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A RATIONAL DECISION CRITERION, THE ITERATED MINIMAX REGRET CRITERION[1]

BY EARL NORDBROCK

University of Alberta

We feel that a criterion for selecting optimal decision rules in a statistical decision problem should be selected rationally. Specifically, we would like a criterion to satisfy eight properties which we have given. It is known that none of the commonly used criteria satisfy these eight properties. We give sufficient conditions on the decision problem so that the iterated minimax regret criterion does satisfy all eight properties, and give examples to show that these sufficient conditions cannot be removed.

1. Introduction. In this paper we take the viewpoint that statistical decision problems should be approached rationally. This means that we would like a criterion for selecting optimal decision rules to satisfy certain basic properties, e.g. the optimal set should not be empty.

Milnor (1954) and Chernoff (1954) approached finite decision problems rationally, and each gave a list of properties they wanted a criterion to satisfy. While Chernoff showed that his properties were contradictory, Milnor was able to give a criterion which he claimed satisfied his properties. Atkinson, Church, and Harris (1964) modified the properties and criterion of Milnor and then proved that this modified criterion satisfied their properties. Efron (1965) extended the results of Atkinson, Church, and Harris to closed and bounded S-games (see Blackwell and Girshick (1954) for definition of S-games). In this paper we extend the results to the general decision problem.

In Section 1 we give the mathematical description of the general decision problem. In Section 2 we list eight basic properties we would like a criterion for selecting optimal decision rules to satisfy. We also define the criterion of Atkinson, Church, and Harris, which has been called the iterated minimax regret criterion. In Section 3 we give sufficient conditions on the decision problem so that the iterated minimax regret criterion satisfies our eight basic properties. In Section 4 we give examples where the sufficient conditions of Section 3 are weakened and where the iterated minimax regret criterion no longer satisfies our eight basic properties. We also give an example which shows that Theorem 2 of Efron is incorrect, and we prove some characteristics of the iterated minimax regret criterion.

ASSUMPTIONS 1.1. We characterize a decision problem by $Q = (\Theta, F)$ where

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(1) Θ is a nonempty set; and
(2) F is a set of functions $f: \Theta \to E_1$,

where E_1 is the real line. We assume F is nonempty, convex, and pointwise bounded below (i.e. for every θ, $\inf_{f \in F} f(\theta) > -\infty$).

Θ is the set of possible states of nature, while F is the set of risk functions determined by the set of decision rules. Thus instead of considering (Θ, D^*, R) (or $(\Theta, \mathscr{D}, R)$) as Ferguson (1967) does, we suppress the D^* (or $\mathscr{D}$) and write the risk given by the decision rule δ as $f_\delta(\theta)$, where $f_\delta(\theta) = R(\theta, \delta)$. Note that there is a natural 1–1 correspondence between F and equivalence classes in $D^*(\mathscr{D})$.

For a given decision problem $Q = (\Theta, F)$, we let $B = \{f: f: \Theta \to E_1\}$, and for $f \in B$ we define $\|f\| = \sup_{\theta \in \Theta} |f(\theta)|$. We also make the following definitions.

DEFINITION 1.2. $f \in F$ is better than $g \in F$ if $f(\theta) \leqq g(\theta)$ for all θ, while for some θ_0 we have $f(\theta_0) < g(\theta_0)$.

DEFINITION 1.3. $f \in F$ is admissible if there is no $g \in F$ better than f.

DEFINITION 1.4. $C \subset F$ is a complete class in Q if corresponding to every $f \in F$ which is not in C there exists a $g \in C$ which is better than f.

DEFINITION 1.5. For G and H subsets of B (B is defined above),

$$d(G, H) = \max\{\sup_{g \in G} \inf_{h \in H} d(g, h), \sup_{h \in H} \inf_{g \in G} d(g, h)\},$$

where $d(g, h) = \|g - h\|$.

2. Rationality properties. If K is a criterion for selecting an optimal decision rule in a decision problem, then K must partially order the set of decision rules into a set of optimal decision rules and a set of non-optimal decision rules. Hence for a decision problem $Q = (\Theta, F)$, we denote by $K(Q)$ that subset of F which is selected as optimal by criterion K.

Then the basic properties that we want a criterion K to satisfy are the following:

PROPERTY 1. For every decision problem $Q = (\Theta, K)$, $K(Q)$ is nonempty.

PROPERTY 2. For every decision problem $Q = (\Theta, F)$, $K(Q)$ is convex.

