{-1})\\&\quad\times(I+S+\cdots+S^{m-1})X\\&=0.\end{aligned}$$

It follows that for all $X$, $Y,\in\mathscr{T}^1$

$$S\big(\nabla^2_{YX}S^m\big)(I-S^{-m})X=\big(\nabla^2_{SY\,SX}S^m\big)(I-S^{-m})SX. \tag{2.15}$$

Next, we have

$$g(S^mZ,S^mZ)=g(Z,Z)\quad\text{for all } Z\in\mathscr{T}^1$$

so

$$g\big((\nabla_Y S^m)Z, S^m Z\big) = 0 \quad \text{for all } Y, Z, \in \mathscr{T}^1.$$

Hence

$$(2.16) \qquad g\big((\nabla^2_{XY} S^m)Z, S^m Z\big) = g\big((\nabla^2_{SX\,SY} S^m)SZ, S^{m+1}Z\big).$$

The remainder of the proof depends only on (2.15) and (2.16) and is analogous to that for Lemma 2.9.

## 3. $S$-invariance of $R$ and $\nabla^2_S$

We show that the curvature tensor field $R$ satisfies $\bar{R} = 0$ by considering the restriction of $R$ to all possible combinations of distributions $\mathscr{D}_0$ and $\mathscr{D}_i$, $i \in [r]$. Combinations which include $\mathscr{D}_0$ are easier to study and are left until the end of each case. The $S$-invariance of $\nabla^2 S$ is proved at the end of the section. It should be noted that $\bar{R}$ satisfies the same standard algebraic identities as those of $R$, including the first Bianchi identity.

*Case* 1. $\bar{R}(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i) = 0$.
Let $X_i$, $Y_i \in \mathscr{D}_i$. Then from Lemma 2.9,

$$\bar{R}(JX_i, Y_i, X_i, Y_i) + \bar{R}(X_i, JY_i, X_i, Y_i) + \bar{R}(X_i, Y_i, JX_i, Y_i) \\ + \bar{R}(X_i, Y_i, X_i, JY_i) = 0.$$

Hence, from Lemma 2.4 (ii),

$$\bar{R}(SX_i, SY_i, SX_i, SY_i) = \bar{R}(X_i, Y_i, X_i, Y_i)$$

and then

$$\bar{R}(X_i, Y_i, X_i, Y_i) = 0$$

from Lemma 2.3. It follows easily that $\bar{R}(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i) = 0$.

*Case* 1′. $\bar{R}(\mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0) = 0$.
This is obvious since $SX_0 = -X_0$ for $X_0 \in \mathscr{D}_0$.

*Case* 2. $\bar{R}(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_j) = 0$; $i \neq j$.
Suppose $\cos\theta_j \neq \cos 2\theta_i$. Then from Lemma 2.5, $I_i(\nabla_{X_i} S)X_j = 0$ for all $X_i, X_j$ so, from Lemma 2.7(i), $\bar{R}(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_j) = 0$. Next, suppose $\cos\theta_j = \cos 2\theta_i$. Then $\theta_j = 2\theta_i$ or $2\pi - 2\theta_i$ and, by applying Lemmas 2.5 and 2.8 to

the pair $(I_i(\nabla_{X_i}S)X_j, I_i(\nabla_{Y_i}S)X_j)$, we obtain for all $X_i, Y_i, W_i, X_j$,

$$\bar{R}(JW_i, X_j, X_i, Y_i) - \alpha_{iji}\bar{R}(W_i, JX_j, X_i, Y_i) = 0. \tag{3.1}$$

Also, by writing $X = X_i$ and $Y = aY_i + bY_j$ in Lemma 2.9 and equating to zero the coefficient of $ab$ we have

$$s_i(1 - c_i)\bar{R}(X_i, JY_i, X_i, Y_j) + s_j(1 - c_i)\bar{R}(X_i, JY_j, X_i, Y_i) + s_i(1 - c_i)\bar{R}(JX_i, Y_i, X_i, Y_j) + s_i(1 - c_j)\bar{R}(JX_i, Y_j, X_i, Y_i) = 0.$$

From (3.1) and the relations $\cos\theta_j = \cos 2\theta_i$ and $\sin\theta_j = -\alpha_{iji}\sin 2\theta_i$, this equation reduces to

$$\bar{R}(X_i, JY_i, X_i, Y_j) + \bar{R}(JX_i, Y_i, X_i, Y_j) + 2\bar{R}(X_i, Y_i, JX_i, Y_j) = 0. \tag{3.2}$$

Next, from Lemma 2.7(ii), we have $\bar{R}(X_j, X_i, X_i, JX_i) = 0$ for all $X_i$, $X_j$ and linearisation gives

$$\bar{R}(JX_i, X_i, Y_i, X_j) + \bar{R}(JX_i, Y_i, X_i, X_j) + \bar{R}(JY_i, X_i, X_i, X_j) = 0. \tag{3.3}$$

Also, from (3.2) and the first Bianchi identity,

$$\bar{R}(JY_i, X_i, X_i, X_j) + 2\bar{R}(JX_i, X_i, Y_i, X_j) - 3\bar{R}(JX_i, Y_i, X_i, X_j) = 0.$$

This equation and (3.3) imply

$$\bar{R}(Y_i, X_i, JX_i, X_j) + 4\bar{R}(Y_i, JX_i, X_i, X_j) = 0$$

and by replacing $X_i$ by $JX_i$ we have $\bar{R}(Y_i, X_i, JX_i, X_j) = 0$. It then follows from (3.3) that $\bar{R}(Y_i, X_i, X_i, X_j) = 0$ for all $X_i, Y_i, X_j$. From this equation and the first Bianchi identity we see that $\bar{R}(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_j) = 0$ as required.

*Case* 2′. $\bar{R}(\mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_i) = 0$.

Since $\nabla S$ is $S$-invariant we have $I_0(\nabla_{X_0}S)X_i = 0$ for all $X_0$, $X_i$, $i \in [r]$. It follows that $\bar{R}(SX_0, SX_i, Y_0, Z_0) = \bar{R}(X_0, X_i, Y_0, Z_0)$ so

$$\bar{R}(X_0, (S + I)X_i, Y_0, Z_0) = 0$$

for all $X_0, Y_0, Z_0, X_i$; hence $\bar{R}(\mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_i) = 0$.

*Case* 3. $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_j) = 0$; $i \neq j$.

First, suppose $\cos\theta_i \neq \cos 2\theta_j$ and $\cos\theta_j \neq \cos 2\theta_i$. Then from Lemma 2.5, we have $I_i(\nabla_{X_i}S)Y_j = I_i(\nabla_{X_j}S)Y_j = 0$ for all $X_i, X_j, Y_j$ so, from Lemma 2.7(i), $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_j) = 0$.

