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REMARKS ON HNN EXTENSIONS IN OPERATOR ALGEBRAS

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Dedicated to Mariko, Rio, and Mimi

Abstract. It is shown that any HNN extension is precisely a compression by a projection of a certain amalgamated free product in the framework of operator algebras. As its applications several questions for von Neumann algebras or C^*-algebras arising as HNN extensions are considered.

1. Introduction

In [19], we introduced the notion of reduced HNN extensions in the framework of von Neumann algebras as well as that of C^*-algebras, which naturally includes the group von Neumann algebras or the reduced group C^*-algebras associated with HNN extensions of groups. This paper is its continuation and provides some improvements and several new results.

First, with a minor change made in the previous construction in [19], we see that any reduced HNN extension is precisely a compressed algebra of a certain reduced amalgamated free product. It is also pointed out that the same fact still holds true for universal HNN extensions of C^*-algebras. The observation is new even for the group von Neumann algebras and the (both reduced and universal) group C^*-algebras associated with HNN extensions of groups, and indeed it seems that there is no explicit counterpart in the framework of group theory. However, a similar observation was already pointed out by Gaboriau [7] (also Paulin [11]) for equivalence relations. Indeed, we missed it when we did [19], and comparing it with our construction of reduced HNN extensions

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is a starting point of this paper. Based on the observation, we obtain several results on HNN extensions of von Neumann algebras or those of C^*-algebras. We first improve the previous factoriality and type-classification results in [19], which lead to a satisfactory answer to the questions of factoriality and type classification of HNN extensions of von Neumann algebras, say $N\bigstar_D\theta$, when both D and $\theta(D)$ are (not necessarily inner conjugate) Cartan subalgebras in N. Here, we note that the inner conjugate case was already treated in [19, Remark 3.7(1)] based on its particularity, and one should remind that all Cartan subalgebras in a fixed von Neumann algebra are isomorphic, which allows to take an HNN extension by a bijective $*$-homomorphism between those. We also consider the questions of simplicity and K-theory of (reduced or universal) HNN extensions of C^*-algebras. Some of the consequences here for C^*-algebras should be read as improvements of previous arguments for the group C^*-algebras associated with HNN extensions of groups, but some others are new. We also give some supplements to the recent work [4] on free entropy dimension due to Brown, Dykema, and Jung.

2. Preliminaries

Throughout this paper, we follow the notational conventions in [19], which are summarized here for the reader's convenience.

2.1. von Neumann algebra setup. Let $N \supseteq D$ be σ-finite von Neumann algebras and $\theta : D \to N$ be an injective normal unital $*$-homomorphism. Assume that there are faithful normal conditional expectations $E_D^N : N \to D$, $E_{\theta(D)}^N : N \to \theta(D)$. The *HNN extension* of *base algebra* N by θ with respect to E_D^N, $E_{\theta(D)}^N$ is a unique triple $(M, E_N^M : M \to N, u(\theta))$ of a von Neumann algebra containing N, a faithful normal conditional expectation and a unitary in M (called the *stable unitary*) satisfying the following conditions:

(A) $u(\theta)\theta(d)u(\theta)^* = d$ for every $d \in D$;
(M) $E_N^M(w) = 0$ for every reduced word w in N and $u(\theta)$.

Here, a given word $w = u(\theta)^{\varepsilon_0} n_1 u(\theta)^{\varepsilon_1} n_2 \cdots n_\ell u(\theta_\ell)^{\varepsilon_\ell}$ in N and $u(\theta)$ (with $n_1, \dots, n_\ell \in N$, $\varepsilon_0, \dots, \varepsilon_\ell \in \{1, -1\}$) is said to be reduced (or of reduced form) if $\varepsilon_{j-1} \neq \varepsilon_j$ implies that

$$n_j \in N_\theta^\circ \overset{\text{def}}{:=} \operatorname{Ker} E_{\theta(D)}^N \quad \text{when } \varepsilon_{j-1} = 1, \varepsilon_j = -1;$$
$$n_j \in N^\circ \overset{\text{def}}{:=} \operatorname{Ker} E_D^N \quad \text{when } \varepsilon_{j-1} = -1, \varepsilon_j = 1.$$

We write the triple in the following way:

$$(M, E_N^M, u(\theta)) := (N, E_D^N) \underset{D}{\bigstar} \big(\theta, E_{\theta(D)}^N\big).$$

Let ψ be a faithful normal semifinite weight on D. Then the modular automorphism $\sigma_t^{\psi\circ E_D^N\circ E_N^M}$ $(t\in\mathbf{R})$ is completely determined by

$$\sigma_t^{\psi\circ E_D^N\circ E_N^M}(u(\theta))=u(\theta)\big[D\psi\circ\theta^{-1}\circ E_{\theta(D)}^N : D\psi\circ E_D^N\big]_t \tag{1}$$

(see [19, Theorem 4.1]). In particular, if N has a faithful normal trace τ satisfying that (i) both E_D^N and $E_{\theta(D)}^N$ are τ-preserving and (ii) $\tau|_{\theta(D)}\circ\theta=\tau|_D$, then the new positive functional or weight $\tau\circ E_N^M$ $(=\tau|_D\circ E_D^N\circ E_N^M=\tau|_{\theta(D)}\circ E_{\theta(D)}^N\circ E_N^M)$ becomes again a trace on M. Let $\widetilde{M}=M\rtimes_{\sigma^{\psi\circ E_D^N\circ E_N^M}}\mathbf{R}\supseteq\widetilde{N}=N\rtimes_{\sigma^{\psi\circ E_D^N}}\mathbf{R}\supseteq\widetilde{D}=D\rtimes_{\sigma^\psi}\mathbf{R}$ be the inclusions of (continuous) *cores* with common canonical generators $\lambda(t)$ $(t\in\mathbf{R})$, and then the *canonical liftings* $\widehat{E}_N^M:\widetilde{M}\to\widetilde{N}$, $\widehat{E}_D^N:\widetilde{N}\to\widetilde{D}$ are provided in such a way that $\widehat{E}_N^M|_M=E_N^M$ and $\widehat{E}_D^N|_N=E_D^N$. Also, let $\widetilde{\theta}:\widetilde{D}\to\widetilde{N}$ be the *canonical extension* of θ defined by $\widetilde{\theta}|_D=\theta$ and $\widetilde{\theta}(\lambda(t))=[\psi\circ\theta^{-1}\circ E_{\theta(D)}^N:\psi\circ E_D^N]_t\lambda(t)$ for $t\in\mathbf{R}$, and hence $\widetilde{\theta}(\widetilde{D})=\widetilde{\theta(D)}:=\theta(D)\rtimes_{\sigma^{\psi\circ\theta^{-1}}}\mathbf{R}$ so that we have the canonical lifting $\widehat{E}_{\theta(D)}^N:\widetilde{N}\to\widetilde{\theta}(\widetilde{D})$ too as before. Then $(\widetilde{M},\widehat{E}_N^M,u(\theta))$ is naturally identified with the HNN extension $(\widetilde{N},\widehat{E}_D^N)\bigstar_{\widetilde{D}}(\tilde{\theta},\widehat{E}_{\theta(D)}^N)$ (see [19, Section 4] for details).

2.2. C^*-algebra setup. Let $B\supseteq C$ be a unital inclusion of C^*-algebras, $\theta:C\to B$ be an injective unital $*$-homomorphism, and $E_C^B:B\to C$, $E_{\theta(C)}^B:B\to\theta(C)$ be conditional expectations. Assume, as a natural and usual requirement, that E_C^B and $E_{\theta(C)}^B$ are nondegenerate (or equivalently have the faithful GNS representations), which ensures that B is embedded in the reduced HNN extension faithfully. The *reduced HNN extension* of base algebra B by θ with respect to E_C^B, $E_{\theta(C)}^B$ is constructed and defined as a triple $(A,E_B^A:A\to B,u(\theta))$ in the exactly same manner as in the von Neumann algebra case, and it is indeed characterized by the same conditions (A), (M) under the additional assumption that E_B^A are nondegenerate (see [19, Section 7.2] for the details). (An important issue about the characterization [19, Proposition 7.1] will be discussed in Remark 3.2.) In the C^*-algebra setup, another kind of HNN extension is available, and it is the *universal HNN extension* $B\bigstar_C^{\mathrm{univ}}\theta$, i.e., the universal C^*-algebra generated by B and a single unitary $u(\theta)$ with subject to only the algebraic relations $u(\theta)\theta(c)u(\theta)^*=c$ for all $c\in C$.

3. Observation

Let $(M,E_N^M,u(\theta))=(N,E_D^N)\bigstar_D(\theta,E_{\theta(D)}^N)$ be an HNN extension of von Neumann algebras, and

$$(\mathcal{M},\mathcal{E}):=\big(N\otimes M_2(\mathbf{C}),E_\theta:\iota_\theta\big)\underset{D\oplus D}{\bigstar}\big(D\otimes M_2(\mathbf{C}),E_1:\iota_1\big)$$

be the amalgamated free product von Neumann algebra over $D\oplus D$ via the distinguished embedding maps

$$\iota_\theta(d_1\oplus d_2):=\begin{bmatrix} d_1 & 0 \\ 0 & \theta(d_2)\end{bmatrix}, \qquad \iota_1(d_1\oplus d_2):=\begin{bmatrix} d_1 & 0 \\ 0 & d_2\end{bmatrix}$$

with respect to the conditional expectations

$$E_\theta:=\begin{bmatrix} E_D^N & 0 \\ 0 & E_{\theta(D)}^N\end{bmatrix}, \qquad E_1:=\begin{bmatrix} \mathrm{Id}_D & 0 \\ 0 & \mathrm{Id}_D\end{bmatrix}.$$

Then we denote by λ, λ_θ, λ_1 the canonical embedding maps of $D\oplus D$, $N\otimes M_2(\mathbf{C})$, $D\otimes M_2(\mathbf{C})$ into $\mathcal{M}$, respectively, which satisfy $\lambda=\lambda_\theta\circ\iota_\theta=\lambda_1\circ\iota_1$ (see [19, Section 2] for the construction and terminologies). Let us denote by $\mathcal{E}_\theta$ the conditional expectation from $\mathcal{M}$ onto $\lambda_\theta(N\otimes M_2(\mathbf{C}))$ that satisfies $\mathcal{E}\circ\mathcal{E}_\theta=\mathcal{E}$.

