ON COMPLEMENTARY AUTOMORPHIC FORMS AND SUPPLEMENTARY FOURIER SERIES

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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1. Let Γ be a discontinuous group of linear transformations of the upper half-plane $\mathcal{H}$ on itself. If F, G are automorphic forms belonging to Γ, we say that F and G are *complementary forms* provided FG is a differential, i.e., provided

$$FG \in \{\Gamma, -2, 1\},$$

where by $\{\Gamma, -r, v\}$ we mean the complex vector space of automorphic forms of dimension $-r$ belonging to Γ and the multiplier system v. If $F \in \{\Gamma, -r, v\}$, $G \in \{\Gamma, -r', v'\}$, then F is complementary to G if and only if[2]

$$r + r' = 2, \qquad vv' = 1. \tag{1}$$

In particular, assume $r < 0$, μ a positive integer, and let $F_\mu(\tau)$ be that form in $\{\Gamma, -r, v\}$ which is regular in $\mathcal{H}$, has a pole of order μ at ∞, and has the Fourier expansion

$$e(-\kappa\tau/\lambda)F_\mu(\tau) = t^{-\mu} + \textstyle\sum_{m=0}^{\infty} a_m t^m, \qquad t = e(\tau/\lambda), \tag{2}$$

where

$$e(z) = e^{2\pi i z}.$$

Here λ, κ are defined in (3), (4). That is, we assume such a form exists. Petersson has defined a system of forms belonging to the complementary class $\{\Gamma, -r', v'\}$, namely, the Poincaré series $G(\tau, -r', v', \mu) = G_\mu$; cf. (8). In §3, Theorem 1, we shall exhibit a connection between F_μ and G: *If there exists a form $F_\mu \in \{\Gamma, -r, v\}$ satisfying* (2), *then $G_{\mu-1} \equiv 0$ when $\kappa > 0$ and $G_\mu \equiv 0$ when $\kappa = 0$.* This result is applied to the modular group in §4 (cf. Petersson [3, p. 432]).

If $F_\mu \in \{\Gamma, -r, v\}$, the coefficients a_m have convergent series representations given in (5). But a_m can be defined by (5) *whether F_μ is an automorphic form or not.* Write

$$a_m = a_m(\mu, -r, v)$$

to express the fact that a_m is determined by the data in parentheses even though there may not exist a form of type F_μ.

In §5 we restrict $-r$ to positive integral values. We regard F_μ as a Fourier

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[2] It is known that v^{-1} is a multiplier system for $[\Gamma, -r']$ if v is one for $[\Gamma, -r]$.

series with the expansion (2) and make no assumption whatever about its automorphic character. We define a new Fourier series $\hat{F}_{\mu'}$, called the series *supplementary* to F_μ, by setting

$$e(-\kappa'\tau/\lambda)\hat{F}_{\mu'}(\tau) = t^{-\mu'} + \sum_{m=0}^{\infty} \hat{a}_m t^m$$

with

$$\hat{a}_m = a_m(\mu', -r, v')$$

and κ', μ' defined in (7) and (15), respectively. It turns out that F_μ *does belong to* $\{\Gamma, -r, v\}$ if and only if $\hat{F}_{\mu'} \equiv 0$ when $\kappa > 0$ and if and only if $\hat{F}_{\mu'} \equiv \hat{a}_0$ when $\kappa = 0$ (§6, Theorem 3).

From these results we can deduce easily (§8, Theorem 4) that $F_\mu \in \{\Gamma, -r, v\}$ *if and only if* $G_{\mu-1} \equiv 0$ $(\kappa > 0)$ *or* $G_\mu \equiv 0$ $(\kappa = 0)$, provided we keep the requirement that $-r$ be a positive integer. This is a strengthening of Theorem 1.

We owe the idea for the present investigation to a paper of Rademacher [5] in which he considers the partition coefficients $p(n)$ for negative n.

2. It is doubtless possible to treat rather general groups by the methods of this paper. We prefer to restrict the group as well as the automorphic form. Let Γ be a group which is discontinuous in $\mathfrak{H}$, is not discontinuous at any point of the real axis, and contains exactly one class of parabolic transformations, namely, translations generated by an element of period λ, say $(\lambda > 0)$. Γ has a fundamental region with a cusp at ∞ and no other real cusp. We represent Γ as a 2×2 unimodular matrix group. For convenience $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is assumed to belong to Γ; the matrices $\pm V$, $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, are identified with the transformation $Vz = (az + b)/(cz + d)$.