Next, suppose $\cos\theta_i \neq \cos 2\theta_j$ and $\cos\theta_j = \cos 2\theta_i$. Then $I_j(\nabla_{X_j} S)X_i = 0$ for all $X_i$, $X_j$ so, from Lemma 2.7(iii),

$$\overline{R}(Y_j, X_i, X_i X_j) = \overline{R}(Y_j, JX_i, X_i, X_j) = 0$$

for all $X_i$, $X_j$, $Y_j$. Hence

$$(3.4) \qquad \overline{R}(Y_j, X_i, Y_i, X_j) + \overline{R}(Y_j, Y_i, X_i, X_j) = 0$$

and

$$(3.5) \qquad \overline{R}(Y_j, JX_i, Y_i, X_j) + \overline{R}(Y_j, JY_i, X_i, X_j) = 0$$

for all $X_i, Y_i, X_j, Y_j$. Also, since $I_j(\nabla_{X_j} S)X_i = 0$ then, from Lemma 2.6, $I_i(\nabla_{X_i} S)X_j = 0$ for all $X_i$, $X_j$ and it follows from Lemma 2.7(i) that for all $X_i, Y_i, X_j, Y_j$,

$$(3.6) \qquad \overline{R}(JY_j, X_j, X_i, Y_i) + \overline{R}(Y_j, JX_j, X_i, Y_i) = 0.$$

Next, we apply Lemma 2.9 by writing $X = \alpha_i X_i + \alpha_j X_j$, $Y = \beta_i Y_i + \beta_j Y_j$ and equating to zero the coefficient of $\alpha_i \alpha_j \beta_i \beta_j$. This gives

$$\begin{aligned}(3.7) \quad (1 - c_i)s_i\big(\overline{R}(X_i, JY_i, X_j, Y_j) + \overline{R}(JX_i, Y_i, X_j, Y_j)\big)& \\ +(1 - c_i)s_j\big(\overline{R}(X_i, JY_j, X_j, Y_i) + \overline{R}(JX_j, Y_i, X_i, Y_j)\big)& \\ +(1 - c_j)s_i\big(\overline{R}(X_j, JY_i, X_i, Y_j) + \overline{R}(JX_i, Y_j, X_j, Y_i)\big)& \\ +(1 - c_j)s_j\big(\overline{R}(X_j, JY_j, X_i, Y_i) + \overline{R}(JX_j, Y_j, X_i, Y_i)\big)&=0.\end{aligned}$$

After simplification using (3.4), (3.5) and (3.6) it follows that

$$s_i(3 - c_i - c_j)\overline{R}(X_j, X_i, Y_i, Y_j) = 0 \quad \text{for all } X_i, Y_i, X_j, Y_j.$$

Then $\overline{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_j) = 0$ since $s_i(3 - c_i - c_j) = 2s_i(1 - c_i)(2 + c_i) \neq 0$.

Finally, suppose $\cos\theta_i = \cos 2\theta_j$ and $\cos\theta_j = \cos 2\theta_i$. Equivalently, suppose $2\theta_i = \theta_j = 4\pi/5$, since Case 3 is symmetric in $i$ and $j$. It follows that $\alpha_{iji} = -\alpha_{jij} = -1$ so, for all $X_i$, $X_j$,

$$J_i(\nabla_{X_i} S)X_j - I_i(\nabla_{X_i} S)JX_j = 0$$

and

$$J_j(\nabla_{X_j} S)X_i + I_j(\nabla_{X_j} S)JX_i = 0.$$

Then it results from Lemma 2.8 (ii) that

$$\bar{R}(JX_i, X_j, X_j, Y_i) + \bar{R}(X_i, JX_j, X_j, Y_i) = 0 \tag{3.8}$$

and

$$\bar{R}(JX_j, X_i, X_i, Y_j) - \bar{R}(X_j, JX_i, X_i, Y_j) = 0. \tag{3.9}$$

For all $X_i, Y_i, X_j, Y_j$. Next we consider Lemma 2.9 with $X = X_i$ and $Y = Y_j$ to obtain

$$c_i\bar{R}(X_i, JY_j, X_i, Y_j) + (1 + c_i)\bar{R}(JX_i, Y_j, X_i, Y_j) = 0. \tag{3.10}$$

Then by comparing (3.9) and (3.10) with $X_j = Y_j$, it follows that

$$\bar{R}(X_i, JX_j, X_i, X_j) = \bar{R}(X_j, JX_i, X_j, X_i) = 0 \tag{3.11}$$

for all $X_i, X_j$. Thus from (3.8), (3.9) and (3.11),

$$\bar{R}(JX_j, X_i, X_i, Y_j) = \bar{R}(X_j, JX_i, X_i, Y_j) = -\bar{R}(JY_j, X_i, X_i, X_j) \tag{3.12}$$

and

$$\bar{R}(JX_i, X_j, X_j, Y_i) = -\bar{R}(X_i, JX_j, X_j, Y_i) = -\bar{R}(JY_i, X_j, X_j, X_i) \tag{3.13}$$

for all $X_i, Y_i, X_j, Y_j$. By applying (3.12) and (3.13) to (3.7) we obtain, after some calculation,

$$(c_i + 2)A + c_i(4c_i + 5)B = 0 \tag{3.14}$$

where

$$A = \bar{R}(X_j, Y_i, JX_i, Y_j) + \bar{R}(X_j, JY_i, X_i, Y_j)$$

and

$$B = \bar{R}(X_j, Y_i, X_i, JY_j) + \bar{R}(JX_j, Y_i, X_i, Y_j).$$

Also, by Lemma 2.10, we may replace $\theta_i, \theta_j$ by $2\theta_i, 2\theta_j$ in (3.7) which implies

$$(c_j + 2)A + c_j(4c_j + 5)B = 0. \tag{3.15}$$

It results from (3.14) and (3.15) that $A = B = 0$. Hence, we see from Lemma 2.4(ii) that for all $X_i, Y_i, X_j, Y_j$,

$$\bar{R}(X_j, SY_i, SX_i, Y_j) = \bar{R}(SX_j, Y_i, X_i, SY_j) = \bar{R}(X_j, Y_i, X_i, Y_j).$$

This clearly implies $\bar{R}(SX_j, SY_i, SX_i, SY_j) = \bar{R}(X_j, Y_i, X_i, Y_j)$ or, equivalently, $\bar{R}(S^m X_j, S^m Y_i, S^m X_i, S^m Y_j) = \bar{R}(X_j, Y_i, X_i, Y_j)$ for all positive integers $m$. Consequently, from Lemma 2.1, we have $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_j) = 0$ and the proof for Case 3 is complete.

*Case* 3′. $\bar{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_0, \mathscr{D}_i) = 0$.