PROPOSITION 3.1. *There is a bijective $*$-homomorphism $\Phi:\mathcal{M}\to M\otimes M_2(\mathbf{C})$ such that $\Phi(\lambda_\theta(N\otimes M_2(\mathbf{C})))=N\otimes M_2(\mathbf{C})\subseteq M\otimes M_2(\mathbf{C})$, and moreover*

$$\Phi\circ\mathcal{E}_\theta=(E_N^M\otimes\mathrm{Id})\circ\Phi. \tag{2}$$

The above bijective $*$-homomorphism Φ is precisely given by

$$\Phi:\begin{cases}\lambda_1\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\lambda_\theta\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)\longmapsto\begin{bmatrix}u(\theta)&0\\0&0\end{bmatrix},\\ \lambda_\theta\left(\begin{bmatrix}n&0\\0&0\end{bmatrix}\right)\longmapsto\begin{bmatrix}n&0\\0&0\end{bmatrix},\\ \lambda_\theta\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\longmapsto\begin{bmatrix}0&1\\0&0\end{bmatrix}.\end{cases} \tag{3}$$

Proof. Let us first recall (and improve) the construction of reduced HNN extensions given in [19]. Let $(\mathcal{M},\,\mathcal{E})$ be as above, and the HNN extension $(M,E_N^M,u(\theta))$ is realized in the compressed algebra $p\mathcal{M}p$ with $p:=\lambda(1\oplus 0)$ as follows. (Note that another algebra slightly larger than this $\mathcal{M}$ was used in [19], but it is clear that $\mathcal{M}$ is sufficiently large to construct the desired algebra.) Identify $n\in N$ with $\lambda_\theta\left(\begin{bmatrix}n&0\\0&0\end{bmatrix}\right)$ and set $u(\theta):=\lambda_1\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\lambda_\theta\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)$. Then the desired algebra M is generated by N and $u(\theta)$ inside $p\mathcal{M}p$, and the conditional expectation E_N^M is obtained as the restriction of $\mathcal{E}_\theta$ to M.

Let $\Phi:\mathcal{M}\to p\mathcal{M}p\otimes M_2(\mathbf{C})$ be the bijective normal $*$-isomorphism determined by the 2×2 matrix unit system

$$p=\lambda_\theta\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right), \qquad \lambda_\theta\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right), \qquad \lambda_\theta\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right), \qquad \lambda_\theta\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right).$$

Then we get

$$\Phi\left(u(\theta)\right)=\begin{bmatrix}u(\theta)&0\\0&0\end{bmatrix} \quad\text{and}$$

$$\Phi\left(\lambda_\theta\left(\begin{bmatrix}n&0\\0&0\end{bmatrix}\right)\right)=\begin{bmatrix}n&0\\0&0\end{bmatrix}; \qquad \Phi\left(\lambda_\theta\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\right)=\begin{bmatrix}0&1\\0&0\end{bmatrix},$$

where the right-hand sides are considered in $M_2(p\mathcal{M}p) = p\mathcal{M}p \otimes M_2(\mathbf{C})$. This implies that Φ sends $M \vee \{\lambda_\theta(\left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right])\}''$ to $M_2(M) = M \otimes M_2(\mathbf{C})$. Note that

$$u(\theta) = \lambda_1\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)\lambda_\theta\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right), \qquad \lambda_\theta\left(\begin{bmatrix}n & 0\\ 0 & 0\end{bmatrix}\right), \qquad \lambda_\theta\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)$$

(considered in $\mathcal{M}$) generate the whole $\mathcal{M}$ since

$$u(\theta)\lambda_\theta\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) = \lambda_1\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)\lambda_\theta\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)\lambda_\theta\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) = \lambda_1\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)$$

inside $\mathcal{M}$. Since $\Phi(\mathcal{M}) = p\mathcal{M}p \otimes M_2(\mathbf{C})$, we conclude that $p\mathcal{M}p \otimes M_2(\mathbf{C}) = M \otimes M_2(\mathbf{C})$ and $M = p\mathcal{M}p$. The equality (2) is easily verified. □

Remark that the above argument clearly works well even in the general case where the θ is replaced by a family of injective normal unital $*$-homomorphisms from D into N, where the 2×2 matrix algebra in the both free components should be replaced by $B(\ell^2(\Theta_1))$ with $\Theta_1 := \{\mathrm{Id}_D\} \sqcup \Theta$.

The same observation as Proposition 3.1 clearly holds true even in the reduced C^*-algebra setup, and we call it "the C^*-version of Proposition 3.1" in what follows. We also remark that it still holds true even when the θ in a reduced HNN extension $(A, E^A_B) = (B, E^B_C)\bigstar_C(\theta, E^B_{\theta(C)})$ is replaced by a family Θ of injective unital $*$-homomorphisms from C into B. However, when Θ is an infinite family, one has to replace the larger amalgamated free product C^*-algebra $\mathcal{A}$ constructed in the same way as in the von Neumann algebra setup by the C^*-subalgebra generated by $\lambda_\Theta(B \otimes \mathbb{K}(\ell^2(\Theta_1)))$ and $\lambda_1(C \otimes \mathbb{K}(\ell^2(\Theta_1)))$ with the notations in [19, Section 7], where $\mathbb{K}(\mathcal{H})$ denotes the algebra of all compact operators on a Hilbert space $\mathcal{H}$. The proof of Proposition 3.1 still works without any change when Θ is a finite family. The case when Θ is infinite needs to pass through the inductive limit by finite subfamilies $\Xi \nearrow \Theta$ with the aid of [3, Theorem 1.3].

Remark 3.2. There is an insufficient point related to the characterization of reduced HNN extensions ([19, Proposition 7.1]); in fact, we did not prove that the reduced HNN extensions constructed in [19] actually satisfy the condition (ii) (the nondegeneracy condition) there. Of course, this is not an issue in several cases including reduced group C^*-algebras associated with HNN extensions of groups. However, it is certainly necessary to prove it for the justification of our definition. One easy way to do so is provided by the C^*-version of Proposition 3.1 as follows. Let $(A, E^A_B) = (B, E^B_C)\bigstar_C(\theta, E^B_{\theta(C)})$ be a reduced HNN extension, and $(\mathcal{A}, \mathcal{E})$ be the larger reduced amalgamated free product and $\mathcal{E}_\theta$ the unique conditional expectation from $\mathcal{A}$ onto the first free component with $\mathcal{E} \circ \mathcal{E}_\theta = \mathcal{E}$, both appeared in the C^*-version of Proposition 3.1. Notice that the proof of Proposition 3.1 was done based on the construction. Thus, the C^*-version of Proposition 3.1 shows, in particular, that $E^A_B \otimes \mathrm{Id} : A \otimes M_2(\mathbf{C}) \to B \otimes M_2(\mathbf{C})$ is nondegenerate (since so is $\mathcal{E}_\theta$ by the amalgamated free product construction), which immediately implies that so is E_B^A. Note that we used in [19] a reduced amalgamated free product larger than the above $\mathcal{A}$ to construct the reduced HNN extension A, and thus it is necessary to prove that this A is the same as that constructed there without the use of [19, Proposition 7.1]. However, this is clearly true because $\mathcal{A}$ is naturally embedded into the larger one faithfully thanks to [3, Theorem 1.3].

Let $B \supseteq C$ be a unital inclusion of C^*-algebras with an injective unital $*$-homomorphism $\theta : C \to B$ as above, and $A = B\bigstar_C^{\text{univ}}\theta$ be the universal HNN extension of C^*-algebras. Let

$$\mathcal{A} := \big(B \otimes M_2(\mathbf{C}) : \iota_\theta\big) \underset{C\oplus C}{\bigstar^{\text{univ}}} \big(C \otimes M_2(\mathbf{C}) : \iota_1\big)$$

be the universal amalgamated free product C^*-algebra over $C \oplus C$ via the distinguished embedding maps

$$\iota_\theta(c_1 \oplus c_2) := \begin{bmatrix} c_1 & 0 \\ 0 & \theta(c_2) \end{bmatrix}, \qquad \iota_1(c_1 \oplus c_2) := \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

Let us denote by j, j_θ and j_1 the canonical embedding maps of $C \oplus C$, $B \otimes M_2(\mathbf{C})$ and $C \otimes M_2(\mathbf{C})$ into $\mathcal{A}$, respectively, which satisfy $j = j_\theta \circ \iota_\theta = j_1 \circ \iota_1$.

Proposition 3.3. *There is a unique bijective $*$-homomorphism $\Phi : \mathcal{A} \to A \otimes M_2(\mathbf{C})$ determined by the same correspondence among generators as* (3).