Automorphic forms on such groups have been treated in a previous paper [2]. We shall make use of the results of that paper. The notation used there is the notation of the Rademacher school, whereas forms of negative dimension have been elaborated extensively by Petersson in a different notation (cf. [4], for example). In order to make a comparison, it is necessary to settle on one notation. We shall rewrite the results of [2] in Petersson's notation since we have already used it in §1.

Notice particularly that Petersson calls the dimension of a form $-r$, rather than r, as in [2]. A form $F \in \{\Gamma, -r, v\}$ has the transformation equation

$$F(M\tau) = v(M)(c\tau + d)^r F(\tau), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tau \in \mathfrak{H},$$

for each $M \in \Gamma$. We require $|v(M)| = 1$ and set

$$-\pi < \arg(c\tau + d) \leqq \pi.$$

Let

(3) $$U^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad \lambda > 0,$$

generate the cyclic subgroup of Γ consisting of these elements that fix ∞. Define κ by

$$(4) \qquad e(\kappa) = v(U^{\lambda}), \qquad 0 \leqq \kappa < 1.$$

$F(\tau)$ has the Fourier expansion

$$e(-\kappa\tau/\lambda)F(\tau) = \textstyle\sum_{m=-\infty}^{\infty} a_m\, e(m\tau/\lambda) = f(t), \qquad t = e(\tau/\lambda).$$

Suppose now F is a form of dimension $-r > 0$, and F is *regular* in $\mathcal{H}$ but admits a pole of order μ at ∞ with principal part $t^{-\mu}$. We write $F = F_\mu$; then the above expansion reads:

$$(2) \qquad e(-\kappa\tau/\lambda)F_\mu(\tau) = f(t) = t^{-\mu} + \textstyle\sum_{m=0}^{\infty} a_m\, t^m,$$

the series converging in $\mathcal{H}$, i.e., for $\operatorname{Im} \tau > 0$ or $|t| < 1$. We write more explicitly

$$a_m = a_m(\mu, -r, v).$$

The Fourier coefficients a_m are given by Theorem 1 of [2]. When we transcribe this formula in the new notation, we get

$$(5) \quad a_m(\mu, -r, v) = e(-r/4)(2\pi/\lambda) \textstyle\sum_{c \in C^+} c^{-1} W_c(m, -\mu, v) L_c(m, \mu, -r, \kappa)$$

where

$$C^+ = \left\{ c \,\middle|\, \exists \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix} \in \Gamma, c > 0 \right\},$$

$$D_c = \left\{ d \,\middle|\, \exists \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma, c \neq 0, 0 \leqq -d < |c| \lambda \right\},$$

$$W_c(m, \mu, v) = \textstyle\sum_{d \in D_c} \bar{v}(M) e\{[(m + \kappa)d + (\mu + \kappa)a]/c\lambda\}, \qquad c \neq 0,$$

$$L_c(m, \mu, r, \kappa) = \left(\frac{\mu - \kappa}{m + \kappa}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi}{c\lambda}(\mu - \kappa)^{1/2}(m + \kappa)^{1/2}\right), \quad m + \kappa \neq 0.$$

Here $I_r(z)$ is the Bessel function

$$I_r(z) = e(-r/4)J_r(iz) = \sum_{p=0}^{\infty} \frac{(z/2)^{r+2p}}{p!\,\Gamma(r + p + 1)}.$$

For each integer m we have from (2),

$$a_m = \frac{1}{2\pi i} \int_C \frac{f(t)}{t^{m+1}}\, dt,$$

C being a circle interior to the unit circle and enclosing the origin. It follows that

$$a_m = 0 \quad \text{for} \quad m < 0, \quad m \neq -\mu.$$

Now an examination of the proof of Theorem 1 in [2] reveals that it does not depend on the nonnegativity of m. That is, (5) *holds for all integral m.*

Hence we obtain the remarkable identity:

$$\begin{aligned} a_{-m}(\mu, -r, v) \\ = e(-r/4)(2\pi/\lambda)\textstyle\sum_{c\epsilon C^+} c^{-1}W_c(-m, -\mu, v)L_c(-m, \mu, r, \kappa) = 0 \end{aligned} \tag{6}$$

for $m > 0$, $m \neq \mu$. This may be called an expansion of zero.

