AVERAGE ORDER OF ARITHMETIC FUNCTIONS

BY
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1. Introduction and an elementary lemma

The author has given a theorem [8] by which it is possible to find an asymptotic formula for the summatory function of the convolution of two arithmetic functions if such a formula is known for these functions. By the convolution of arithmetic functions a and b we mean

$$(a * b)(n) = \sum_{d \mid n} a(d)\, b(n/d).$$

If $A(x) = \sum_{n<x} a(n)$ and $B(x) = \sum_{n<x} b(n)$, we have used the term *Stieltjes resultant* for the function

$$C(x) = \sum_{n<x} (a * b)(n)$$

due to the fact that for almost all x

$$C(x) = \int_1^x A(x/u)\, dB(u).$$

However, the term *convolution* is just as natural, and so we have two convolutions, $*$ and $\times$, where for $x \geqq 1$

$$(A \times B)(x) = \sum_{n<x} (a * b)(n).$$

In the present paper we shall apply the theorem of [8] to some interesting arithmetic functions and then apply the following elementary lemma to some of these results and also to some known nonelementary asymptotic formulae to find estimates for sums $\sum_{n<x} a(n)/n$.

LEMMA. *Given an arithmetic function a, if for $x \geqq 1$*

$$A(x) = \sum_{n<x} a(n) = R(x) + O(x^{\alpha} L(x)),$$

where R is continuous on $[1, \infty)$, *α is real, L slowly oscillating (see below), then*

$$\sum_{n<x} a(n)/n = \int_1^x R(t) t^{-2}\, dt + R(x) x^{-1} + c + O(x^{\alpha-1} L_1(x)),$$

where $c = 0$ if $\alpha \geqq 1$,

$$c = \int_1^{\infty} t^{-2}(A(t) - R(t))\, dt$$

if $\alpha < 1$, $L_1(x) = L(x)$ if $\alpha \neq 1$, and

$$L_1(x) = \int_1^x t^{-1} L(t)\, dt$$

if $\alpha = 1$.

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A function L is said to be slowly oscillating if it is continuous and positive valued on $[x_0, \infty)$ for some x_0, and if for every $c > 0$

$$\lim_{x\to\infty} L(cx)/L(x) = 1.$$

Such a function is characterized by the form [5]

$$L(x) = \rho(x)\rho_0 \exp\left(\int_{x_0}^{x} t^{-1}\delta(t)\, dt\right),$$

where ρ and δ are continuous, $\rho_0 > 0$, ρ is positive valued, and $\rho(x) \to 1$ and $\delta(x) \to 0$ as $x \to \infty$. (x_0 will be taken as 1 in this paper.)

Note that as $x \to \infty$, $L(x)$ is asymptotic to

$$J(x) = \rho_0 \exp\left(\int_{x_0}^{x} t^{-1}\delta(t)\, dt\right),$$

where J is differentiable. Thus the use of l'Hospital's rule is justified in the following proof.

Proof of lemma. Let $E(x) = A(x) - R(x) = O(x^\alpha L(x))$. Then

$$\begin{aligned}\textstyle\sum_{n<x} a(n)/n &= \int_1^x t^{-1}\, dA(t)\\ &= A(x)/x + \int_1^x t^{-2}A(t)\, dt\\ &= R(x)/x + O(x^{\alpha-1}L(x)) + \int_1^x t^{-2}R(t)\, dt + \int_1^x t^{-2}E(t)\, dt.\end{aligned}$$

Now if $\alpha > 1$, then

$$\int_1^x t^{-2}E(t)\, dt = O\left(\int_1^x t^{\alpha-2}L(t)\, dt\right) = O(x^{\alpha-1}L(x)),$$

for one can use l'Hospital's rule to prove that

$$\int_1^x t^{\alpha-2}L(t)\, dt \sim x^{\alpha-1}L(x)/(\alpha - 1).$$

If $\alpha = 1$, then

$$\int_1^x t^{\alpha-2}L(t)\, dt = \int_1^x t^{-1}L(t)\, dt.$$

This is readily seen to be a slowly oscillating function with the aid of l'Hospital's rule; further, it can be shown that it dominates $L(x)$. If $\alpha < 1$,

$$\begin{aligned}\int_1^x t^{-2}E(t)\, dt &= \int_1^\infty t^{-2}E(t)\, dt - \int_x^\infty t^{-2}E(t)\, dt\\ &= c + O\left(\int_x^\infty t^{\alpha-2}L(t)\, dt\right)\\ &= c + O(x^{\alpha-1}L(x))\end{aligned}$$

by l'Hospital's rule. This completes the proof of the lemma.