Proof. Let us first define two $*$-homomorphisms $\Phi_\theta : B \otimes M_2(\mathbf{C}) \to A \otimes M_2(\mathbf{C})$, $\Phi_1 : C \otimes M_2(\mathbf{C}) \to A \otimes M_2(\mathbf{C})$ by

$$\Phi_\theta\left(j_\theta\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)\right) := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix};$$

$$\Phi_1\left(j_1\left(\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right)\right) := \begin{bmatrix} c_{11} & c_{12}u(\theta) \\ u(\theta)^* c_{21} & \theta\left(c_{22}\right) \end{bmatrix}$$
$$\left(= \begin{bmatrix} 1 & 0 \\ 0 & u(\theta)^* \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u(\theta) \end{bmatrix} \right).$$

Then we have

$$\Phi_\theta\big(j_\theta \circ \iota_\theta((c_1, c_2))\big) = \Phi_\theta\left(j_\theta\left(\begin{bmatrix} c_{11} & 0 \\ 0 & \theta\left(c_{22}\right) \end{bmatrix}\right)\right) = \begin{bmatrix} c_{11} & 0 \\ 0 & \theta\left(c_{22}\right) \end{bmatrix},$$
$$\Phi_1\big(j_1 \circ \iota_1((c_1, c_2))\big) = \Phi_1\left(j_1\left(\begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix}\right)\right) = \begin{bmatrix} c_{11} & 0 \\ 0 & \theta\left(c_{22}\right) \end{bmatrix}.$$

Thus, the universality of $\mathcal{A} = (B \otimes M_2(\mathbf{C}) : \iota_\theta) \underset{C\oplus C}{\bigstar} (C \otimes M_2(\mathbf{C}) : \iota_1)$ ensures that there is a unique unital $*$-homomorphism $\Phi := \Phi_\theta \bigstar \Phi_1 : \mathcal{A} \to A \otimes M_2(\mathbf{C})$ extending both Φ_θ and Φ_1. Since Φ agrees with the required correspondence among generators, it remains only to show that Φ is bijective. To do so, we will construct the inverse of Φ in what follows. By the universality of

$A = B\bigstar_C^{\mathrm{univ}}\theta$, we can construct the unital $*$-homomorphism $\Psi_0 : A \to p\mathcal{A}p$ with $p := j(1\oplus 0)$ in such a way that

$$\Psi_0(b) := j_\theta\left(\begin{bmatrix} b & 0 \\ 0 & 0\end{bmatrix}\right), \qquad \Psi_0(u(\theta)) := j_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\right) j_\theta\left(\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\right)$$

since $j_\theta\big(\begin{bmatrix} c & 0 \\ 0 & 0\end{bmatrix}\big)$ coincides with

$$j_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\right) j_\theta\left(\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\right) j_\theta\left(\begin{bmatrix} \theta(c) & 0 \\ 0 & 0\end{bmatrix}\right) j_\theta\left(\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\right) j_1\left(\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\right)$$

for all $c \in C$. Consider the following two 2×2 matrix unit systems inside $\mathbf{C}1\otimes M_2(\mathbf{C}) \subseteq A\otimes M_2(\mathbf{C})$ and $\mathcal{A}$

$$e_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, \qquad e_{12} := \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}, \qquad e_{21} := \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}, \qquad e_{22} := \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix};$$

$$f_{11} := j_\theta\left(\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}\right), \qquad f_{12} := j_\theta\left(\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\right),$$

$$f_{21} := j_\theta\left(\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\right), \qquad f_{22} := j_\theta\left(\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}\right),$$

respectively, with $f_{11} = p$. Clearly, Ψ_0 is extended to a $*$-homomorphism $\Psi : A\otimes M_2(\mathbf{C}) \to \mathcal{A}$ by $\Psi(x) := \sum_{i,j=1}^2 f_{i1}\Psi_0(e_{1i}xe_{j1})f_{1j}$ for $x \in A\otimes M_2(\mathbf{C})$. Then one immediately observes that $\Psi\circ\Phi = \mathrm{id}_{\mathcal{A}}$ and $\Phi\circ\Psi = \mathrm{id}_{A\otimes M_2(\mathbf{C})}$. $\square$

The statement of Proposition 3.3 still holds true even when the θ is replaced by a family Θ of injective unital $*$-homomorphisms from C into B, but the same care as in the reduced construction setting is required. Also, it should be pointed out that the above proof says that the matrix trick we employ provides a simple way to construct universal HNN extensions of C^*-algebras.

In [7] (see also a bit earlier work due to Paulin [11]) Gaboriau introduced the notion of HNN equivalence relations and derive a formula of costs for them from that for amalgamated free product equivalence relations. It is not hard to see that any HNN equivalence relation can be regarded as a particular case of HNN extensions of von Neumann algebras via Feldman–Moore's construction [6]. In this point of view, Proposition 3.1 is nothing less than the von Neumann algebra analog of the observation due to Gaboriau [7, lines 12–26 in p. 66]. Gaboriau's observation consists of the "converse" assertion too, and we then examine its operator algebra counterpart.

Let P_1, P_2, Q be σ-finite von Neumann algebras with two embeddings $\iota_1 : Q \hookrightarrow P_1$, $\iota_2 : Q \hookrightarrow P_2$. Suppose that there are two faithful normal conditional expectations $E_1 : P_1 \to \iota_1(Q)$, $E_2 : P_2 \to \iota_2(Q)$. Then let $(P, E) := (P_1, E_1 : \iota_1)\bigstar_Q(P_2, E_2 : \iota_2)$ be the amalgamated free product von Neumann algebra. Set $N := P_1 \oplus P_2 \supseteq D := \iota_1(Q)\oplus\iota_2(Q)$, and define the bijective $*$-homomorphism $\theta : (\iota_1(x), \iota_2(y)) \in D \mapsto (\iota_1(y), \iota_2(x)) \in D$. Also, define $E_D^N = E_{\theta(D)}^N := E_1 \oplus E_2 : N \to D = \theta(D)$. Then let $(M, E_N^M, u(\theta)) =$

$(N, E_D^N)\bigstar_D(\theta, E_{\theta(D)}^N)$ be the HNN extension. Set $p := 1_{P_1} \oplus 0 \in D$, and denote by M_0 the von Neumann subalgebra generated by N and $v := pu(\theta)$ (a partial isometry with $v^*v = \theta(p) = 1-p$, $vv^* = p$). It is plain to see that $e_{11} := p$, $e_{12} := v$, $e_{21} := v^*$, $e_{22} := 1-p$ form a 2×2 matrix unit system in M_0, and moreover, that $e_{11}M_0e_{11}$ is generated by $e_{11}Ne_{11} = P_1 \oplus 0$ and $e_{12}Ne_{21} = v(0\oplus P_2)v^* = u(\theta)(0\oplus P_2)u(\theta)$ (see e.g., [20, Lemma 5.2.1]). The restriction $F := E_D^N \circ E_N^M|_{e_{11}M_0e_{11}}$ clearly gives a faithful normal conditional expectation from $e_{11}M_0e_{11}$ onto $e_{11}D = \iota_1(Q)\oplus 0$. It is trivial that the restriction of F to $e_{11}Ne_{11} = P_1\oplus 0$ is given by $E_1 \oplus 0$. Also, the characterization of HNN extensions enables us to compute

$$\begin{aligned}
F\big(u(\theta)(0\oplus x)u(\theta)^*\big) &= E_D^N \circ E_N^M\big(u(\theta)\big(0\oplus \big(E_2(x) + \big(x - E_2(x)\big)\big)\big)u(\theta)^*\big)\\
&= E_D^N \circ E_N^M\big(u(\theta)\theta\big(\iota_1(\iota_2^{-1}(E_2(x)))\oplus 0\big)u(\theta)^*\big)\\
&\quad + E_D^M \circ \underbrace{E_N^M\big(u(\theta)\big(0\oplus\big(x-E_2(x)\big)\big)u(\theta)^*\big)}_{=0}\\
&= E_D^N \circ E_N^M\big(\iota_1\circ\iota_2^{-1}(E_2(x))\oplus 0\big)\\
&= \iota_1\circ\iota_2^{-1}(E_2(x))\oplus 0
\end{aligned}$$

for $x \in P_2$. Define $\lambda : x \in Q \mapsto \iota_1(x)\oplus 0 \in e_{11}De_{11} \subseteq e_{11}Me_{11}$, $\lambda_1 : x \in P_1 \mapsto x\oplus 0 \in e_{11}Ne_{11} \subseteq e_{11}Me_{11}$, $\lambda_2 : x\in P_2 \mapsto u(\theta)(0\oplus x)u(\theta)^* \in e_{12}Ne_{21} \subseteq e_{11}Me_{11}$. Then we have

$$\begin{aligned}
&\lambda_1\circ\iota_1(x) = \iota_1(x)\oplus 0 = \lambda(x),\\
&\lambda_2\circ\iota_2(x) = u(\theta)\big(0\oplus\iota_2(x)\big)u(\theta)^* = u(\theta)\theta\big(\iota_1(x)\oplus 0\big)u(\theta)^* = \iota_1(x)\oplus 0 = \lambda(x)
\end{aligned}$$

for $x\in Q$. Since

$$\begin{aligned}
\operatorname{Ker} F \cap (P_1\oplus 0) &= \operatorname{Ker} E_1 \oplus 0 \subseteq \operatorname{Ker} E_D^N,\\
\operatorname{Ker} F\cap u(\theta)(0\oplus P_2)u(\theta)^* &= u(\theta)(0\oplus \operatorname{Ker} E_2)u(\theta)^* \subseteq u(\theta)\operatorname{Ker} E_{\theta(D)}^N u(\theta)^*,
\end{aligned}$$

one easily derives, from the condition (M) in Section 2.1, that $\lambda_1(P_1) = P_1\oplus 0$ and $\lambda_2(P_2) = u(\theta)(0\oplus P_2)u(\theta)^*$ are free with respect to F. Summarizing the discussion so far, we conclude the following.