PROPERTY 3. If $Q = (\Theta, F)$, $Q' = (\Theta', F')$, $h: \Theta' \to \Theta$ is 1–1 onto, and if $F' = F \circ h$, then $K(Q') = K(Q) \circ h$.

PROPERTY 4. If $Q = (\Theta, F)$, $Q' = (\Theta, F')$, λ is any fixed positive number, $c: \Theta \to E_1$ is any fixed function, and if $F' = \lambda F + c$, then $K(Q') = \lambda K(Q) + c$.

PROPERTY 5. If $Q' = (\Theta', F)$ and $Q = (\Theta, F \mid \Theta)$ where $\Theta \subset \Theta'$, if for every $f \in F$, $f \mid \Theta$ is measurable with respect to a σ-algebra $\mathscr{B}$ of subsets of Θ, and if corresponding to every $\theta_0 \in \Theta'$ there is a probability measure μ_{θ_0} on $(\Theta, \mathscr{B})$ such that for all f in F we have $f(\theta_0) = \int_\Theta f(\theta)\, d\mu_{\theta_0}(\theta)$, then $K(Q') \mid \Theta = K(Q)$.

PROPERTY 6. If $Q = (\Theta, F)$, $Q^{(n)} = (\Theta, F^{(n)})$ for $n = 1, 2, \cdots$, $d(F^{(n)}, F) \to 0$ as $n \to \infty$, if $f_n \in K(Q^{(n)})$ for all n, and $\{f_n\}$ is such that $d(f_n, f) \to 0$, then $f \in K(Q)$.

PROPERTY 7. Every f in $K(Q)$ is admissible.

PROPERTY 8. If $Q = (\Theta, F)$, $Q' = (\Theta, F')$, and if C is a complete class in both Q and Q', then $K(Q) = K(Q')$.

Property 3 requires that if we relabel the states of nature we should not change our mind about which decision rules are optimal. Property 4 says, in effect, that the scale and origin used in measuring the loss is irrelevant. Property 5 contains the special case of nature duplication. The notion that a criterion should not select drastically different optimal sets in problems which resemble each other very much is formalized in Property 6. Property 8 says that the optimal set selected by a criterion should not depend on unimportant decision rules.

A good discussion of rational properties is found in Luce and Raiffa (1957). In his paper, Milnor (1954) was able to characterize minimax, minimax regret, and the principle of insufficient reason (i.e. use a uniform prior) using his basic properties. At the same time, he showed that none of these criteria satisfied all of his basic properties. He was able to give a criterion, however, which he claimed satisfied his basic properties. This criterion was modified by Atkinson, Church, and Harris (1964), and we now give their criterion.

DEFINITION 2.1. The iterated minimax regret criterion. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a monotone non-increasing sequence of positive numbers converging to zero. Let $Q = (\Theta, F)$ be a decision problem satisfying Assumptions 1.1. The iterated minimax regret criterion corresponding to the sequence $\{\varepsilon_n\}$ selects as optimal the set $\mathrm{IMR}\{\varepsilon_n\}(Q)$ defined below.

Define $F_1 = F$.

Define $v_1(\theta) = \inf_{f \in F_1} f(\theta)$ and $z_1 = \inf_{f \in F_1} d(v_1, f)$.

If z_1 is infinite, define $\mathrm{IMR}\{\varepsilon_n\}(Q) = F_1$.

If z_1 is finite, inductively define

$$v_n(\theta) = \inf_{f \in F_n} f(\theta)\,, \qquad z_n = \inf_{f \in F_n} d(v_n, f) \qquad \text{and}$$
$$F_{n+1} = \{f \in F_n : d(v_n, f) \leqq z_n + \varepsilon_n z_1\}$$

for $n = 1, 2, 3, \cdots$. Then define $\mathrm{IMR}\{\varepsilon_n\}(Q) = \bigcap_{n=1}^{\infty} F_n$.

We see that z_1 is the minimax regret value of (Θ, F). When z_1 is finite, F_2 consists precisely of the decision rules which are within $\varepsilon_1 z_1$ of being minimax regret in (Θ, F). Similarly, z_2 is the minimax regret value of (Θ, F_2) and F_3 consists of those decision rules in F_2 which are within $\varepsilon_2 z_1$ of being minimax regret in (Θ, F_2). Thus at each iteration we keep only those rules within $\varepsilon_n z_1$ of being minimax regret in (Θ, F_n).