3. We are now going to compare the expression for a_{-m} with the Fourier coefficients of certain Poincaré series G (cf. [4, p. 469 ff.]) belonging to $\{\Gamma, -r', v'\}$. Since we want G to be complementary to F, we shall choose r' and v' as in (1). Now $|v| = 1$, so $v' = v^{-1} = \bar{v}$. Defining κ' by $e(\kappa') = v'(U^\lambda)$, $0 \leqq \kappa' < 1$, we see that

$$e(\kappa') = \bar{v}(U^\lambda) = e(-\kappa) = e(1 - \kappa);$$

hence

$$\begin{aligned} \kappa' &= 1 - \kappa, \qquad \kappa > 0, \\ \kappa' &= 0, \qquad\quad\;\; \kappa = 0. \end{aligned} \tag{7}$$

Let

$$G(\tau, -r', v', \mu) = G_\mu = \sum_M \frac{e((\mu + \kappa')M\tau/\lambda)}{v'(M)(c\tau + d)^{r'}}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{8}$$

where $r' > 2$, $\mu \geqq 0$ is an integer, and M runs over a complete set of matrices of Γ with different lower row. G_μ is an automorphic form in $\{\Gamma, -r', v'\}$ and is regular in $\mathfrak{H}$. If $\mu + \kappa' > 0$, G_μ vanishes at all parabolic cusps of Γ, in other words, G_μ is an entire cusp form. Furthermore, G_μ has the Fourier series

$$e(-\kappa'\tau/\lambda)G_\mu = 2e(\mu\tau/\lambda) + \textstyle\sum_{m+\kappa'>0} c_m\, e(m\tau/\lambda).$$

The Fourier coefficients c_m, which we write more explicitly as

$$c_m = c_m(\mu, -r', v')$$

are found in [4, p. 474]:

$$\begin{aligned} c_m(\mu, -r', v') = (2\pi/\lambda)e(-r'/4)\{ &\textstyle\sum_{c\epsilon C^+} c^{-1}W_c(m, \mu, v')M_c(m, \mu, r', \kappa') \\ &+ \textstyle\sum_{c\epsilon C^+} e(r'/2)c^{-1}W_{-c}(m, \mu, v')M_c(m, \mu, r', \kappa')\}, \end{aligned}$$

with

$$M_c(m, \mu, r, \kappa) = \left(\frac{m + \kappa}{\mu + \kappa}\right)^{(r-1)/2} J_{r-1}\left(\frac{4\pi}{c\lambda}(\mu + \kappa)^{1/2}(m + \kappa)^{1/2}\right),$$

$$c \neq 0, m + \kappa \neq 0.$$

Since, for $c > 0$,

$$W_{-c}(m, \mu, v) = e(-r'/2)W_c(m, \mu, v),$$

the above expression reduces to

$$c_m(\mu, -r', v') = 2(2\pi/\lambda)e(-r'/4)\textstyle\sum_{c\epsilon C^+} W_c(m, \mu, v')M_c(m, \mu, r', \kappa'). \tag{9}$$

Consider $c_{m-1}(\mu - 1, -r', v')$. Recalling (1) and (7) we note:

$$M_c(m-1, \mu-1, r', \kappa') = e\left(\frac{r-1}{2}\right)\left(\frac{-m+\kappa}{\mu-\kappa}\right)^{-r+1} L_c(-m, \mu, -r, \kappa)$$

and

$$W_c(m-1, \mu-1, v') = \overline{W}_c(-m, -\mu, v).$$

Since $L_c(-m, \mu, -r, \kappa)$ is real for $m, \mu \geqq 1$, we get

$$\text{(10a)} \quad c_{m-1}(\mu-1, -r', v') = 2e(r/2)\left(\frac{-m+\kappa}{\mu-\kappa}\right)^{-r+1} \bar{a}_{-m}(\mu, -r, v), \qquad \mu \geqq 1, m \geqq 1, \kappa' > 0,$$

$$\text{(10b)} \quad c_m(\mu, -r', v') = 2e(r/2)\left(\frac{-m}{\mu}\right)^{-r+1} \bar{a}_{-m}(\mu, -r, v), \qquad m > 0, \kappa' = 0.$$