PROPOSITION 3.4. *Let $(M, E_N^M, u(\theta)) = (N, E_D^N)\bigstar_D(\theta, E_{\theta(D)}^N)$ be the HNN extension with $N := P_1\oplus P_2 \supseteq D := \iota_1(Q)\oplus\iota_2(Q)$, $\theta : (\iota_1(x),\iota_2(y)) \in D \mapsto (\iota_1(y),\iota_2(x)) \in D$, $E_D^N = E_{\theta(D)}^N := E_1\oplus E_2$, and $p(= e_{11}) := 1_{P_1}\oplus 0 \in N$. Then, by letting M_0 be the von Neumann subalgebra of M generated by N and $v := pu(\theta)$, the compressed system $(pM_0p, F = E_D^N\circ E_N^M|_{pM_0p})$ is identified with the amalgamated free product $(P,E) = (P_1, E_1 : \iota_1)\bigstar_Q(P_2, E_2 : \iota_2)$.*

The triple consisting of M_0, the conditional expectation $E_N^{M_0} := E_N^M|_{M_0}$, and the partial isometry v can be characterized, similarly as in the case of $(M, E_D^M, u(\theta))$, by the following two conditions: (A) $v\theta(d)v^* = d$ for every

$d \in pD$; (M) $E_N^{M_0}(w) = 0$ for every nonzero word $w = v^{\varepsilon_0} n_1 v^{\varepsilon_1} n_2 \cdots n_\ell v^{\varepsilon_\ell}$ (with $n_1, \dots, n_\ell \in N$, $\varepsilon_0, \dots, \varepsilon_\ell \in \{\cdot, *\}$) satisfying that $\varepsilon_{j-1} \neq \varepsilon_j$ implies that

$$n_j \in \mathrm{Ker}\big(E^N_{\theta(D)}|_{\theta(p)N\theta(p)}\big) \quad \text{when } \varepsilon_{j-1} = \cdot, \varepsilon_j = *;$$
$$n_j \in \mathrm{Ker}(E^N_D|_{pNp}) \quad \text{when } \varepsilon_{j-1} = *, \varepsilon_j = \cdot.$$

Hence, the triple $(M_0, E_N^{M_0}, v)$ depends only on the restrictions $\theta|_{pD}$ and $E^N_D|_{pNp}, E^N_{\theta(D)}|_{\theta(p)N\theta(p)}$ so that it should be called the generalized HNN extension by the partial $*$-isomorphism $\theta|_{pD} : pD \to \theta(p)N\theta(p)$ with respect to $E^N_D|_{pNp}, E^N_{\theta(D)}|_{\theta(p)N\theta(p)}$. Here, a partial $*$-isomorphism means an injective unital $*$-homomorphism from a subalegebra, whose unit is different from a given algebra, into a compressed algebra of the given one. We should also remark that the same assertion as Proposition 3.1 still holds true for $(M_0, E_N^{M_0}, v)$. Namely, $M_0 \otimes M_2(\mathbf{C})$ can be identified with the amalgamated free product of

$$\left(\begin{bmatrix} N & N \\ N & N \end{bmatrix}, \begin{bmatrix} E^N_D & 0 \\ 0 & E^N_{\theta(D)} \end{bmatrix} : \begin{bmatrix} \mathrm{Id}_D \\ \theta \end{bmatrix} \right)$$
$$\mathop{\bigstar}_{D \oplus D} \left(\begin{bmatrix} D & pD \\ pD & D \end{bmatrix}, \begin{bmatrix} \mathrm{Id}_D & 0 \\ 0 & \mathrm{Id}_D \end{bmatrix} : \begin{bmatrix} \mathrm{Id}_D & \\ & \mathrm{Id}_D \end{bmatrix} \right)$$

in the same way as in Proposition 3.1.

The same facts as the above (including Proposition 3.4) is still valid in the C^*-algebra settings. The reduced construction setting is treated in the exactly same way, but the universal construction setting needs to use the universality similarly to Proposition 3.3. In the course of the proof, one easily observes the following fact:

FACT 3.5. *Let $B \supseteq C$ be unital C^*-algebras, $\theta : C \to B$ be an injective unital $*$-homomorphism, and p be a (nonzero) central projection in C. Write $C_0 := pC$ and $\theta_0 := \theta|_{C_0} : C_0 \to \theta(p)B\theta(p)$. Let*

$$\mathcal{A}_0 := \left(\begin{bmatrix} B & B \\ B & B \end{bmatrix} : \begin{bmatrix} \mathrm{Id}_C & \\ & \theta \end{bmatrix} \right) \mathop{\bigstar}^{\mathrm{univ}}_{C \oplus C} \left(\begin{bmatrix} C & C_0 \\ C_0 & C \end{bmatrix} : \begin{bmatrix} \mathrm{Id}_C & \\ & \mathrm{Id}_C \end{bmatrix} \right)$$

be the universal amalgamated free product C^*-algebra with the canonical embedding maps j (of the amalgamated subalgebra into $\mathcal{A}_0$), j_θ (of the first free component into $\mathcal{A}_0$), j_1 (of the second free component into $\mathcal{A}_0$). Then the C^*-subalgebra A_0 (inside the compressed algebra of $\mathcal{A}_0$ by $j(1 \oplus 0)$) generated by $j_\theta\big(\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\big)$, $b \in B$, and $v := j_1\big(\begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}\big) j_\theta\big(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\big)$ is universal with subject to the algebraic equations $v\theta_0(c)v^* = c$ for all $c \in C_0$. Moreover, $A_0 \otimes M_2(\mathbf{C})$ is identified with $\mathcal{A}_0$ in the same way as in Proposition 3.3.

Hence, the matrix trick we employ also provides the precise construction of “universal HNN extensions by partial $*$-isomorphisms” (cf. the comment after Proposition 3.3).

4. Results

4.1. Factoriality and type classification. Let $N \supseteq D$ be σ-finite von Neumann algebras with an injective normal unital $*$-homomorphism $\theta : D \to N$, and then two faithful normal conditional expectations $E_D^N : N \to D$, $E_{\theta(D)}^N : N \to \theta(D)$ are given. We introduce the assumption below.

ASSUMPTION 4.1. There are two unitaries $v_1, v_\theta \in N$ and two faithful normal states $\varphi_1, \varphi_\theta$ on D such that
(a) $E_D^N(v_1^m) = E_{\theta(D)}^N(v_\theta^m) = 0$ as long as $m \neq 0$;
(b) $v_1 \in N_{\varphi_1 \circ E_D^N}$ and $v_\theta \in N_{\varphi_\theta \circ \theta^{-1} \circ E_{\theta(D)}^N}$.

In what follows, we will use the notational rule around Proposition 3.1. Namely, $(M, E_N^M, u(\theta))$ is the HNN extension of N by θ with respect to E_D^N, $E_{\theta(D)}^N$; and $\mathcal{M}$ is the associated amalgamated free product as in Proposition 3.1, so that $\mathcal{M} \cong M \otimes M_2(\mathbf{C})$. In what follows, we use the usual notations for ultraproducts of von Neumann algebras. Namely, for a von Neumann algebra L and a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, L^ω denotes the ultraproduct of L with respect to ω. If a von Neumann subalgebra $K \subseteq L$ is the range of a faithful normal conditional expectation from L, then K^ω can be naturally regarded as a von Neumann subalgebra of L^ω. Moreover, for a bijective normal $*$-homomorphism $\alpha : L_1 \to L_2$ between two von Neumann algebras induces a unique bijective normal $*$-homomorphism $\alpha^\omega : L_1^\omega \to L_2^\omega$.