It is apparent that different $\{\varepsilon_n\}$ sequences will, in general, give different optimal sets. However, for convenience we will often suppress the $\{\varepsilon_n\}$ notation and write $\mathrm{IMR}(Q)$ for $\mathrm{IMR}\{\varepsilon_n\}(Q)$. If $\mathrm{IMR}(Q)$ is nonempty, we define $v(\theta) = \inf(f(\theta) : f \in \mathrm{IMR}(Q))$ and $z = \inf(d(v, f) : f \in \mathrm{IMR}(Q))$. Since the sequence $\{F_n\}$ is nested, it follows that for each θ the sequence $\{v_n(\theta)\}$ is monotone non-decreasing and that $v_n(\theta) \leqq v(\theta)$ for all n. It also follows that $\{z_n\}$ is a monotone non-increasing sequence and that $z_n \geqq z$ for all n.

The following example shows that the iterated minimax regret criterion does, in general, give an optimal set which is different from the optimal sets given by other commonly used criteria. In this example, minimax, minimax regret, and the principle of insufficient reason (i.e. use a uniform prior) all give the same optimal set, but the iterated minimax regret criterion selects a different set as optimal.

EXAMPLE 2.1. Let $\Theta = \{1, 2\}$ and let f_1, f_2, and f_3 be given by the following matrix

$$\begin{array}{cc|ccc} & & f_1 & f_2 & f_3 \\ \hline \Theta & 1 & 0 & 1 & 6 \\ & 2 & 21 & 1 & 0 \end{array}\ .$$

Let F be the convex hull of f_1, f_2, and f_3. The decision problem of interest is then $Q = (\Theta, F)$. It is easily verified that minimax, minimax regret, and the principle of insufficient reason all select $\{f_2\}$ as the optimal set. Letting $\varepsilon_n = 100^{1-n}$ for $n = 1, 2, \cdots$, and proceeding to find IMR (Q), we find that $f_2 \notin F_3$, and therefore $f_2 \notin$ IMR (Q). Thus the optimal set selected by IMR differs from the optimal sets selected by the other criteria.

3. Main results. We now give sufficient conditions so that the iterated minimax regret criterion satisfies our eight basic properties. These sufficient conditions are weak for Properties 1 (nonempty) and 6 (continuity), very strong for Properties 7 and 8 (admissibility and some complete class), while no assumptions other than 1.1 are needed for all the other properties.

It is obvious that without further conditions (other than Assumptions 1.1) on the decision problem, IMR (Q) may be empty. For example, take $Q = (\Theta, F)$ where $\Theta = \{1, 2\}$ and $F = \{pf_1 + (1-p)f_2 \colon 1 \geqq p > 0\}$ and $f_1(1) = 2, f_1(2) = 1$, $f_2(\theta) \equiv 1$. Then any choice of $\{\varepsilon_n\}$ will give IMR (Q) as the empty set. We have found that the concept of weak intrinsic compactness given by Wald (1950) is sufficient to insure that IMR (Q) is nonempty.

DEFINITION 3.1. Let $Q = (\Theta, F)$ be a decision problem. F is weak intrinsically compact if for every sequence $\{f_n\}$ in F there is a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ and an f in F such that $\liminf_{k\to\infty} f_{n(k)}(\theta) \geqq f(\theta)$ for all θ.

THEOREM 3.2. *Let $Q = (\Theta, F)$ satisfy Assumptions* 1.1 *and also have F weak intrinsically compact. Then* IMR (Q) *is nonempty and weak intrinsically compact.*

PROOF. If $z_1 = \infty$, then IMR $(Q) = F$ and the theorem is true. So assume $z_1 < \infty$. We use induction to show that F_n is nonempty and weak intrinsically compact for all n. The induction assumption is that F_N is nonempty and weak intrinsically compact. Clearly F_{N+1} is nonempty, so let $\{f_n\}_{n=1}^{\infty}$ be a sequence in F_{N+1}. Then there is a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ and an f in F_N such that

$\liminf f_{n(k)}(\theta) \geqq f(\theta)$ for all θ. Since $f_{n(k)} \in F_{N+1}$, this implies $f(\theta) \leqq v_N(\theta) + z_N + \varepsilon_N z_1$ for all θ, and thus $f \in F_{N+1}$. Therefore F_{N+1} is weak intrinsically compact. Thus by induction, F_n is nonempty and weak intrinsically compact for all n.