By similar methods one obtains another symmetry formula involving an interchange of m and μ:

$$\text{(11a)} \quad c_{\mu-1}(m-1, -r', v') = 2e(r/2)a_{-m}(\mu, -r, v), \qquad \mu \geqq 1, m \geqq 1, \kappa' > 0,$$

$$\text{(11b)} \quad c_\mu(m, -r', v') = 2e(r/2)a_{-m}(\mu, -r, v), \qquad m > 0, \kappa' = 0.$$

We apply formulas (10). Since in F_μ we have $a_{-m} = 0$ for $m > 0, m \neq \mu$, and $a_{-\mu} = 1$, it follows that for $\kappa' > 0$,

$$\text{(12a)} \quad \begin{aligned} c_m(\mu-1, -r', v') &= 0, \qquad m > 0, m \neq \mu - 1, \mu \geqq 1, \\ c_{\mu-1}(\mu-1, -r', v') &= 2e(r/2)(-1)^{-r+1} = -2. \end{aligned}$$

For $\kappa' = 0$ we have

$$\text{(12b)} \quad \begin{aligned} c_m(\mu, -r', v') &= 0, \qquad m > 0, m \neq \mu, \\ c_\mu(\mu, -r', v') &= -2. \end{aligned}$$

Hence we have proved

THEOREM 1. *If $F_\mu \in \{\Gamma, -r, v\}, r < 0, \mu \geqq 1$, and has the expansion (2), then $G(\tau, -r', v', \mu - 1)$ is identically zero when $\kappa' > 0$, and $G(\tau, -r', v', \mu)$ is identically zero when $\kappa' = 0$. Here G belongs to $\{\Gamma, r-2, v^{-1}\}$ and is defined by* (8).

4. We present an application of the above theorem to the modular group $\Gamma(1)$. The function

$$\text{(13)} \quad \eta(\tau) = e(\tau/24) \textstyle\prod_{m=1}^{\infty} (1 - e(m\tau)), \qquad \tau \in \mathcal{H},$$

is the well-known Dedekind modular function; it belongs to $\{\Gamma(1), -\frac{1}{2}, v_0\}$, where v_0 is the classical multiplier of Dedekind and Hermite. Here $\lambda = 1, t = e(\tau)$. It is known that $v_0^{24} \equiv 1$.

Consider $\eta^{2r}(\tau) \in \{\Gamma(1), -r, v_0^{2r}\}$, with $-12 \leqq r < 0$. Its Fourier expansion is

$$\eta^{2r}(\tau) = e(r\tau/12)[1 + O(e(\tau))].$$

Setting $\kappa = r/12 - [r/12]$, we have

$$e(-\kappa\tau)\eta^{2r}(\tau) = t^{-1} + \cdots,$$

and $0 \leqq \kappa < 1$. Hence $F_1 = \eta^{2r}$ satisfies the conditions of Theorem 1, and we conclude that

$$G(\tau, r-2, v_0^{-2r}, 0) \equiv 0 \qquad \text{for } r > -12,$$

$$G(\tau, -14, 1, 1) \equiv 0.$$

Now let μ be arbitrary. Consider

$$H_k(\tau) = J^k(\tau)\eta^{2r}(\tau),$$

where $J(\tau)$ is the absolute modular invariant of Klein with Fourier series

$$J(\tau) = e(-\tau) + b_0 + b_1 e(\tau) + \cdots.$$

Since $J \in \{\Gamma(1), 0, 1\}$, H_k belongs to the same class of automorphic forms as η^{2r}, and has the Fourier series

$$e(-\kappa\tau)H_k(\tau) = t^{-k-1} + \cdots.$$

A suitable linear combination of $\{H_k, k = 0, 1, \cdots, \mu - 1\}$ will be of the form

$$e(-\kappa\tau)F_\mu(\tau) = t^{-\mu} + d_0 + d_1 t + \cdots,$$

and to this we can apply Theorem 1.

We collect these results in a theorem (cf. [3, p. 432]).

THEOREM 2. *Let* $-12 \leqq r < 0$. *Then*

$$G(\tau, r-2, v_0^{-2r}, \mu) \equiv 0, \qquad \mu = 0, 1, 2, \cdots, \quad r > -12$$

and

$$G(\tau, -14, 1, \mu) \equiv 0, \qquad \mu = 1, 2, \cdots.$$

Furthermore, there are no cusp forms except 0 *in the spaces* $\{\Gamma(1), r-2, v_0^{-2r}\}$, $0 > r > -12$, *or in* $\{\Gamma(1), -14, 1\}$.