LEMMA 4.1. *Under Assumption 4.1, we have*

$$\left\{ \begin{bmatrix} v_1 & 0 \\ 0 & v_\theta \end{bmatrix}, \begin{bmatrix} 0 & u(\theta) \\ u(\theta)^* & 0 \end{bmatrix} \right\}' \cap \big(M \otimes M_2(\mathbf{C})\big)^\omega \subseteq \begin{bmatrix} D & 0 \\ 0 & \theta(D) \end{bmatrix}^\omega .$$

In particular,

$$\big(M \otimes M_2(\mathbf{C})\big)' \cap \big(M \otimes M_2(\mathbf{C})\big)^\omega = \big(M \otimes M_2(\mathbf{C})\big)' \cap \begin{bmatrix} D & 0 \\ 0 & \theta(D) \end{bmatrix}^\omega . \tag{4}$$

Proof. Via the bijective $*$-homomorphism Φ in Proposition 3.1

$$\big(M \otimes M_2(\mathbf{C}) \supseteq N \otimes M_2(\mathbf{C}), E_N^M \otimes \mathrm{Id}\big)$$

is identified with

$$\big(\mathcal{M} \supseteq \lambda_\theta\big(N \otimes M_2(\mathbf{C})\big), \mathcal{E}_\theta\big),$$

and correspondingly,

$$\begin{bmatrix} v_1 & 0 \\ 0 & v_\theta \end{bmatrix}, \begin{bmatrix} 0 & u(\theta) \\ u(\theta)^* & 0 \end{bmatrix} \quad \text{with } V := \lambda_\theta\left(\begin{bmatrix} v_1 & 0 \\ 0 & v_\theta \end{bmatrix}\right), W := \lambda_1\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right),$$

respectively. Hence, it suffices to show that

$$\{V, W\}' \cap \mathcal{M}^\omega \subseteq \lambda(D \oplus D)^\omega = \lambda_\theta\big(\iota_\theta(D \oplus D)\big)^\omega . \tag{5}$$

With letting $\psi((d_1,d_2)) := \frac{1}{2}(\varphi_1(d_1)+\varphi_\theta(d_2))$, a faithful normal state on $D\oplus D$, Assumption 4.1(b) implies that

$$\sigma_t^{\psi\circ\iota_\theta^{-1}\circ E_\theta}(V)=V$$

for $t\in\mathbf{R}$, and hence $\begin{bmatrix} v_1 & 0 \\ 0 & v_\theta\end{bmatrix}\in(N\otimes M_2(\mathbf{C}))_{\psi\circ\iota_\theta^{-1}\circ E_\theta}$. Since

$$E_\theta\left(\begin{bmatrix} v_1 & 0 \\ 0 & v_\theta\end{bmatrix}^m\right)=0 \quad (m\neq 0), \qquad E_1\left(\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\right)=0$$

by Assumption 4.1(a), we can apply [18, Proposition 5] (note here that the assumption "$uDu^*=D=wDw^*$" there is never used in the proof as remarked in [19, p. 400] so that we can apply it) to

$$(\mathcal{M},\mathcal{E})=\big(N\otimes M_2(\mathbf{C}),E_\theta:\iota_\theta\big)\mathop{\bigstar}_{D\oplus D}\big(D\otimes M_2(\mathbf{C}),E_1:\iota_1\big)$$

with V,W, and thus for all $X\in\{V\}'\cap\mathcal{M}^\omega$ we get

$$\big\|W\big(X-\mathcal{E}^\omega(X)\big)\big\|_{(\psi\circ\lambda^{-1}\circ\mathcal{E})^\omega}\le\|WX-XW\|_{(\psi\circ\lambda^{-1}\circ\mathcal{E})^\omega}.$$

This inequality immediately implies (5). □

Here is a simple and well-known lemma.

LEMMA 4.2 (E.g., [13, Lemma 2.1]). *Let $P\supseteq Q$ be von Neumann algebras and $e\in Q$ be a projection. Then $(eQe)'\cap ePe=Q'e\cap ePe=(Q'\cap P)e$.*

PROPOSITION 4.3. *Under Assumption 4.1, we have*

$$\mathcal{Z}(M)=\{x\in D\cap\theta(D)\cap N' : \theta(x)=x\}, \tag{6}$$

$$M'\cap M^\omega=\{x\in D^\omega\cap\theta^\omega(D^\omega)\cap N' : \theta(x)=x\}. \tag{7}$$

Moreover, the core $\widetilde{M}$ satisfies that

$$\mathcal{Z}(\widetilde{M})=\{x\in\widetilde{D}\cap\widetilde{\theta}(\widetilde{D})\cap\widetilde{N}' : \widetilde{\theta}(x)=x\}, \tag{8}$$

where we use the notations in Section 2.1.

Proof. Applying Lemma 4.2 to (4) in Lemma 4.1 with $e=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$ we get, respectively,

$$M'\cap M^\omega\subseteq D^\omega,\qquad M'\cap M^\omega\subseteq\theta(D)^\omega=\theta^\omega(D^\omega).$$

Then the desired assertions immediately follow since M is generated by N and $u(\theta)$, and also $u(\theta)\theta^\omega(x)u(\theta)^*=x$ for all $x\in D^\omega$. (For more details, we refer to [19, pp. 406–409].) □

In the next remark, we use the notations in Section 2.1.

REMARK 4.4. *The dual action $\{\vartheta_t^M\}_{t\in\mathbf{R}}$ on $\widetilde{M}$ is defined in such a way that $\vartheta_t^M|_M=\mathrm{Id}_M$ and $\vartheta_t^M(\lambda(s))=e^{-its}\lambda(s)$ for $s,t\in\mathbf{R}$. Then ϑ_t^M commutes with $\widetilde{\theta}$ for every $t\in\mathbf{R}$. In particular, (8) implies (6).*

Proof. The commutativity between ϑ_t^M and $\widetilde{\theta}$ is clear from their definitions. If (8) was true, then it would follow that

$$\begin{aligned}\mathcal{Z}(M) &= \mathcal{Z}(\widetilde{M})^{\vartheta^M} \\ &= \{x \in \widetilde{D} \cap \widetilde{\theta}(\widetilde{D}) \cap \widetilde{N}' : \widetilde{\theta}(x) = x, \vartheta_t^M(x) = x (t \in \mathbf{R})\} \\ &= \{x \in D \cap \theta(D) \cap N' : \theta(x) = x\}.\end{aligned}$$

Here, we use [15, Theorem XII. 1.1] for $\widetilde{M}$ and $\widetilde{D}$ twice. Note that $D \cap \theta(D) \cap N' = D \cap \theta(D) \cap \widetilde{N}'$ thanks to the fact that $\operatorname{Ad}\lambda(s)$ acts on the center $\mathcal{Z}(N)$ trivially for every $s \in \mathbf{R}$. □

We then consider and discuss a particular case; *both D and $\theta(D)$ are assumed to be Cartan subalgebras in N throughout the rest of this subsection.* Since any MASA in a von Neumann algebra contains its center, we note that both the domains of θ and $\widetilde{\theta}$ must contain $\mathcal{Z}(N)$ and $\mathcal{Z}(\widetilde{N})$, respectively.

Theorem 4.5. *If N has no type* I *direct summand, then*

$$\mathcal{Z}(M) = \{x \in \mathcal{Z}(N) : \theta(x) = x\}, \quad \mathcal{Z}(\widetilde{M}) = \{x \in \mathcal{Z}(\widetilde{N}) : \widetilde{\theta}(x) = x\}. \tag{9}$$

Moreover, if N is either of type II *or a nontype* I *factor, then*

$$M' \cap M^\omega = \{x \in D^\omega \cap \theta^\omega(D^\omega) \cap N' : \theta^\omega(x) = x\}. \tag{10}$$

Proof. Since, the core $\widetilde{N}$ is of type II (under the hypothesis of the first assertion), i.e., a direct sum of von Neumann algebras of type II_1 and type II_∞, the argument of [17, Lemma 4.2] enables us to confirm that Assumption 4.1 holds for $\widetilde{M} = \widetilde{N} \bigstar_{\widetilde{D}} \widetilde{\theta}$, and hence Proposition 4.3 with the aid of Remark 4.4 shows (9) since $D \cap \theta(D) \cap N' = \mathcal{Z}(N)$ and $\widetilde{D} \cap \widetilde{\theta}(\widetilde{D}) \cap \widetilde{N}' = \mathcal{Z}(\widetilde{N})$. The last assertion is also shown similarly by combining Proposition 4.3 with the argument of [17, Lemma 4.2]. □

Remarks 4.6. *Theorem 4.5 implies the following facts:*

1. *If N is a non-type* I *factor, then so is M thanks to the first equality in* (9).
2. *If N is a type* III_1 *factor, then so is M thanks to the second in* (9).
3. *When N is a non-type* I *factor, if M is a factor of type* III_0, *then so must be N thanks to the second in* (9).
4. *If N is a non-type* I *factor, then $M_\omega = M' \cap M^\omega \subseteq D^\omega$ thanks to* (10) *and the argument given in* [18, Theorem 8]. *Hence, M is never strongly stable, i.e., $M \ncong M \otimes R$ with the hyperfinite* II_1 *factor R.*

Proposition 4.7. *If N is a factor of type* II_1 *or of type* III_λ *with $\lambda \neq 0$, then there is a faithful normal state φ on D with $(N_{\varphi\circ E_D^N})' \cap N = \mathbf{C}1$ $(\varphi \circ E_D^N$

should be the unique tracial state in the type II_1 *case), and moreover,*

$$T(M)=\{t\in T(N):\,[D\varphi\circ\theta^{-1}\circ E^N_{\theta(D)} : D\varphi\circ E^N_D]_t=1\}. \tag{11}$$

Proof. The first part of the assertion holds true thanks to [17, Lemma 4.2]; more precisely, one can construct two faithful normal states φ and φ_θ on D in such a way that

- there are unitaries $v_1\in N_{\varphi\circ E^N_D}$, $v_\theta\in N_{\varphi_\theta\circ\theta^{-1}\circ E^N_{\theta(D)}}$ with $E^N_D(v_1^m)=E^N_{\theta(D)}(v_\theta^m)=0$ as long as $m\neq 0$;
- $(N_{\varphi\circ E^N_D})'\cap N=\mathbf{C}1$ and $(N_{\varphi_\theta\circ\theta^{-1}\circ E^N_D})'\cap N=\mathbf{C}1$.