Write $X = X_0$ and $Y = Y_i$ in Lemma 2.9 to obtain $\bar{R}(X_0, JY_i, X_0, Y_i) = 0$ from which it follows that

$$\bar{R}(X_0, JX_i, Y_i, Y_0) + \bar{R}(X_0, JY_i, X_i, Y_0) + \bar{R}(X_0, X_i, JY_i, Y_0) + \bar{R}(X_0, Y_i, JX_i, Y_0) = 0$$

for all $X_0, Y_0, X_i, Y_i$. Similarly, write $X = aX_0 + bX_i$, $Y = cY_0 + dY_i$ in Lemma 2.9 and equate to zero the coefficient of $abcd$ to obtain, after simplification,

$$(3 - c_i)\bar{R}(X_0, JY_i, X_i, Y_0) + (3 - c_i)\bar{R}(X_0, Y_i, JX_i, Y_0) - (1 - c_i)\bar{R}(X_0, X_i, JY_i, Y_0) - (1 - c_i)\bar{R}(X_0, JX_i, Y_i, Y_0) = 0.$$

From these two equations we have

$$\bar{R}(X_0, X_i, JY_i, Y_0) + \bar{R}(X_0, JX_i, Y_i, Y_0) = 0$$

and then from Lemma 2.4 (ii),

$$\bar{R}(SX_0, SX_i, SY_i, SY_i, SY_0) = \bar{R}(X_0, X_i, Y_i, Y_0).$$

But this implies $\bar{R}(X_0, X_i, Y_i, Y_0) = 0$ from Lemma 2.3. Hence $\bar{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_0, \mathscr{D}_i) = 0$ as required.

*Case* 4. $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_k) = 0$; $i, j, k$ distinct.

Suppose $I_i(\nabla_{X_i} S)X_j = I_i(\nabla_{X_k} S)X_j = 0$ for all $X_i, X_j, X_k$. Then by Lemma 2.9, $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_k) = 0$. Thus, by Lemma 2.5, we may assume $\cos\theta_j = \cos 2\theta_i$ or $\cos\theta_i = \cos(\theta_j + \alpha_{jki}\theta_k)$. Similarly, we may assume $\cos\theta_k = \cos 2\theta_i$ or $\cos\theta_i = \cos(\theta_j + \alpha_{jki}\theta_k)$. Now since $\cos\theta_j \neq \cos\theta_k$ then we must have $\cos\theta_i = \cos(\theta_j + \alpha_{jki}\theta_k)$. Also, by Lemma 2.5, either $I_i(\nabla_{X_j} S)X_i = I_i(\nabla_{X_k} S)X_i = 0$ for all $X_i, X_j, X_k$ or $\cos\theta_j = \cos 2\theta_i$ or $\cos\theta_k = \cos 2\theta_i$. It follows that we may assume either

(i) $\cos\theta_i = \cos(\theta_j + \alpha_{jki}\theta_k)$ and $I_i(\nabla_{X_j} S)X_i = I_i(\nabla_{X_k} S)X_i = 0$ for all $X_i, X_j, X_k$, or

(ii) $\cos\theta_i = \cos(\theta_j + \alpha_{jki}\theta_k)$ and $\cos\theta_j = \cos 2\theta_i$

since Case 4 is symmetric in $j$ and $k$.

We first consider (i) and note that from Lemma 2.7 (i),

$$\overline{R}(JX_i, Y_i, X_j, X_k) + \overline{R}(X_i, JY_i, X_j, X_k) = 0 \tag{3.16}$$

for all $X_i, Y_i, X_j, X_k$. Also, by applying Lemmas 2.5 and 2.8 (i) to the pairs

$$\left(I_j(\nabla_{X_i}S)X_k, I_j(\nabla_{Y_i}S)X_k\right), \left(I_i(\nabla_{X_j}S)X_k, I_i(\nabla_{X_i}S)X_k\right),$$

and

$$\left(I_i(\nabla_{X_k}S)X_j, I_i(\nabla_{X_i}S)X_j\right)$$

we have

$$\overline{R}(JX_j, X_k, X_i, Y_i) - \alpha_{jki}\overline{R}(X_j, JX_k, X_i, Y_i) = 0, \tag{3.17}$$

$$\overline{R}(JX_i, X_k, Y_i, X_j) - \alpha_{ikj}\overline{R}(X_i, JX_k, Y_i, X_j) = 0 \tag{3.18}$$

and

$$\overline{R}(JX_i, X_j, Y_i, X_k) - \alpha_{ijk}\overline{R}(X_i, JX_j, Y_i, X_k) = 0. \tag{3.19}$$

We apply the first Bianchi identity to (3.17) and use (3.18) and (3.19) to obtain

$$\begin{aligned}\overline{R}(X_j, JY_i, X_i, X_k) &- \overline{R}(X_j, JX_i, Y_i, X_k)\\ &- \overline{R}(X_j, Y_i, JX_i, X_k) + \overline{R}(X_j, X_i, JY_i, Y_k) = 0.\end{aligned}$$

Then from (3.16), this reduces to

$$\overline{R}(X_j, Y_i, JX_i, X_k) = \overline{R}(X_j, X_i, JY_i, X_k) \tag{3.20}$$

and from (3.18) we also have

$$\overline{R}(X_j, X_i, Y_i, X_k) = \overline{R}(X_j, Y_i, X_i, X_k) \tag{3.21}$$

for all $X_i, Y_i, X_j, X_k$. Clearly (3.21) implies

$$\overline{R}(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k) = 0. \tag{3.22}$$

Next, we write $X = \alpha_i X_i + \alpha_j X_j$ and $Y = \beta_i Y_i + \beta_k Y_k$ in Lemma 2.9 and equate to zero the coefficient of $\alpha_i\alpha_j\beta_i\beta_k$. This gives

$$(1-c_i)s_k\overline{R}(X_i, JX_k, X_j, Y_i) + (1-c_i)s_j\overline{R}(JX_j, Y_i, X_i, X_k)$$
$$+(1-c_k)s_i\overline{R}(JX_i, X_k, X_j, Y_i) + (1-c_j)s_i\overline{R}(X_j, JY_i, X_i, X_k) = 0,$$

where we have used (3.22). A further simplification follows using (3.18)–(3.22) to give

$$\big((1-c_j)s_i + \alpha_{ijk}(1-c_i)s_j + (1-c_k)s_i + \alpha_{ikj}(1-c_i)s_k\big)$$
$$\overline{R}(X_j, JY_i, X_i, X_k) = 0$$

and this in turn reduces to

$$\big(\sin\tfrac{1}{2}(\theta_i\alpha_{ijk} + \theta_j)\sin\tfrac{1}{2}\theta_j + \sin\tfrac{1}{2}(\theta_i\alpha_{ikj} + \theta_k)\sin\tfrac{1}{2}\theta_k\big)$$
$$\overline{R}(X_j, JY_i, X_i, X_k) = 0.$$

From the definition of $\alpha_{ijk}$ and $\alpha_{ikj}$, the left hand side of this equation is just $\pm 2\sin 1/2\theta_j \sin 1/2\theta_k \overline{R}(X_j, JY_i, X_i, X_k)$ so it follows that

$$\overline{R}(X_j, JY_i, X_i, X_k) = 0.$$

Thus $\overline{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_k) = 0$ as required.