S. A. Amitsur [2] has used the arithmetic linear transformations of K. Yamamoto [9] to find some formulae for sums $\sum_{n \leqq x} a(n)/n$. His technique involves the method of convolutions applied directly to these sums. It is interesting to note that with the aid of the above lemma we are able to get a better estimate in his formulae even in some cases where we used only the convolution method to get a formula for $\sum_{n<x} a(n)$. In fact, the theorem by which he derives his formulae can easily be derived as a special case of the theorem of [7] which is a special case of [8].

2. Statement of results

We begin with the assumption that for $x \geqq 1$,

$$M(x) = \sum_{n<x} \mu(n) = O(x^{\theta} L_0(x)), \tag{2.1}$$

and

$$D_k(x) = \sum_{n<x} d_k(n) = xP_k(\log x) + O(x^{\alpha_k} L_k(x)) \qquad (k \geqq 2), \tag{2.2}$$

where μ is the Möbius function, $d_k(n)$ is the number of ordered positive integral solutions of $x_1 x_2 \cdots x_k = n$, L_0 and L_k are slowly oscillating functions, P_k a polynomial function of degree $k - 1$ (which is known explicitly),

$$\tfrac{1}{2} \leqq \theta \leqq 1 \quad \text{and} \quad (k-1)/(2k) \leqq \alpha_k \leqq (k-1)/(k+1).$$

(See [6], Chapter 12 and [4] for estimates of α_k.) We further assume that if $\theta = 1$, then for $x \geqq 1$

$$L_0(x) = O(\exp\{-c(\log x)^{4/7}/(\log\log x)^{3/7}\}), \tag{2.3}$$

for suitable $c > 0$. (This follows by standard arguments from the information on p. 114 of [6]. See [6], p. 316 for the case $\theta = \frac{1}{2}$. Of course it is not yet known whether one can take $\theta < 1$.) Under these assumptions we shall prove the following:

$$\sum_{n<x} \mu_k(n) = x/\zeta(k) + O(x^{1/(k+1-\theta)} L_0^*(x)) \qquad (k \geqq 2), \tag{2.4}$$

$$\sum_{n<x} \mu_k(n)/n = (\log x)/\zeta(k) + c_1 + O(x^{-(k-\theta)/(k+1-\theta)} L_0^*(x)), \tag{2.5}$$

$$\sum_{n<x} 2^{\nu(n)} = xP_2^*(\log x) + O(x^{\theta_2} L_2^*(x)), \tag{2.6}$$

$$\sum_{n<x} 2^{\nu(n)}/n = P_2^{**}(\log x) + O(x^{\theta_2 - 1} L_2^*(x)), \tag{2.7}$$

$$\sum_{n<x} d(n^2) = xP_3^*(\log x) + O(x^{\theta_3} L_3^*(x)), \tag{2.8}$$

$$\sum_{n<x} d(n^2)/n = P_3^{**}(\log x) + O(x^{\theta_3 - 1} L_3^*(x)), \tag{2.9}$$

$$\sum_{n<x} d(n)^2 = xP_4^*(\log x) + O(x^{\theta_4} L_4^*(x)), \tag{2.10}$$

$$\sum_{n<x} d(n)^2/n = P_4^{**}(\log x) + O(x^{\theta_4 - 1} L_4^*(x)), \tag{2.11}$$

$$\sum_{n<x} d_k(n)/n = P_k^{\#}(\log x) + O(x^{\alpha_k - 1} L_k(x)) \qquad (k \geqq 2), \tag{2.12}$$

where μ_k is the characteristic function of the k^{th}-power-free integers (thus $\mu_2(n) = |\mu(n)|$), $\nu(n)$ is the number of distinct prime factors of n,