Let us define the faithful normal conditional expectation $\mathcal{E}: M\otimes M_2(\mathbf{C})\to D\oplus\theta(D)=\left[\begin{smallmatrix}D & \\ & \theta(D)\end{smallmatrix}\right]$ and the faithful state ψ on $D\oplus\theta(D)$ by

$$\mathcal{E}\left(\begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\right):=\begin{bmatrix} E^N_D\circ E^M_N(m_{11}) & \\ & E^N_{\theta(D)}\circ E^M_N(m_{22})\end{bmatrix},$$
$$\psi\left(\begin{bmatrix} d_{11} & \\ & \theta(d_{22})\end{bmatrix}\right):=\frac{1}{2}\big(\varphi(d_{11})+\varphi_\theta(d_{22})\big).$$

Clearly, $V:=\left[\begin{smallmatrix}v_1 & 0\\ 0 & v_\theta\end{smallmatrix}\right]$ is in the centralizer $(M\otimes M_2(\mathbf{C}))_{\psi\circ\mathcal{E}}$, and the proof of Lemma 4.1 shows that all $X\in\{V\}'\cap(M\otimes M_2(\mathbf{C}))$ and $W_1,W_2\in\operatorname{Ker}\mathcal{E}$ must satisfy that

$$\big\|W_1\big(X-\mathcal{E}(X)\big)\big\|_{\psi\circ\mathcal{E}}\le\|W_1X-XW_2\|_{\psi\circ\mathcal{E}}. \tag{12}$$

Let t_0 be a real number such that $\sigma^{\psi\circ\mathcal{E}}_{t_0}=\operatorname{Ad}U$ for some unitary $U\in M\otimes M_2(\mathbf{C})$, and set $W:=\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. Since $\sigma^{\psi\circ\mathcal{E}}_t(W)\in\operatorname{Ker}\mathcal{E}$, (12) shows that

$$\big\|\sigma^{\psi\circ\mathcal{E}}_{t_0}(W)\big(U-\mathcal{E}(U)\big)\big\|_{\psi\circ\mathcal{E}}\le\|\sigma^{\psi\circ\mathcal{E}}_{t_0}(W)U-UW\|_{\psi\circ\mathcal{E}}=0.$$

Hence, $U=\mathcal{E}(U)=\left[\begin{smallmatrix}u & 0\\ 0 & u_\theta\end{smallmatrix}\right]$ with some unitaries $u\in D$, $u_\theta\in\theta(D)$. It is plain to see that

$$\sigma^{\psi\circ\mathcal{E}}_t\left(\begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\right)=\begin{bmatrix}\sigma^{\varphi_1\circ E^N_D\circ E^M_N}_t(m_{11}) & \sigma^{\varphi_1\circ E^N_D\circ E^M_N}_t(m_{12})u_t\\ u_t^*\sigma^{\varphi_1\circ E^N_D\circ E^M_N}_t(m_{21}) & u_t^*\sigma^{\varphi_1\circ E^N_D\circ E^M_N}_t(m_{22})u_t\end{bmatrix}$$

for $m_{ij}\in M$, $i,j=1,2$, and $t\in\mathbf{R}$ with letting $u_t:=[D\varphi_1\circ E^N_D : D\varphi_\theta\circ\theta^{-1}\circ E^N_{\theta(D)}]_t$. In particular, we see that $\sigma^{\varphi\circ E^M_N\circ E^M_N}_{t_0}=\operatorname{Ad}u$. Since $N_{\varphi\circ E^N_D}$ sits in $M_{\varphi\circ E^N_D\circ E^M_N}$, we have $u\in(N_{\varphi\circ E^N_D})'\cap D\subseteq(N_{\varphi\circ E^N_D})'\cap N=\mathbf{C}1$ so that $\sigma^{\varphi\circ E^N_D\circ E^M_N}_{t_0}=\mathrm{Id}$. Consequently, $t\in T(M)$ if and only if $\sigma^{\varphi\circ E^N_D\circ E^M_N}_t=\mathrm{Id}$, which is equivalent to that $t\in T(N)$ and $\sigma^{\varphi\circ E^N_D\circ E^M_N}_t(u(\theta))=u(\theta)$ since $M=\{N,u(\theta)\}''$ and $\sigma^{\varphi\circ E^N_D\circ E^M_N}_t|_N=\sigma^{\varphi\circ E^N_D}_t$. Hence, the desired assertion immediately follows thanks to (1) in Section 2.1. $\square$

When N is a type II_1 factor, the T-set $T(M)$ can be described more explicitly as follows. Let τ be the unique tracial state on N. Since $(\tau|_D)\circ E_D^N = \tau = (\tau|_{\theta(D)})\circ E_{\theta(D)}^N$ must hold, we have $[D(\tau|_D)\circ\theta^{-1}\circ E_{\theta(D)}^N : D(\tau|_D)\circ E_D^N]_t = [D(\tau|_D)\circ\theta^{-1} : D(\tau|_{\theta(D)})]_t$, and hence (11) in Proposition 4.7 is rewritten as

$$T(M) = \left\{t\in\mathbf{R} : \left[D(\tau|_D)\circ\theta^{-1} : D\big(\tau|_{\theta(D)}\big)\right]_t = 1\right\}.$$

Thus, M is of type II_1 if and only if θ "preserves the trace".

4.2. Simplicity. Here, we will give a partial answer to the question of simplicity of reduced HNN extensions of C^*-algebras. Our method is to derive from a result on the simplicity of reduced amalgamated free products of C^*-algebras due to McClanahan [10] with the aid of the C^*-version of Proposition 3.1 (see the comment after the proposition).

Let us first briefly review the above-mentioned result of McClanahan (which essentially comes from a technique due to Avitzour [2]). Let P_1, P_2, Q be unital C^*-algebras and $\eta_1 : Q\hookrightarrow P_1$, $\eta_2 : Q\hookrightarrow P_2$ be embeddings. Assume that there are two conditional expectations $F_1 : P_1\to\eta_2(Q)$, $F_2 : P_2\to\eta_2(Q)$. Let $(P,F) := (P_1, F_1 : \eta_1)\bigstar_Q(P_2, F_2 : \eta_2)$ be the reduced amalgamated free product of C^*-algebras, where the canonical maps are denoted by $\rho : Q\to P$, $\rho_1 : P_1\to P$, $\rho_2 : P_2\to P$, which satisfy that $\rho=\rho_1\circ\eta_1=\rho_2\circ\eta_2$ and $F : P\to\rho(Q)$ is a conditional expectation. Let us introduce two conditions:

1°. There are unitaries $u,v\in P_1$, $w\in P_2$ such that $u\cdot\mathrm{Ker}\,F_1\cdot u^*\subseteq\mathrm{Ker}\,F_1$, $F_1(u^*v)=0$, $w\cdot\mathrm{Ker}\,F_2\cdot w^*\subseteq\mathrm{Ker}\,F_2$;

2°. For every $x\in Q$ and every $j\in\mathbb{Z}\setminus\{0\}$, there is an increasing sequence $\{m_k\}_{k=1,2,\dots}$ of natural numbers such that

$$[\rho(x),(\rho_1(u)\rho_2(w))^{m_k}\rho_1(v)\rho_2(w)\rho_1(v)(\rho_2(w)\rho_1(u))^j]=0$$

for all $k\geq k_0$ with some $k_0\in\mathbb{N}$,

and then the subsets of P_i, $i=1,2$:

$$\mathcal{N}^{(2)}(F_i) := \{(x,y)\in P_i\times P_i : x\cdot\mathrm{Ker}\,F_i\cdot y\subseteq\mathrm{Ker}\,F_i, x\cdot\eta_i(Q)\cdot y\subseteq\eta_i(Q)\},$$

which act on P by left–right multiplication. (Note that two more kinds of subsets are used in [10] to formulate the assertion, but they are nothing less than Q and thus meaningless since Q is unital.) It is not so difficult to see that for any $(x,y)\in\mathcal{N}^{(2)}(F_i)$, one has $\rho_i(x)F(z)\rho_i(y) = F(\rho_i(x)z\rho_i(y))$ for every $z\in P$.

The next lemma is shown by a simple calculation.

LEMMA 4.8. *Assume that the unitaries u,v,w in the condition 1° satisfy that $u,v\in\eta_1(Q)'\cap P_1$ and that $w^2=1$, i.e., w is a self-adjoint unitary, and moreover $w\cdot\eta_2(Q)\cdot w=\eta_2(Q)$. Then the condition 2° automatically holds true with $m_k := 2k-j-1$, $k\geq\frac{j+2}{2}$.*

Lemma 4.8 apparently gives the following variant of [10, Proposition 3.10].

LEMMA 4.9. *Assume that there are unitaries* $u, v \in \eta_1(Q)' \cap P_1$ *and* $w = w^* \in P_2$ *such that* (i) $u \cdot \mathrm{Ker}\, F_1 \cdot u^* \subseteq \mathrm{Ker}\, F_1$, (ii) $F_1(u^* v) = 0$, (iii) $w \cdot \mathrm{Ker}\, F_2 \cdot w \subseteq \mathrm{Ker}\, F_2$, (iv) $w \cdot \eta_2(Q) \cdot w = \eta_2(Q)$, *and* (v) (Q) *has no non-trivial* C^**-ideal under the actions of* $\mathcal{N}^{(2)}(F_i)$ *via* $\eta_i, i = 1, 2$. *Then* P *must be simple.*

We are now in position to apply McClanahan's result to the case of reduced HNN extensions. In what follows, $(A, E_B^A, u(\theta)) = (B, E_C^B) \bigstar_C (\theta, E_{\theta(C)}^B)$ is a reduced HNN extension of C^*-algebras. Let $\mathcal{N}^{(2)}(E_C^B)$, $\mathcal{N}^{(2)}(E_{\theta(C)}^B)$ be defined as before, and they act on B by left–right multiplication. We apply Lemma 4.9 to the associated larger reduced amalgamated free product $(\mathcal{A}, \mathcal{E})$ appeared in the C^*-version of Proposition 3.1 with letting $Q := C \oplus C$, $P_1 := B \otimes M_2(\mathbf{C})$, $P_2 := C \otimes M_2(\mathbf{C})$, $P := \mathcal{A}$, etc., and then get the following proposition.