Next, we consider (ii) above and note from its conditions that $\cos\theta_k = \cos 3\theta_i$. Thus $I_i(\nabla_{X_k}S)X_i = 0$ for all $X_i, X_k$. Then by applying Lemmas 2.5 and 2.8 to each of the pairs

$$\big(I_i(\nabla_{X_j}S)X_k, I_i(\nabla_{X_i}S)X_k\big), \big(I_j(\nabla_{X_i}S)X_k, I_j(\nabla_{Y_i}S)X_k\big)$$

and

$$\big(I_j(\nabla_{X_k}S)X_i, I_j(\nabla_{X_i}S)X_i\big)$$

we have

$$\overline{R}(JY_i, X_k, X_i, X_j) - \alpha_{ikj}\overline{R}(Y_i, JX_k, X_i, X_j) = 0, \tag{3.23}$$

$$R(JX_j, X_k, X_i, Y_i) - \alpha_{jki}\overline{R}(X_j, JX_k, X_i, Y_i) = 0, \tag{3.24}$$

and

$$\overline{R}(JX_i, X_i, X_i, X_k) - \alpha_{ijk}\overline{R}(X_j, JX_i, X_i, X_k) = 0 \tag{3.25}$$

for all $X_i, Y_i, X_j, X_k$. Clearly (3.24) and (3.25) imply

$$(3.26)\quad \bar{R}(JX_j, X_i, Y_i, X_k) - \bar{R}(JX_j, Y_i, X_i, X_k) - \alpha_{jki}\bar{R}(X_j, X_i, Y_i, JX_k) + \alpha_{jki}\bar{R}(X_j, Y_i, X_i, JX_k) = 0$$

and

$$(3.27)\quad \bar{R}(JX_j, X_i, Y_i, X_k) + \bar{R}(JX_j, Y_i, X_i, X_k) - \alpha_{ijk}\bar{R}(X_j, JX_i, Y_i, X_k) - \alpha_{ijk}\bar{R}(X_j, JY_i, X_i, X_k) = 0.$$

By adding (3.26) and (3.27) and using (3.23) we obtain

$$(3.28)\quad \begin{aligned} 2\bar{R}(JX_j, X_i, Y_i, X_k) &- \alpha_{ijk}\bar{R}(X_j, X_i, JY_i, X_k) \\ &+ \alpha_{ijk}\bar{R}(X_j, Y_i, JX_i, X_k) - \alpha_{ijk}\bar{R}(X_j, JX_i, Y_i, X_k) \\ &- \alpha_{ijk}\bar{R}(X_j, JY_i, X_i, X_k) = 0. \end{aligned}$$

Now write $X = \alpha_i X_i + \alpha_j X_j$ and $Y = \beta_i Y_i + \beta_k Y_k$ in Lemma 2.9 and equate to zero the coefficient of $\alpha_i \alpha_j \beta_i \beta_k$ to obtain

(3.29)

$$\begin{aligned} (1 - c_i)s_i\bar{R}(X_i, JY_i, X_j, X_k) &+ (1 - c_i)s_i\bar{R}(JX_i, Y_i, X_j, X_k) \\ +(1 - c_j)s_i\bar{R}(X_j, JY_i, X_i, X_k) &+ (1 - c_i)s_j\bar{R}(JX_j, Y_i, X_i, X_k) \\ +(1 - c_j)s_k\bar{R}(X_j, JX_k, X_i, Y_i) &+ (1 - c_k)s_j\bar{R}(JX_j, X_k, X_i, Y_i) \\ +(1 - c_i)s_k\bar{R}(X_i, JX_k, X_j, Y_i) &+ (1 - c_k)s_i\bar{R}(JX_i, X_k, X_j, Y_i) = 0. \end{aligned}$$

By choosing $Y_i = X_i$ in (3.29) and using (3.23), (3.25) and the first Bianchi identity we have

$$(3.30)\quad \big((1 - c_j)s_i + (1 - c_i)s_j\alpha_{ijk}\big)\bar{R}(X_j, JX_i, X_i, X_k) + \big((1 - c_k)s_i + (1 - c_i)s_k\alpha_{ikj}\big)\bar{R}(X_j, X_i, JX_i, X_k) = 0.$$

It is easily verified that if any $\theta_u, \theta_v, \theta_w$ are related by $\cos\theta_u = \cos(\theta_v + \alpha_{vwu}\theta_w)$ then

$$(3.31)\quad (1 - c_v)s_w + (1 - c_w)s_v\alpha_{vwu} = 4\lambda \sin\tfrac{1}{2}\theta_u \sin\tfrac{1}{2}\theta_v \sin\tfrac{1}{2}\theta_w$$

where $\lambda = -1$ if $\theta_w = \theta_u + \theta_v$ and $\lambda = 1$ otherwise. Hence, (3.30) simplifies to

$$\bar{R}(X_j, JX_i, X_i, X_k) + \bar{R}(X_j, X_i, JX_i, X_k) = 0,$$

or equivalently,

$$\overline{R}(X_j, JX_i, Y_i, X_k) + \overline{R}(X_j, JY_i, X_i, X_k) + \overline{R}(X_j, X_i, JY_i, X_k) + \overline{R}(X_j, Y_i, JX_i, X_k) = 0. \tag{3.32}$$

Then from (3.28) and (3.32) we obtain

$$\overline{R}(JX_j, X_i, Y_i, X_k) + \alpha_{ijk}\overline{R}(X_j, Y_i, JX_i, X_k) = 0 \tag{3.33}$$

for all $X_i, Y_i, X_j, X_k$.

Next, we use Lemma 2.10 which implies that (3.29) remains valid with $\theta_i, \theta_j, \theta_k$ replaced by $m\theta_i, m\theta_j, m\theta_k$ for any positive integer $m$. Then it follows from (3.23), (3.32) and (3.33) that the generalised form of (3.29) reduces to

$$A_m\overline{R}(X_j, JY_i, X_i, X_k) + B_m\overline{R}(X_j, Y_i, JX_i, X_k) - C_m\overline{R}(X_j, X_i, JY_i, X_k) = 0 \tag{3.34}$$

where

$$\begin{aligned} A_m &= 2(1 - \cos m\theta_i)\sin m\theta_i + (1 - \cos m\theta_j)\sin m\theta_i, \\ B_m &= 2(1 - \cos m\theta_i)\sin m\theta_i + (1 - \cos m\theta_k)\sin m\theta_i \\ &\quad + \alpha_{ikj}(1 - \cos m\theta_i)\sin m\theta_k + \alpha_{ikj}(1 - \cos m\theta_j)\sin m\theta_k \\ &\quad + \alpha_{ijk}(1 - \cos m\theta_k)\sin m\theta_j, \\ C_m &= \alpha_{ikj}(1 - \cos m\theta_j)\sin m\theta_k + \alpha_{ijk}(1 - \cos m\theta_k)\sin m\theta_j \\ &\quad + \alpha_{ijk}(1 - \cos m\theta_i)\sin m\theta_j. \end{aligned}$$