The last statement follows from the known result that the above Poincaré series generate the space of cusp forms of the appropriate dimension and multiplier system.

5. From now on we restrict our attention to forms of positive integral dimension $-r$ $(r < 0)$ and change our point of view somewhat. Previously we had begun with a function F which was assumed to be an automorphic form, that is, we assumed $F \in \{\Gamma, -r, v\}$. Now we lift this restriction and think of $F = F_\mu$ as a Fourier series *defined* by (2), with $a_m = a_m(\mu, -r, v)$

given by (5). As before, μ is a positive integer. As was shown in [2], if $F_\mu \in \{\Gamma, -r, v\}$, then F_μ has such a Fourier expansion. The converse, however, is not true, even with the present assumption that $-r$ is an integer. That is, an F_μ defined by (2) may or may not be in $\{\Gamma, -r, v\}$.

On the other hand, it is known [1] that any function F_μ defined by (2) with a_m given by (5) satisfies the transformation equation

$$(14)\quad F_\mu(M\tau) = v(M)(c\tau + d)^r F_\mu(\tau) - v(M)(c\tau + d)^r p_M(\tau; \mu, v), \quad \tau \in \mathcal{H},$$

for each $M = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$, where $p_M(\tau;\ \mu,\ v)$ is a polynomial in τ of degree at most $-r$. Then, $F_\mu \in \{\Gamma, -r, v\}$ is equivalent to $p_M(\tau;\ \mu,\ v) \equiv 0$ for all $M \in \Gamma$. The method of [1] is based on a technique introduced by Rademacher [6].

We now define a Fourier series that we shall call the series *supplementary* to F_μ. Let $-r$ be *unchanged*, so that $-r > 0$, and let v' be defined as in (1), that is, $vv' \equiv 1$ on Γ. As before we have κ' connected with v', with κ' defined by (7). In addition, let μ' be

$$(15)\quad \begin{aligned} \mu' &= 1 - \mu, &&\text{if } \kappa > 0,\\ \mu' &= -\mu, &&\text{if } \kappa = 0. \end{aligned}$$

The *supplementary series* $\hat{F}_{\mu'}$ is the series

$$(16)\quad e(-\kappa'\tau/\lambda)\hat{F}_{\mu'}(\tau) = t^{-\mu'} + \sum_{m=0}^{\infty} \hat{a}_m t^m, \qquad t = e(\tau/\lambda),$$

where

$$\hat{a}_m = a_m(\mu',\ -r,\ v').$$

The method of [1] shows that $F_{\mu'}$ has the transformation property

$$(17)\quad \hat{F}_{\mu'}(M\tau) = v'(M)(c\tau + d)^r \hat{F}_{\mu'}(\tau) - v'(M)(c\tau + d)^r \hat{p}_M(\tau; \mu', v'), \qquad \tau \in \mathcal{H},$$

for each $M = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$, where $\hat{p}_M(\tau;\ \mu',\ v')$ is again a polynomial in τ of degree at most $-r$.

6. The cases $\kappa > 0$ and $\kappa = 0$ are somewhat different in detail, and it is convenient to treat them separately. Assume first that $\kappa > 0$. In this case the computations of [1] show that

$$(18)\quad \hat{p}_M(\tau;\ \mu',\ v') = \overline{p_M(\bar{\tau};\ \mu,\ v)} \qquad \text{for all } M \in \Gamma.$$

Furthermore a comparison of $a_m(\mu,\ -r,\ v)$ with $\hat{a}_m$ shows that

$$(19)\quad \hat{a}_m = e(-r/2)\bar{a}_{-m-1}(\mu,\ -r,\ v).$$

(Note that $L_c(m,\ 1 - \mu,\ -r,\ \kappa')$, $L_c(-m - 1,\ \mu,\ -r,\ \kappa)$ are real since $r =$ integer.) Now, by (14), $F_\mu \in \{\Gamma, -r, v\}$ if and only if $p_M(\tau; \mu, v) \equiv 0$ for all