$d(n) = d_2(n)$ the number of divisors of n,

$$\theta_k = (1 - \theta\alpha_k)/\lambda_k, \qquad \lambda_k = 3 - \theta - 2\alpha_k, \qquad \text{if} \quad \alpha_k \leqq \tfrac{1}{2},$$

$$\theta_k = \alpha_k \qquad \text{if} \quad \alpha_k \geqq \tfrac{1}{2}.$$

The P's are polynomial functions which can be explicitly calculated by an Abelian argument (see [3]) or by (4.2) below. The L's are slowly oscillating functions satisfying

$$L_0^*(x) = (1 + o(1))L_0(x^{1/(k+1-\theta)})^{(1+o(1))/(k+1-\theta)} \quad \text{as} \quad x \to \infty,$$

$$L_k^*(x) = \{L_k(x^{(1-\theta)/\lambda_k})^{2-\theta}L_0(x^{(1-\alpha_k)/\lambda_k})^{1-2\alpha_k}\log^{(k-1)(1-2\alpha_k)}(x+1)\}^{(1+o(1))/\lambda_k}$$

$$\text{as} \quad x \to \infty \quad \text{if} \quad \alpha_k < \tfrac{1}{2} \quad (k \geqq 2),$$

$$L_k^*(x) = \int_1^x u^{-1}L_k(u)\,du \quad \text{if} \quad \alpha_k = \tfrac{1}{2} \qquad (k \geqq 2),$$

$$L_k^*(x) = L_k(x) \qquad \text{if} \quad \alpha_k > \tfrac{1}{2} \qquad (k \geqq 2).$$

Note that all the arithmetic functions a in the above formulae satisfy $a(n) = O(n^{\varepsilon})$ for each $\varepsilon > 0$ and hence although we use the sum $\sum_{n<x} a(n)$ in the text, the formulae are unchanged by replacing this sum by $\sum_{n \leqq x} a(n)$.

3. Proof of (2.4)

We observe that $\sum_{n=1}^{\infty} \mu_k(n)/n^s = \zeta(s)/\zeta(ks)$, and hence if

$$A_k(x) = \sum_{n^k<x} \mu(n), \qquad B(x) = \sum_{n<x} 1, \quad \text{and} \quad M_k(x) = \sum_{n<x} \mu_k(n),$$

then $M_k = A_k \times B$. Thus we apply [8] to the formulae

$$A_k(x) = O(x^{\theta/k}L_0(x^{1/k})), \tag{3.1}$$

$$B(x) = x + O(1), \tag{3.2}$$

$$V_{A_k}(x) = O(x^{1/k}), \qquad V_B(x) = O(x). \tag{3.3}$$

Here $V_A(x)$ denotes the total variation of the function A over the interval $[1, x]$.

Formulae (5) and (6) of [8] applied to A^k and B give us

$$M_k(x) = \int_1^x M((x/u)^{1/k})\,du + O(x^{\theta/k}L_0(x^{1/k})) + O(z^{\theta/k}yL_0(z^{1/k})) + O(z^{1/k})$$

uniformly for $1 \leqq y \leqq x$, $z = x/y$. The main term is

$$\int_1^x M((x/u)^{1/k})\,du = x\int_1^x u^{-2}M(u^{1/k})\,du$$

$$= x\int_1^{\infty} u^{-2}M(u^{1/k})\,du + O\left(x\int_x^{\infty} u^{(\theta/k)-2}L_0(u^{1/k})\,du\right)$$

$$= x/\zeta(k) + O(x^{\theta/k}L_0(x^{1/k}))$$

by l'Hospital's rule. We choose, with $\eta = k + 1 - \theta$,

$$z = x^{k/\eta} L_0(x^{1/\eta})^{k/\eta} = x^{k/\eta} L^*(x)^k,$$

and the error term becomes

$$O\{x^{1/\eta} L_0(x^{1/\eta} L^*(x)) L^*(x)^{\theta - k}\} + O(x^{1/\eta} L^*(x)) = O(x^{1/\eta} L_0^*(x)),$$

where

$$L_0^*(x) = \max\{L^*(x),\ L_0(x^{1/\eta} L^*(x)) L^*(x)^{\theta-k}\} = (1 + o(1)) L_0(x^{1/\eta})^{(1+o(1))/\eta}$$

as $x \to \infty$. The term $O(x^{\theta/k} L_0(x^{1/k}))$ is neglected, for if $\theta < 1$, then

$$\theta/k < 1/(k + 1 - \theta),$$

and if $\theta = 1$, then $L_0(x) \leqq 1$ for large x, and so

$$L_0(x^{1/k}) \leqq L_0(x^{1/k})^{1/k} = L^*(x).$$

Thus we have (2.4):

$$\textstyle\sum_{n<x} \mu_k(n) = x/\zeta(k) + O(x^{1/\eta} L_0^*(x)).$$

A simple application of our lemma now yields (2.5).