We note that for any $\theta_a, \theta_b, \theta_c$, the relation $\cos\theta_a = \cos(\theta_b + \alpha_{bca}\theta_c)$ implies that, for $m$ as above, $\cos m\theta_a = \cos(m\theta_b + \alpha_{bca}m\theta_c)$ and then

$$\begin{aligned} (1 - \cos m\theta_b)\sin m\theta_a &+ \alpha_{abc}(1 - \cos m\theta_a)\sin m\theta_b \\ &= 4\lambda_m \sin\tfrac{1}{2}m\theta_a \sin\tfrac{1}{2}m\theta_b \sin\tfrac{1}{2}m\theta_c \end{aligned}$$

where $\lambda_m = -1$ if $\theta_a = \theta_b + \theta_c$ or if $\theta_a + \theta_b + \theta_c = 2\pi$ and $m$ is even, and otherwise $\lambda_m = 1$. From this, it follows that $A_m = B_m - C_m$ for all $m$ so (3.34) becomes

$$B_m\big(\overline{R}(X_j, Y_i, JX_i, X_k) + \overline{R}(X_j, JY_i, X_i, X_k)\big) - C_m\big(\overline{R}(X_j, X_i, JY_i, X_k) + \overline{R}(X_j, JY_i, X_i, X_k)\big) = 0. \tag{3.35}$$

Also, by writing $X_i, Y_i$ as $JX_i, JY_i$ in (3.35) and using (3.32) we obtain

$$(3.36)\quad B_m\big(\bar{R}(X_j, Y_i, JX_i, X_k) + \bar{R}(X_j, JY_i, X_i, X_k)\big) + C_m\big(\bar{R}(X_j, X_i, JY_i, X_k) + \bar{R}(X_j, JY_i, X_i, X_k)\big) = 0.$$

Now it is easy to verify that

$$(3.37)\quad B_m = 2(1 - \cos m\theta_i)\sin m\theta_i + 8\lambda_m \sin \frac{1}{2} m\theta_i \sin \tfrac{1}{2} m\theta_j \sin \tfrac{1}{2} m\theta_k$$

and

$$(3.38)\quad C_m = \alpha_{ijk}(1 - \cos m\theta_i)\sin m\theta_j + 4\lambda_m \sin \tfrac{1}{2} m\theta_i \sin \tfrac{1}{2} m\theta_j \sin \tfrac{1}{2} m\theta_k$$

where $\lambda_m = -1$ if $\theta_i = \theta_j + \theta_k$ or $\theta_i + \theta_j + \theta_k = 2\pi$ and $m$ is even, and otherwise $\lambda_m = 1$. Then $B_1 = 0$ only if $\lambda_1 = -1$ in which case $\theta_j = 2\pi - 2\theta_i$ and $\theta_k = 3\theta_i - 2\pi$. It follows from (3.37) that $B_1 = 0$ only if $\cos \theta_i = -3/4$. In particular, $\pi/2 < \theta_i < \pi$. Hence if $B_1 = B_2 = 0$ then $\lambda_2 = 1$ which is impossible since $\theta_i = \theta_j + \theta_k$. Similarly, $C_1 = 0$ only if $\theta_j = \theta_i + \theta_k$ and then $\cos \theta_i = -1/3$. Hence $\pi/2 < \theta_j < \pi$ and $C_1 = C_2 = 0$ only if $\alpha_{ijk}\lambda_2 = 1$ which is impossible since $\theta_j = \theta_i + \theta_k$. It now follows from (3.35) and (3.36) that for all $X_i, Y_i, X_j, X_k$,

$$(3.39)\quad \bar{R}(X_j, Y_i, JX_i, X_k) + \bar{R}(X_j, JY_i, X_i, X_k) = 0$$

and

$$(3.40)\quad \bar{R}(X_j, X_i, Y_i, X_k) + \bar{R}(X_j, Y_i, X_i, X_k) = 0.$$

Then from (3.27), (3.39) and (3.40) we obtain

$$(3.41)\quad \bar{R}(X_j, X_i, Y_i, X_k) - \bar{R}(X_j, Y_i, X_i, X_k) = 0$$

and (3.40), (3.41) imply $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_k) = 0$. This completes the proof for Case 4.

*Case* 4′. $\bar{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_0, \mathscr{D}_j) = 0$; $i \neq j$.

First suppose $c_i + c_j \neq 0$. Then $I_0(\nabla_{X_j} S)X_i = 0$ from Lemma 2.4(i). Also, $I_0(\nabla_{X_0} S)X_i = 0$ and it follows easily that $\bar{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_0, \mathscr{D}_j) = 0$. Next, suppose $c_i + c_j = 0$. By Lemma 2.4(d),

$$I_0(\nabla_{Y_j} S)JX_i - I_0(\nabla_{JY_j} S)X_i = 0.$$

Also, $I_0(\nabla_{Y_0}S)JX_i = I_0(\nabla_{Y_0}S)X_i = 0$ which gives us

$$\begin{aligned}\bar{R}(SX_0, SJX_i, Y_j, Y_0) - \bar{R}(X_0, JX_i, Y_j, Y_0) - \bar{R}(SX_0, SX_i, JY_j, Y_0) \\ + \bar{R}(X_0, X_i, JY_j, Y_0) = 0\end{aligned}$$

for all $X_0, Y_0, X_i, Y_j$. This implies

$$\bar{R}(X_0, X_i, JY_j, Y_0) = \bar{R}(X_0, JX_i, Y_j, Y_0).$$

Next, write $X = aX_0 + bX_i$, $Y = cY_0 + dX_j$ in Lemma 2.9 and equate to zero the coefficient of $abcd$ to obtain

$$\begin{aligned}(3 - c_i)\bar{R}(X_0, JY_j, X_i, Y_0) + (3 + c_i)\bar{R}(X_0, Y_j, JX_i, Y_0) \\ -(1 + c_i)\bar{R}(X_0, JX_i, Y_j, Y_0) - (1 - c_i)\bar{R}(X_0, X_i, JY_j, Y_0) = 0.\end{aligned}$$

From these two equations we have $\bar{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_0, \mathscr{D}_j) = 0$ as required.

*Case* $\bar{5}$. $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_1) = 0$; $i, j, k, l$ distinct.

For any given $i, j, k, l$, we show that $\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_l) = 0$ for all permutations of $i, j, k, l$; equivalently, we show that

$$\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_l) = \bar{R}(\mathscr{D}_i, \mathscr{D}_k, \mathscr{D}_j, \mathscr{D}_l) = 0.$$

In this way the number of apparently different cases is considerably reduced. First note that if

$$I_i(\nabla_{X_k}S)X_j = I_i(\nabla_{X_l}S)X_k = I_i(\nabla_{X_j}S)X_l = 0$$

for all $X_j, X_k, X_l$ then, by Lemma 2.7(i),

$$\bar{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_l) = \bar{R}(\mathscr{D}_i, \mathscr{D}_k, \mathscr{D}_j, \mathscr{D}_l) = 0.$$