PROPOSITION 4.10. *Assume that there are unitaries* $a \in C' \cap B$, $b \in \theta(C)' \cap B$ *such that* $E_C^B(a) = E_{\theta(C)}^B(b) = 0$*, and moreover either* $a \cdot \mathrm{Ker}\, E_C^B \cdot a^* \subseteq \mathrm{Ker}\, E_C^B$*; or* $b \cdot \mathrm{Ker}\, E_{\theta(C)}^B \cdot b^* \subseteq \mathrm{Ker}\, E_{\theta(C)}^B$ *holds. If* C *has no* C^**-ideal invariant under the actions of* $\mathcal{N}^{(2)}(E_C^B)$, $\mathcal{N}^{(2)}(E_{\theta(C)}^B)$ (*by left–right multiplication and via* Θ *for the latter*), *then* A *must be simple.*

Proof. Since $\mathcal{A} \cong A \otimes M_2(\mathbf{C})$ by the C^*-version of Proposition 3.1, it suffices to show that $\mathcal{A}$ is simple. We use Lemma 4.9, and thus, need to specify the unitaries u, v, w there in this setting. By symmetry, we may and do assume that $E_C^B(a) = E_{\theta(C)}^B(b) = 0$ and $a \cdot \mathrm{Ker}\, E_C^B \cdot a^* \subseteq \mathrm{Ker}\, E_C^B$. Then it is clear that the unitaries

$$u := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \qquad v := \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}, \qquad w := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

satisfy the first four conditions in Lemma 4.9. Note that $(w, w) \in \mathcal{N}^2(F_2)$, and it is clear that any C^*-ideal in $Q = C \oplus C$ invariant under $\mathrm{Ad}\, w$ (via $\eta_1 = \iota_\theta$) must be of the form $C_0 \oplus C_0$ with C^*-ideal $C_0 \lhd C$. Note also that $\mathcal{N}^{(2)}(E_C^B) \times \mathcal{N}^{(2)}(E_{\theta(C)}^B)$ are embedded into $\mathcal{N}^{(2)}(F_1)$ by

$$\begin{aligned} &((x_1, y_1), (x_2, y_2)) \in \mathcal{N}^{(2)}(E_C^B) \times \mathcal{N}^{(2)}\big(E_{\theta(C)}^B\big) \\ &\quad \mapsto \quad \left(\begin{bmatrix} x_1 & \\ & x_2 \end{bmatrix}, \begin{bmatrix} y_1 & \\ & y_2 \end{bmatrix} \right) \in \mathcal{N}^{(2)}(F_1), \end{aligned}$$

respectively. Therefore, one easily observes that any C^*-ideal in $C \oplus C$ (considered inside P_1 via the embedding) invariant under the action of $\mathcal{N}^{(2)}(F_1)$ must be of the form $C_0 \oplus C_0$ with a C^*-ideal $C_0 \lhd C$ invariant under both the actions of $\mathcal{N}^{(2)}(E_C^B), \mathcal{N}^{(2)}(E_{\theta(C)}^B)$. By the assumption here, there is no such nontrivial C^*-ideal $C_0 \lhd C$, and hence $\mathcal{A}$ is simple by Lemma 4.9. $\square$

Following [5], we say a (discrete) group to be C^*-simple if its reduced group C^*-algebra is simple. The next corollary immediately follows from Proposition 4.10.

Corollary 4.11. *Let G be a discrete group and H be its subgroup with an injective homomorphism $\theta : H \to G$. If the centralizers $C_G(H)$, $C_G(\theta(H))$ satisfy $C_G(H) \cap (G \setminus H) \neq \emptyset \neq C_G(\theta(H)) \cap (G \setminus \theta(H))$ and moreover if H is C^*-simple, then the HNN extension $G \bigstar_H \theta$ is C^*-simple.*

The assumption of C^*-simplicity of H can be replaced by, for example, a certain "relative Powers property" for $H \subseteq G$.

4.3. K-theory of HNN extensions. The C^*-versions of Proposition 3.1 and Proposition 3.3 assert that the computation of K-theory (also KK- or E-theory) of (universal or reduced) HNN extensions of C^*-algebras is reduced to that of the corresponding amalgamated free products. Here, we illustrate how to derive by obtaining the six terms exact sequence for K-groups associated with universal HNN extensions, which is exactly of the same kind of that given in [1].

Here, we use (and keep) the setting and notations in Proposition 3.3. Let us denote

$$\mathcal{A}_1 := B \otimes M_2(\mathbf{C}), \qquad \mathcal{A}_2 := C \otimes M_2(\mathbf{C}), \qquad \mathcal{B} := C \oplus C$$

and also the embedding map from a C^*-algebra $X = C$ or B to another $Y = B$ or $A = B \bigstar_C^{\mathrm{univ}} \theta$ by $\iota_{X \hookrightarrow Y}$. Under a certain mild condition on $\mathcal{A}_1 \overset{\iota_\theta}{\hookleftarrow} \mathcal{B} \overset{\iota_1}{\hookrightarrow} \mathcal{A}_2$, it is known that the six terms exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \overset{(\iota_{\theta *}, \iota_{1*})}{\longrightarrow} & K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2) & \overset{j_{\theta *} - j_{1*}}{\longrightarrow} & K_0(\mathcal{A}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}_1) & \underset{j_{\theta *} - j_{1*}}{\longleftarrow} & K_1(\mathcal{A}_1) \oplus K_1(\mathcal{A}_2) & \underset{(\iota_{\theta *}, \iota_{1*})}{\longleftarrow} & K_1(\mathcal{B}) \end{array} \tag{13}$$

holds true. In fact, the most general result of this type was provided by Thomsen [16], where he assumed that $\mathcal{B}$ is nuclear or the existence of conditional expectations from $\mathcal{A}_1$, $\mathcal{A}_2$ onto $\iota_\theta(\mathcal{B})$, $\iota_1(\mathcal{B})$, respectively. Note that his conditions are apparently translated in our setup to the nuclearity of C or the existence of conditional expectations from B onto C, $\theta(C)$.

Notice here that we have the following isomorphisms:

$$K_0(\mathcal{B}) \cong K_0(C) \oplus K_0(C) \quad \text{by } [(p,q)] \leftrightarrow [p] \oplus [q];$$

$$\begin{cases} K_0(\mathcal{A}_1) \cong K_0(B) \\ K_0(\mathcal{A}_2) \cong K_0(C) \end{cases} \quad \text{with } \left[\begin{pmatrix} p & \\ & q \end{pmatrix}\right] \leftrightarrow [p] + [q];$$

$$K_0(\mathcal{A}) \cong K_0\big(A \otimes M_2(\mathbf{C})\big) \quad \text{by } \Phi_*;$$

$$K_0\big(A \otimes M_2(\mathbf{C})\big) \cong K_0(A) \quad \text{with } \left[\begin{pmatrix} p & \\ & q \end{pmatrix}\right] \leftrightarrow [p] + [q].$$

For the description of the second and the fourth isomorphisms, we use the obvious identification $M_n(D \otimes M_2(\mathbb{C})) = M_2(M_n(D))$ with an arbitrary C^*-algebra D, which identifies $M_n(D \otimes \mathbf{C}^2)$ with the diagonal matrices whose entries are from $M_n(D)$. By these facts, we can rewrite the upper horizontal line in (13) as

$$K_0(C) \oplus K_0(C) \xrightarrow{\phi_0} K_0(B) \oplus K_0(C) \xrightarrow{\psi_0} K_0(A),$$

where the left arrow is given by $\phi_0 : [p] \oplus [q] \mapsto ([p] + \theta_*([q])) \oplus ([p] + [q])$ and the right one by $\psi_0 : [p] \oplus [q] \mapsto [p] - [q]$. The same discussion for K_1-groups shows that the lower horizontal arrow in (13) can be rewritten as

$$K_1(A) \underset{\psi_1}{\longleftarrow} K_1(B) \oplus K_1(C) \underset{\phi_1}{\longleftarrow} K_1(C) \oplus K_1(C),$$

where $\phi_1 : [u] \oplus [v] \mapsto ([u] + \theta_*([v])) \oplus ([u] + [v])$ and $\psi_1 : [u] \oplus [v] \mapsto [u] - [v]$. Hence, (13) becomes

$$\begin{array}{ccccc} K_0(C) \oplus K_0(C) & \xrightarrow{\phi_0} & K_0(B) \oplus K_0(C) & \xrightarrow{\psi_0} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \underset{\psi_1}{\longleftarrow} & K_1(B) \oplus K_1(C) & \underset{\phi_1}{\longleftarrow} & K_1(C). \end{array} \tag{14}$$