$M \in \Gamma$. By (18) this happens if and only if $\hat{p}_M(\tau;\, \mu',\, v') \equiv 0$ for all $M \in \Gamma$. Therefore, according to (17), $F_\mu \in \{\Gamma,\, -r,\, v\}$ if and only if $\hat{F}_{\mu'} \in \{\Gamma,\, -r,\, v'\}$. But

$$\begin{aligned}\hat{F}_{\mu'}(\tau) &= e[(\kappa' - \mu')\tau/\lambda] + \textstyle\sum_{m=0}^{\infty} \hat{a}_m\, e[(m + \kappa')\tau/\lambda] \\ &= e[(\mu - \kappa)\tau/\lambda] + \textstyle\sum_{m=0}^{\infty} \hat{a}_m\, e[(m + 1 - \kappa)\tau/\lambda],\end{aligned}$$

so that $\hat{F}_{\mu'}$ is bounded at ∞. Hence, by [2; p. 274, Theorem 4], $F_\mu \in \{\Gamma, -r, v'\}$ if and only if $\hat{F}_{\mu'} \equiv 0$. If we now apply (19), we see that we have derived the following result for the case $\kappa > 0$:

THEOREM 3. *Let F_μ be the Fourier series given by* (2), *and $\hat{F}_{\mu'}$ the supplementary series defined in* (16). *Here $-r$ is a positive integer, and a_m is defined in* (5). *Then when $\kappa > 0$,*

$$F_\mu \in \{\Gamma,\, -r,\, v\} \quad \textit{if and only if} \quad \hat{F}_{\mu'} \equiv 0;$$

and when $\kappa = 0$,

$$F_\mu \in \{\Gamma\ -r,\, v\} \quad \textit{if and only if} \quad \hat{F}_{\mu'} \equiv \hat{a}_0 .$$

This condition is equivalent to

$$\begin{aligned} a_{-m}(\mu,\, -r,\, v) &= 0, \qquad\qquad m \neq \mu,\, m > 0, \\ a_{-\mu}(\mu,\, -r,\, v) &= e((-r + 1)/2). \end{aligned}$$

7. We now treat the remaining case $\kappa = 0$. In this instance the computations of [1] yield

$$(20) \quad \hat{p}_m(\tau; \mu', v') = \overline{-a_0(\mu, -r, v)}(1 - v'^{-1}(M)(c\tau + d)^{-r}) + \overline{p_M(\bar{\tau}; \mu,}$$

for all $M = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$. We also see that

$$(21) \quad \begin{aligned} \hat{a}_m &= e(-r/2)\bar{a}_{-m}(\mu,\, -r,\, v), \qquad\qquad m \geqq 1, \\ \hat{a}_0 &= -\bar{a}_0(\mu,\, -r,\, v). \end{aligned}$$

We proceed as before to find that $F_\mu \in \{\Gamma,\, -r,\, v\}$ if and only if

$$\hat{F}_{\mu'} + \overline{a_0(\mu,\, -r,\, v)} = \hat{F}_{\mu'} - \hat{a}_0 \in \{\Gamma,\, -r,\, v'\}.$$

Again, since $\hat{F}_{\mu'}$ is bounded at ∞, this occurs if and only if $\hat{F}_{\mu'} \equiv \hat{a}_0$. Thus using (21) we obtain the case $\kappa = 0$ of Theorem 3, which is now completely proved.

8. When r is an integer, we can strengthen Theorem 1.

THEOREM 4. *Let $-r$ be a positive integer, and let F_μ be defined by* (2) *and* (5) *with $\mu \geqq 1$. Then $F_\mu \in \{\Gamma,\, -r,\, v\}$ if and only if*

$$G(\tau,\, -r',\, v',\, \mu - 1) \equiv 0 \qquad \textit{when} \quad \kappa > 0$$

or

$$G(\tau,\, -r',\, v',\, \mu) \equiv 0 \qquad \textit{when} \quad \kappa = 0.$$

Let $\kappa > 0$. The condition $G_{\mu-1} \equiv 0$ is equivalent to

$$c_m(\mu - 1, -r', v') = 0 \qquad \text{for } m > 0, m \neq \mu - 1,$$

and

$$c_{\mu-1}(\mu - 1, -r', v') = -2.$$

By (10a) this implies

$$a_{-m}(\mu, -r, v) = 0 \qquad \text{for } m > 0, m \neq \mu,$$

and

$$a_{-\mu}(\mu, -r, v) = 1.$$

Hence we can apply Theorem 3 to the function $e((r - 1)/2)F_\mu$ which, we conclude, lies in $\{\Gamma, -r, v\}$. The same argument works when $\kappa = 0$, if we use (10b) in place of (10a).

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