4. Proof of (2.6)–(2.11)

If k is an integer $\geqq 2$, set

$$M^{(2)}(x) = M(x^{1/2}) = \textstyle\sum_{n^2<x} \mu(n),$$

$$C_k(x) = (D_k \times M^{(2)})(x) = \textstyle\sum_{mn^2<x} d_k(m)\mu(n).$$

It is easily shown that (see [6], Chapter 1)

$$C_2(x) = \textstyle\sum_{n<x} 2^{\nu(n)}, \qquad C_3(x) = \sum_{n<x} d(n^2),$$

and

$$C_4(x) = \textstyle\sum_{n<x} d(n)^2.$$

Thus we can handle formulae (2.6)–(2.11) in one proof.

With the aid of (2.1), (2.2), and the estimates

$$V_{D_k}(x) = O(x \log^{k-1}(x + 1)), \qquad V_{M^{(2)}}(x) = O(x^{1/2}),$$

the theorem of [8] gives, for $\alpha_k < \frac{1}{2}$, $1 \leqq y \leqq x$, $z = x/y$,

$$\begin{aligned} C_k(x) &= T_k(x) + O(x^{\alpha_k} L_k(x)) + O(x^{\theta/2} L_0(x^{1/2})) \\ &\quad + O(z^{\alpha_k} y^{1/2} L_k(z)) + O(z y^{\theta/2} L_0(y^{1/2}) \log^{k-1}(z + 1)), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} T_k(x) &= \int_1^x M((x/u)^{1/2})\, d(uP_k(\log u)) \\ &= xP_k^*(\log x) + O(x^{\theta/2} L_0(x^{1/2}) \log^{k-1}(x + 1)). \end{aligned} \tag{4.2}$$

Since

$$L_0(x^{1/2}) \log^{k-1}(x + 1) = o(1)$$

if $\theta = 1$, and

$$\theta/2 < (1 - \theta\alpha_k)/(3 - \theta - 2\alpha_k)$$

if $\theta < 1$ ($\alpha_k \leqq \frac{1}{2}$), the above error term is dominated by that found below. The substitution in (4.1) of

$$z = \frac{x^{(1-\theta)/\lambda_k} L_k(x^{(1-\theta)/\lambda_k})^{2/\lambda_k}}{L_0(x^{(1-\alpha_k)/\lambda_k})^{2/\lambda_k} \log^{2(k-1)/\lambda_k}(x+1)}$$

leads to the error term

$$O(x^{(1-\theta\alpha_k)/\lambda_k} L_k^*(x)) \tag{4.3}$$

with λ_k and L_k^* as in Section 2 ($\alpha_k < \frac{1}{2}$).

If $\alpha_k = \frac{1}{2}$, then [8] gives an error term

$$O(x^{\theta/2} L_0(x^{1/2})) + O\left(x^{1/2} \int_1^x u^{-1} L_k(u)\, du\right) = O\left(x^{1/2} \int_1^x u^{-1} L_k(u)\, du\right). \tag{4.4}$$

If $\alpha_k > \frac{1}{2}$, the error is

$$O(x^{\alpha_k} L_k(x)). \tag{4.5}$$

After applying the lemma to these results, we have formulae (2.6)–(2.11). (2.12) is an immediate consequence of the lemma applied to (2.2).

One can easily show that θ_k is a nondecreasing function of α_k and of θ, and thus improvements on α_k and on θ will yield improvements on θ_k. However, since $(1 - \theta\alpha_k)/(3 - \theta - 2\alpha_k) = \frac{1}{2}$ if $\theta = 1$, no improvement on α_k beyond $\alpha_k \leqq \frac{1}{2}$ will improve θ_k by this method until more is known on the Riemann conjecture. Thus at present the best value for θ_k given by this method is $\frac{1}{2}$. Since α_2, α_3, and α_4 can be taken $\leqq \frac{1}{2}$, we have $\theta_2 = \theta_3 = \theta_4 = \frac{1}{2}$. (Hua [4] has a list of the best values of α_k known to date.) Furthermore, if $\alpha_k < \frac{1}{2}$ ($\theta = 1$), then L_k^* is independent of L_k, and so improvements on L_k are of no help unless $\alpha_k \geqq \frac{1}{2}$. However, if L_0 is given by (2.3), then improvements on α_k beyond $\frac{1}{2}$ will improve L_k^*.

If we recall that the best conceivable values for α_k and θ are $(k - 1)/(2k)$ and $\frac{1}{2}$, respectively, then it appears that the best value for θ_k given by this method would be

$$\theta_k = (3k + 1)/(6k + 4).$$

(See [6], p. 273.) It would be interesting to know whether this is indeed a lower bound on the possible values of θ_k.

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