On the other hand, if there exist $X_j, Y_j, X_k, Y_k, Z_k, X_l, Y_l, Z_l$, such that $I_i(\nabla_{X_k}S)X_j$, $I_i(\nabla_{Y_l}S)X_k$, $I_i(\nabla_{Y_j}S)Y_l$ and $I_j(\nabla_{Z_l}S)Z_k$ are non-zero then, by Lemma 2.5,

$$\cos\theta_i = \cos(\theta_j + \alpha_{jki}\theta_k) = \cos(\theta_k + \alpha_{kli}\theta_l) = \cos(\theta_l + \alpha_{lji}\theta_j)$$

and

$$\cos\theta_j = \cos(\theta_k + \alpha_{klj}\theta_l)$$

which is easily seen to be impossible. Thus, because of Lemma 2.5 and the symmetry properties of $\overline{R}$, we need consider only the two cases where

$$\text{(i)} \qquad \cos\theta_l = \cos(\theta_i + \alpha_{ijl}\theta_j) = \cos(\theta_j + \alpha_{jki}\theta_k)$$

and

$$I_i(\nabla_{X_k}S)X_j = I_i(\nabla_{X_l}S)X_k = 0 \quad \text{for all } X_j, X_k, X_l,$$

or

$$\text{(ii)} \quad \cos\theta_l = \cos(\theta_i + \alpha_{ijl}\theta_j) = \cos(\theta_j + \alpha_{jkl}\theta_k) = \cos(\theta_k + \alpha_{kil}\theta_i)$$

and

$$I_i(\nabla_{X_k}S)X_j = 0 \quad \text{for all } X_j, X_k.$$

We first assume (i) and note that $\overline{R}(\mathscr{D}_i, \mathscr{D}_k, \mathscr{D}_j, \mathscr{D}_l) = 0$, from Lemma 2.7(i). Hence,

$$\overline{R}(X_i, X_j, X_k, X_l) + \overline{R}(X_i, X_l, X_j, X_k) = 0 \tag{3.42}$$

for all $X_i$, $X_j$, $X_k$, $X_l$. Next, we apply Lemmas 2.5 and 2.8 to the pairs

$$\left(I_i(\nabla_{X_k}S)X_j, I_i(\nabla_{X_l}S)X_j\right), \left(I_j(\nabla_{X_i}S)X_k, I_j(\nabla_{X_l}S)X_k\right),$$
$$\left(I_i(\nabla_{X_k}S)X_l, I_i(\nabla_{X_j}S)X_l\right) \quad \text{and} \quad \left(I_k(\nabla_{X_i}S)X_l, I_k(\nabla_{X_j}S)X_l\right)$$

to obtain

$$\overline{R}(JX_i, X_j, X_k, X_l) - \alpha_{ijl}\overline{R}(X_i, JX_j, X_k, X_l) = 0,$$
$$\overline{R}(JX_i, X_k, X_i, X_l) - \alpha_{jkl}\overline{R}(X_j, JX_k, X_i, X_l) = 0,$$
$$\overline{R}(JX_i, X_l, X_j, X_k) - \alpha_{ilj}\overline{R}(X_i, JX_l, X_j, X_k) = 0,$$

and

$$\overline{R}(JX_k, X_l, X_i, X_j) - \alpha_{klj}\overline{R}(X_k, JX_l, X_i, X_j) = 0.$$

Then from (3.42) and the above four equations,

$$\begin{aligned}
\overline{R}(JX_i, X_j, X_k, X_l) &= \alpha_{ijl}\overline{R}(X_i, JX_j, X_k, X_l) \\
&= \alpha_{ijl}\alpha_{jkl}\overline{R}(X_i, X_j, JX_k, X_l) \\
&= \alpha_{ijl}\alpha_{jkl}\alpha_{klj}\overline{R}(X_i, X_j, X_k, JX_l) \\
&= \alpha_{ijl}\alpha_{jkl}\alpha_{klj}\alpha_{ilj}\overline{R}(JX_i, X_j, X_k, X_l) \\
&= \alpha_{jli}\alpha_{jlk}\overline{R}(JX_i, X_j, X_k, X_l).
\end{aligned}$$

It is easily seen by inspection that $\alpha_{jli}\alpha_{jlk} = -1$ in all possible cases. Hence $\overline{R}(JX_i, X_j, X_k, X_l) = 0$ and $\overline{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_l) = 0$ as required.

Next, consider (ii). We may assume $\theta_i < \theta_j < \theta_k$ and it then follows from the cosine equalities in (ii) that

$$\theta_j + \theta_k + \theta_l = 2\pi, \theta_j = \theta_l - \theta_i, \theta_k = \theta_l + \theta_i, \theta_l = \frac{2\pi}{3}. \tag{3.43}$$

Also, by applying Lemmas 2.5 and 2.8 to the pairs

$$\big(I_i(\nabla_{X_k}S)X_j, I_i(\nabla_{X_l}S)X_j\big), \big(I_i(\nabla_{X_j}S)X_k, I_i(\nabla_{X_l}S)X_k\big) \quad \text{and}$$
$$\big(I_j(\nabla_{X_i}S)X_k, I_j(\nabla_{X_l}S)X_k\big),$$

and using (3.43), we have

$$R(JX_i, X_j, X_k, X_l) - R(X_i, JX_j, X_k, X_l) = 0, \tag{3.44}$$

$$R(JX_i, X_k, X_j, X_l) + R(X_i, JX_k, X_j, X_l) = 0, \tag{3.45}$$

$$R(JX_j, X_k, X_i, X_l) - R(X_j, JX_k, X_i, X_l) = 0. \tag{3.46}$$

We now use Lemma 2.10 by writing $X = \alpha_i X_i + \alpha_j X_j$ and $Y = \alpha_k X_k + \alpha_l X_l$, and equating to zero the coefficient of $\alpha_i\alpha_j\alpha_k\alpha_l$. It follows that for all $X_i, X_j, X_k, X_1$,

$$\begin{aligned}
&(1 - \cos m\theta_i)\sin m\theta_k \overline{R}(X_i, JX_k, X_j, X_l)\\
&\quad + (1 - \cos m\theta_k)\sin m\theta_i \overline{R}(JX_i, X_k, X_j, X_l)\\
&\quad + (1 - \cos m\theta_i)\sin m\theta_l \overline{R}(X_i, JX_l, X_j, X_k)\\
&\quad + (1 - \cos m\theta_l)\sin m\theta_i \overline{R}(JX_i, X_l, X_j, X_k)\\
&\quad + (1 - \cos m\theta_j)\sin m\theta_k \overline{R}(X_j, JX_k, X_i, X_l)\\
&\quad + (1 - \cos m\theta_k)\sin m\theta_j \overline{R}(JX_j, X_k, X_i, X_l)\\
&\quad + (1 - \cos m\theta_j)\sin m\theta_l \overline{R}(X_j, JX_l, X_i, X_k)\\
&\quad + (1 - \cos m\theta_l)\sin m\theta_j \overline{R}(JX_j, X_l, X_i, X_k) = 0.
\end{aligned} \tag{3.47}$$