Let ϕ_0' be the projection map from $K_0(C) \oplus K_0(C)$ to the second component and set $\psi_0' : = (\mathrm{id}_B)_* - (\iota_{C \hookrightarrow B})_*$. Then we have

$$\begin{array}{ccccccc} & K_0(C) & \xrightarrow{\theta_* - (\iota_{C\hookrightarrow B})_*} & K_0(B) & & & \\ & \phi_0' \uparrow & \circlearrowleft & \uparrow \psi_0' & \circlearrowleft & \searrow & (\iota_{B \hookrightarrow A})_* \\ K_0(C) \oplus K_0(C) & & \underset{\phi_0}{\longrightarrow} & K_1(B) \oplus K_0(C) & \underset{\psi_0}{\longrightarrow} & & K_0(A), \end{array}$$

and $\phi_0'(\mathrm{Ker}\,\phi_0) = \mathrm{Ker}(\theta_* - (\iota_{C \hookrightarrow B})_*)$. Similarly, let ϕ_1' be the projection from $K_1(C) \oplus K_1(C)$ onto the second component and set $\psi_1' := (\mathrm{id}_B)_* - (\iota_{C \hookrightarrow B})_*$. Then we have

$$\begin{array}{ccccc} K_1(A) & \underset{\psi_1}{\longleftarrow} & K_1(B) \oplus K_1(C) & \underset{\phi_1}{\longleftarrow} & K_1(C) \oplus K_1(C) \\ (\iota_{B \hookrightarrow A})_* \quad \nwarrow \quad \circlearrowleft & & \psi_1' \downarrow & \circlearrowleft & \downarrow \phi_1' \\ & & K_1(B) & \underset{\theta_* - (\iota_{C\hookrightarrow B})_*}{\longleftarrow} & K_1(C), \end{array}$$

and $\phi_1'(\mathrm{Ker}\,\phi_1) = \mathrm{Ker}(\theta_* - (\iota_{C \hookrightarrow B})_*)$. From these facts together with (14), we finally arrive at the following proposition.

Proposition 4.12. *If C is nuclear or there are conditional expectations from B onto C and $\theta(C)$, then the universal HNN extension $A = B\bigstar_C^{\mathrm{univ}}\theta$ satisfies*

$$\begin{array}{ccccc} K_0(C) & \xrightarrow{\theta_*-(\iota_{C\hookrightarrow B})_*} & K_0(B) & \xrightarrow{(\iota_{B\hookrightarrow A})_*} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \underset{(\iota_{B\hookrightarrow A})_*}{\longleftarrow} & K_1(B) & \underset{\theta_*-(\iota_{C\hookrightarrow B})_*}{\longleftarrow} & K_1(C). \end{array}$$

Remark 4.13. Note that the above proposition apparently includes the celebrated six terms exact sequence for crossed-products by the integers $\mathbb{Z}$ due to Pimsner and Voiculescu [12]. The work [12] also deals with crossed-products by free groups $\mathbb{F}_n$ whose universal construction version can be also treated in the same way, where one needs what we commented after Proposition 3.3.

4.4. Supplements to the Brown, Dykema, and Jung's work [4]. Here, we give supplementary facts to [4], and thus we refer to that paper for necessary backgrounds including notations and terminologies. Let $(M, E_N^M, u(\theta)) = (N, E_D^N)\bigstar_D(\theta, E_{\theta(D)}^N)$ be an HNN extension of von Neumann algebras. Assume that E_D^N and $E_{\theta(D)}^N$ are τ-preserving with a faithful normal tracial state τ on N, and moreover that $\tau|_{\theta(D)} = (\tau|_D)\circ\theta$ holds. Then [19, Corollary 4.2] shows that τ is extended to a faithful normal tracial state on M by E_N^M, and we still denote it by the same symbol τ.

Proposition 4.14. *Let X and Y be generating finite families* (*of self-adjoint elements*) *of N and D, respectively. If D is hyperfinite and N can be embedded into R^ω with the hyperfinite type* II_1 *factor R, then $\delta_0(X \sqcup Y \sqcup \{u(\theta)\}) = \delta_0(X \sqcup Y \sqcup \theta(Y)) + 1 - \delta_0(D)$.*

Proof. Let $\widehat{X} := \{x\otimes e_{11} : x \in X\} \sqcup \{1\otimes e_{12}\}$ (inside $N\otimes M_2(\mathbf{C})$), $\widehat{Y} := \{y\otimes e_{11} : y \in Y\}\sqcup\{1\otimes e_{12}\}$ (inside $D\otimes M_2(\mathbf{C})$), and $\widetilde{Y} := \{y_1\oplus 0, 0\oplus y_2 : y_1, y_2 \in Y\}$ (inside $D\oplus D$). Note here, that the unital $*$-algebra generated by $\widehat{Y}$ contains $\{y_1\otimes e_{11},\ y_2\otimes e_{22} : y_1,\ y_2\in Y\}$, a generating set of the diagonals in $D\otimes M_2(\mathbf{C})$. Then [4, Corollary 4.5] shows $\delta_0(\lambda_\theta(\widehat{X})\sqcup\lambda_1(\widehat{Y})) = \delta_0(\lambda_\theta(\widehat{X})\sqcup\lambda_1(\widehat{Y})\sqcup\lambda(\widetilde{Y})) = \delta_0(\widehat{X}\sqcup\iota_\theta(\widetilde{Y})) + \delta_0(D\otimes M_2(\mathbf{C})) - \delta_0(D\oplus D)$ with the embedding maps ι_θ, λ, λ_θ, λ_1 as in Proposition 3.1. By applying [4, Proposition 5.1] to the right most side of the above equation with the matrix units $\lambda_\theta(1\otimes e_{ij})$'s, it becomes $\delta_0(X\sqcup Y\sqcup\{u(\theta)\})/4 + 3/4$ via the bijective $*$-homomorphism Φ in Proposition 3.1 (note here that it is trace-preserving thanks to the hypothesis here). Similarly, the same [4, Proposition 5.1] (in the special case when $B = \mathcal{M}_2$ there) enables us to show $\delta_0(\widehat{X}\sqcup\iota_\theta(\widetilde{Y})) = \delta_0(X\sqcup Y\sqcup\theta(Y))/4 + 3/4$. Also, Jung's result [8] allows us to compute that $\delta_0(D\otimes M_2(\mathbf{C})) = 1-(1-\delta_0(D))/4$ and $\delta_0(D\oplus D) = 1-(1-\delta_0(D))/2$ (since the trace-weights on $D \oplus D$ are $1/2$). Then the desired assertion is immediate. $\square$

Let us point out that the above fact shows, in particular, that the class of R^ω-embeddable finite von Neumann algebras with separable preduals is closed under taking HNN extension over hyperfinite algebra. Here, we need, in general, to take an inductive limit arising from an increasing chain $D, \theta(D) \subset N_n \subset N$ with finitely generated N_n's.

The next proposition is proved in a similar way as in [4, Theorem A.1]; indeed a repetition of its proof works with only one change that the G-pushout used there as the initial point should be replaced by that given in [9, Example 4.11, Remark 4.12]. Hence, the details are left to the reader.

Proposition 4.15. *Let $G^* = G\bigstar_H$ be an HNN extenison of groups, and suppose that the first L^2-Betti number $b_1^{(2)}(H)$ vanishes. Then $b_1^{(2)}(G^*) = b_1^{(2)}(G) - |G|^{-1} + |H|^{-1}$ with $1/\infty = 0$.*

We say a group G to be hyperlinear if there is a faithful representation of G into the unitary group of R^ω, which is known to be equivalent to the R^ω-embeddability of $L(G)$ (see e.g., [14, Proposition 2.4]). The above two facts immediately imply:

Corollary 4.16. *Let $G^* = G\bigstar_H$ be an HNN extension of groups, and suppose that G is finitely generated, hyperlinear, and H amenable. If $\delta_0(G) = b_1^{(2)}(G) - b_0^{(2)}(G) + 1$, then $\delta_0(G^*) = b_1^{(2)}(G^*) - b_0^{(2)}(G^*) + 1$. Thus, the class of finitely generated, hyperlinear groups whose δ_0 and $b_1^{(2)} - b_0^{(2)} + 1$ coincide is closed under taking HNN extension over amenable subgroup.*

In closing, we would like to point out three things: (i) The discussion of Proposition 4.14 also works for showing that the class of finitely generated, hyperlinear, and microstates-packing regular groups is closed under taking HNN extension over amenable subgroup. (ii) Proposition 3.4 enables us to obtain a different (from [4, e.g., Corollary 4.7]) formula for δ_0 (of natural generating sets) of amalgamated free products once through HNN extensions. The resulting formula says two things. Firstly the class of R^ω-embeddable von Neumann algebras with separable preduals is closed under taking reduced amalgamated free product over hyperfinite subalgebra. Secondly proving the expected formula for δ_0 of natural generating sets of direct sums (of two algebras with equal trace-weights) is enough to get rid of the microstates-packing regularity assumption from the main result in [4]. However, we encountered the same difficulty, and the formula of δ_0 for such direct sums could be proved only when the same regularity condition is assumed. (iii) We can show that if $(M, \tau_M) = (N \oplus N, (\tau_N + \tau_N)/2)$ with the same (N, τ_N) and $X'' = N$ with $|X| < \infty$, then the expected formula $\delta_0(X \oplus 0 \sqcup 0 \oplus X) = (\delta_0(X) + 1)/2$

($= (\delta_0(X) + \delta_0(X) + 2)/4$) holds without the microstates-packing regularity assumption. Correspondingly the same is true in the amalgamated free product case, and in particular, $\delta_0(G \bigstar_H G) = 2\delta_0(G) - \delta_0(H)$ always holds for any finitely generated, hyperlinear G with amenable $H \subset G$.

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