Since $\theta_l = 2\pi/3$ we replace $m$ by $3m + 1$ in (3.47). Then, using (3.44)–(3.47) and the first Bianchi identity we obtain, after some calculation,

$$
\begin{aligned}
(3.48)\quad & \cos(3m+1)\theta_i\big(2\bar{R}(X_i, JX_k, X_j, X_l) - 2\bar{R}(X_i, JX_l, X_j, X_k) \\
&\qquad + 4\bar{R}(X_j, JX_k, X_i, X_l) + \bar{R}(X_j, JX_l, X_i, X_k) \\
&\quad + 3\bar{R}(JX_j, X_l, X_i, X_k)\big) \\
&\quad - \sqrt{3}\sin(3m+1)\theta_i\big(2\bar{R}(X_i, JX_k, X_j, X_l) - 2\bar{R}(JX_i, X_l, X_j, X_k) \\
&\qquad + \bar{R}(X_j, JX_l, X_i, X_k) \\
&\qquad - \bar{R}(JX_j, X_l, X_i, X_k)\big) \\
&\qquad - 2\big(\bar{R}(X_i, JX_k, X_j, X_l) \\
&\qquad - \bar{R}(X_i, JX_l, X_j, X_k) \\
&\qquad - \bar{R}(X_j, JX_k, X_i, X_l) - \bar{R}(X_j, JX_l, X_i, X_k)\big) = 0.
\end{aligned}
$$

We consider (3.48) for $m = 0, 1, 2$ and note that

$$
\begin{vmatrix} 1 & 1 & 1 \\ \cos\theta_i & \cos 4\theta_i & \cos 7\theta_i \\ \sin\theta_i & \sin 4\theta_i & \sin 7\theta_i \end{vmatrix} = 2\sin 3\theta_i - \sin 6\theta_i \neq 0
$$

since if $\sin 3\theta_i = 0$ then $\theta_i = \pi/3$ or $2\pi/3$ which is impossible by (3.43). Hence, from (3.48), we have

$$
\begin{aligned}
(3.49)\quad & 2\bar{R}(X_i, JX_k, X_j, X_l) - 2\bar{R}(X_i, JX_l, X_j, X_k) + 4\bar{R}(X_j, JX_k, X_i, X_l) \\
& + \bar{R}(X_j, JX_l, X_i, X_k) + 3\bar{R}(JX_j, X_l, X_i, X_k) = 0,
\end{aligned}
$$

$$
\begin{aligned}
(3.50)\quad & 2\bar{R}(X_i, JX_k, X_j, X_l) - 2\bar{R}(JX_i, X_l, X_j, X_k) + \bar{R}(X_j, JX_l, X_i, X_k) \\
& - \bar{R}(JX_j, X_l, X_i, X_k) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.51)\quad & \bar{R}(X_i, JX_k, X_j, X_l) - \bar{R}(X_i, JX_l, X_j, X_k) - \bar{R}(X_j, JX_k, X_i, X_l) \\
& - \bar{R}(X_j, JX_l, X_i, X_k) = 0.
\end{aligned}
$$

Now replace $X_k, X_l$ by $JX_k, JX_l$ in (3.51) to obtain

$$\begin{aligned}(3.52)\quad &\overline{R}(X_i, X_k, X_j, JX_l) - \overline{R}(X_i, X_l, X_j, JX_k) - \overline{R}(X_j, X_k, X_i, JX_l)\\ &-\overline{R}(X_j, X_l, X_i, JX_k) = 0.\end{aligned}$$

Then from (3.51) and (3.52),

$$(3.53)\qquad \overline{R}(X_i, X_l, X_j, JX_k) + \overline{R}(X_i, JX_l, X_j, X_k) = 0$$

and

$$(3.54)\qquad \overline{R}(X_i, X_k, X_j, JX_l) = \overline{R}(X_i, JX_k, X_j, X_l).$$

Using (3.53), (3.54) and the first Bianchi identity, (3.49) and (3.50) reduce to

$$\begin{aligned}(3.55)\quad &3\overline{R}(X_i, JX_k, X_j, X_l) - 2\overline{R}(X_i, X_j, JX_k, X_l) + \overline{R}(X_i, X_k, JX_j, X_l)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(3.56)\quad &5\overline{R}(X_i, JX_k, X_j, X_l) + 2\overline{R}(X_i, JX_j, X_k, X_l) - \overline{R}(X_i, X_k, JX_j, X_l)\\ &= 0.\end{aligned}$$

We now replace $X_j, X_k$ by $JX_j, JX_k$ in (3.55) to obtain

$$\begin{aligned}(3.57)\quad &3\overline{R}(X_i, X_k, JX_j, X_l) - 2\overline{R}(X_i, JX_j, X_k, X_l) + \overline{R}(X_i, JX_k, X_j, X_l)\\ &= 0\end{aligned}$$

and then by adding (3.55) and (3.57) we have

$$(3.58)\qquad 3\overline{R}(X_i, JX_k, X_j, X_l) + \overline{R}(X_i, X_k, JX_j, X_l) = 0.$$

But then from (3.58),

$$(3.59)\qquad 3\overline{R}(X_i, X_k, JX_j, X_l) + \overline{R}(X_i, JX_k, X_j, X_l) = 0$$

so, from (3.58) and (3.59), $\overline{R}(\mathscr{D}_i, \mathscr{D}_k, \mathscr{D}_j, \mathscr{D}_l) = 0$. Hence, from (3.57), $\overline{R}(\mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_l) = 0$. This completes the proof of Case 5.

*Case 5′.* $\overline{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k) = 0$; $i, j, k$ distinct and $\neq 0$.

If $I_0(\nabla_{X_j} S)X_i = I_0(\nabla_{X_j} S)X_k = I_0(\nabla_{X_i} S)X_k = 0$ for all $X_i, X_j, X_k$ then, following the proof of Lemma 2.7(i), we obtain

$$\overline{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k) = \overline{R}(\mathscr{D}_0, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_k) = 0.$$

Hence, from Lemma 2.4(c) and the $S$-invariance of $\nabla S$, we must assume $c_j + c_k = 0$. Then

$$I_0(\nabla_{X_j} S)X_i = I_0(\nabla_{X_k} S)X_i = 0 \quad \text{for all } X_i, X_j, X_k, \tag{3.60}$$

from which

$$\bar{R}(\mathscr{D}_0, \mathscr{D}_i, \mathscr{D}_j, \mathscr{D}_k) = 0. \tag{3.61}$$

We note that (3.60) is true for any permutation of $i, j, 0$ or of $i, k, 0$. Then if $I_i(\nabla_{X_j} S)X_k = 0$ for all $X_j, X_k$, we use the relation $I_i(\nabla_{X_0} S)X_k = 0$ and Lemma 2.4(i) to obtain $\bar{R}(\mathscr{D}_0, \mathscr{D}_j, \mathscr{D}_i, \mathscr{D}_k) = 0$. Thus, we may assume $I_i(\nabla_{X_j} S)X_k \neq 0$ for some $X_j, X_k$ and then

$$\cos\theta_k = \cos(\theta_i + \alpha_{ijk}\theta_j)$$

by Lemma 2.5. From this and the above relation $c_j + c_k = 0$, we may also assume $\theta_i = 2\theta_j - \pi$, $\theta_k = \pi - \theta_j$ and $\theta_j > \pi/2$. Next, we note that for all $X_j, X_k$,

$$J_i(\nabla_{X_k} S)X_j + \alpha_{ijk} I_i(\nabla_{X_k} S)JX_j = J_i(\nabla_{X_0} S)X_j + \alpha_{ijk} I_i(\nabla_{X_0} S)JX_j = 0,$$

where $\alpha_{ijk} = -1$ since $\theta_j = \theta_i + \theta_k$. Then, by following the proof of Lemma 2.8, we, obtain

$$\bar{R}(JX_i, X_j, X_k, X_0) + \bar{R}(X_i, JX_j, X_k, X_0) = 0.$$

Now write $X = X_0 + aX_j$, $Y = bX_i + cX_k$ in Lemma 2.9 and equate to zero the coefficient of $abc$. After some calculation using the above relations, this gives

$$\bar{R}(X_0, X_j, JX_k, X_i) + c_j \bar{R}(X_0, JX_j, X_k, X_i) = 0$$

for all $X_i, X_j, X_k$. Since $c_j^2 \neq 1$, it follows that $\bar{R}(\mathscr{D}_0, \mathscr{D}_j, \mathscr{D}_k, \mathscr{D}_i) = 0$ and this together with (3.61) proves Case 5′.

The above five cases together with the first Bianchi identity are clearly sufficient to establish that $\bar{R} = 0$ so the proof that the curvature tensor field $R$ is $S$-invariant is complete.

Finally in this section we prove the $S$-invariance of $\nabla^2 S$. Thus, from (2.9) we have

$$(\nabla^2_{YX} S)(I - S^{-1})X - (\nabla_X S)(\nabla_Y S^{-1})X = 0 \quad \text{for all } X, Y \in \mathscr{T}^1.$$

Also, from the relation $(\nabla_X S)(I - S^{-1})Y + (\nabla_Y S)(I - S^{-1})X = 0$ we obtain

$$(\nabla^2_{XX} S)(I - S^{-1})Y + (\nabla^2_{XY} S)(I - S^{-1})X - (\nabla_X S)(\nabla_X S^{-1})Y \\ -(\nabla_Y S)(\nabla_X S^{-1})X = 0.$$

Hence

$$(\nabla^2_{XX} S)(I - S^{-1})Y + R(X,Y)(S - I)X - SR(X,Y)(I - S^{-1})X \\ +(\nabla_X S)(\nabla_Y S^{-1})X - (\nabla_X S)(\nabla_X S^{-1})Y - (\nabla_Y S)(\nabla_X S^{-1})X = 0.$$

By linearising this equation and noting that $\nabla^2_{XY} S - \nabla^2_{YX} S$ can be expressed in terms of $R$, it follows that $\nabla^2 S$ is $S$-invariant.

## 4. Conditions for the $S$-invariance of $\nabla R$

For convenience of notation, we define $A \in \mathscr{T}^1_2$ by $A_X Y = A(X,Y) = (\nabla_{(I-S)^{-1}X} S)S^{-1}Y$ for $X$, $Y \in \mathscr{T}^1$. Then from (2.1), $\tilde{\nabla}_X Y = \nabla_X Y - A_X Y$ where we regard $A_X$ as a derivation. Since $\nabla S$ and $\nabla^2 S$ are $S$-invariant, we know that $A$ and $\nabla A$ are $S$-invariant so, from [2], $\tilde{\nabla} A = 0$ and the curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ satisfies

$$(4.1)\quad g(\tilde{R}(Z,W)Y,X) \underset{\text{def}}{=} \tilde{R}(X,Y,Z,W) \\ = R(X,Y,Z,W) + g(A(W,Y),A(Z,X)) \\ - g(A(Z,Y),A(W,X)) \\ + 2g(A(Z,W),A(Y,X))$$

where we have used (2.2) and (2.11) for simplifications.

As noted in §2, $\tilde{\nabla} S = 0$, so

$$\tilde{R}(SX,SY,Z,W) = \tilde{R}(X,Y,Z,W)$$

and

$$(\tilde{\nabla}_V \tilde{R})(SX,SY,Z,W) = (\tilde{\nabla}_V \tilde{R})(X,Y,Z,W)$$

for all $X$, $Y$, $Z$, $W$, $V \in \mathscr{T}^1$. We now define $P \in \mathscr{T}_5$ by

$$(4.2)\quad P(X,Y,Z,W,V) \\ = (\tilde{\nabla}_{SV} \tilde{R})(SX,SY,SZ,SW) - (\tilde{\nabla}_V \bar{R})(X,Y,Z,W) \\ = (\tilde{\nabla}_{SV} R)(SX,SY,SZ,SW) - (\tilde{\nabla}_V R)(X,Y,Z,W).$$

Because of the $S$-invariance of $R$ and $A$ we also have

$$(4.3)\qquad P(X,Y,Z,W,V) = (\nabla_{SV}R)(SX,SY,SZ,SW) - (\nabla_V R)(X,Y,Z,W).$$

Clearly $P$ satisfies all the Riemannian curvature identities including the second Bianchi identity. Moreover,

$$(4.4)\qquad P(SX,SY,Z,W,V) = P(X,Y,Z,W,V) \quad \text{for all } X, Y, Z, W, V \in \mathscr{T}^1$$

and it follows easily that $P(X_h, Y_i, Z_j, W_k, V_l) = 0$ unless $X_h$, $Y_i$, $Z_j$, $W_k$, $V_l \in \mathscr{D}_0$ or $X$, $Y$, $Z$, $W$, $V \in \mathscr{D}_p$ for some $p \in [r]$, where, for the latter case, we use Lemma 2.4(a). Now suppose $\nabla R$ is $S$-invariant, that is $P = 0$. Then clearly $(\nabla_{\mathscr{D}_0} R)(\mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0) = 0$. Also, since $R$ and $\nabla R$ are $S$-invariant then $\tilde{\nabla} R = 0$ [2]. Hence $\nabla_X R = A_X R$ for all $X \in \mathscr{T}^1$. In particular, for all $i \in [r]$ and for all $X_i$

$$(\nabla_{X_i} R)(X_i, JX_i, X_i, JX_i) = (A_{X_i} R)(X_i, JX_i, X_i, JX_i) = 0$$

as follows from (2.2) and Lemma 2.5. Conversely, suppose

$$(\nabla_{\mathscr{D}_0} R)(\mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0, \mathscr{D}_0) = 0 \quad \text{and} \quad (\nabla_{X_i} R)(X_i, JX_i, X_i, JX_i) = 0$$

for all $i \in [r]$ and all $X_i$. From Lemma 2.4(b) and (4.4),

$$(4.5)\qquad P(JX_i, JY_i, Z_i, W_i, V_i) = P(X_i, Y_i, Z_i, W_i, V_i)$$

for all $X_i, Y_i, Z_i, W_i, V_i$. Also, by assumption,

$$(4.6)\qquad P(X_i, JX_i, X_i, JX_i, X_i) = 0 \quad \text{for all } X_i.$$

Then, as is well known, (4.5) and (4.6) imply $P(\mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i, \mathscr{D}_i) = 0$ [11]. Hence $P = 0$ and $\nabla R$ is $S$-invariant. Theorem 2.2 now follows as an immediate consequence of Theorem 2.1.

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