Now let $\{f_n\}_{n=1}^{\infty}$ be a sequence with $f_n \in F_n$. By the weak intrinsic compactness of F_1 there is a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ and a $g \in F_1$ such that $\liminf f_{n(k)}(\theta) \geqq g(\theta)$ for all θ. Assuming that $g \in F_N$, it follows that $g \in F_{N+1}$. So by induction, $g \in F_n$ for all n, and IMR (Q) is therefore nonempty.

To show that IMR (Q) is weak intrinsically compact, let $\{f_n\}_{n=1}^{\infty}$ be a sequence in IMR (Q). Then $f_n \in F_n$ for all n, and as above we can find a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ and a g in IMR (Q) such that $g(\theta) \leqq \liminf f_{n(k)}(\theta)$ for all θ. Thus by definition, IMR (Q) is weak intrinsically compact. End of Theorem 3.2.

The proof that the iterated minimax regret criterion satisfies Property 2 (convexity) in the general decision problem is straightforward.

THEOREM 3.3. *Let* $Q = (\Theta, F)$ *satisfy Assumptions* 1.1. *Then* IMR (Q) *is convex.*

We now show that the iterated minimax regret criterion satisfies Property 3 (relabelling nature) in the general decision problem.

THEOREM 3.4. *Let* $Q' = (\Theta', F')$ *satisfy Assumptions* 1.1. *Let* $h: \Theta' \to \Theta$ *be* 1–1 *onto. Define* F *by* $F' = F \circ h$, *and let* $Q = (\Theta, F)$. *Then* IMR $(Q') =$ IMR $(Q) \circ h$.

PROOF. From Nordbrock (1971), it is apparent that $v_1' = v_1 \circ h$ and $z_1' = z_1$. Thus if $z_1 = z_1' = \infty$, the desired result follows. If $z_1 = z_1' < \infty$, it follows by induction that $F_n' = F_n \circ h$ for all n, and therefore IMR $(Q') =$ IMR $(Q) \circ h$.

We now show that the iterated minimax regret criterion satisfies Property 4 (change of scale and origin) in the general decision problem.

THEOREM 3.5. *Let* $Q = (\Theta, F)$ *satisfy Assumptions* 1.1. *Let* $\lambda > 0$ *be fixed and let* $c: \Theta \to E_1$. *Define* $F' = \lambda F + c$ *and* $Q' = (\Theta, F')$. *Then* Q' *satisfies Assumptions* 1.1 *and* IMR $(Q') = \lambda$ IMR $(Q) + c$.

PROOF. It is easily shown that $v_1' = \lambda v_1 + c$ and $z_1' = \lambda z_1$. Therefore the theorem is true when $z_1 = \infty$. If $z_1 < \infty$, we have by induction that $F_n' = \lambda F_n + c$, from whence the theorem follows.

We now show that the iterated minimax regret criterion satisfies Property 5 (nature duplication) in the general decision problem.

THEOREM 3.6. *Let* $Q' = (\Theta', F')$ *satisfy Assumptions* 1.1. *Let* $\Theta \subset \Theta'$ *and* $Q = (\Theta, F)$ *where* $F = F' \mid \Theta$. *Assume that corresponding to every* $\theta_0 \in \Theta'$ *there is a probability measure* μ_{θ_0} *on* Θ *such that for all* $f \in F'$ *we have* $f(\theta_0) = \int_\Theta f(\theta)\, d\mu_{\theta_0}(\theta)$. *Then* IMR $(Q') \mid \Theta =$ IMR (Q).

PROOF. It can be shown that $d(v_1', f) = d(v_1, f \mid \Theta)$ and that $z_1' = z_1$. Thus when $z_1 = \infty$ the theorem is true. If $z_1 < \infty$, we use induction to show that $F_n = F_n' \mid \Theta$ for all n, from which the theorem follows.

We now show that the iterated minimax regret criterion satisfies Property 6 (continuity) if F is closed. The proof of the continuity property is based on the following theorem.

THEOREM 3.7. *Let $Q = (\Theta, F)$, $Q^N = (\Theta, F^N)$ for $N = 1, 2, \cdots$, be decision problems each of which satisfies Assumptions* 1.1. *Let $d(F, F^N) \to 0$ (see Definition* 1.5). *Assume $z_1 < \infty$. Then*

$$\lim_{N\to\infty} d(v_n{}^N, v_n) = 0 \qquad \textit{for all} \quad n\,, \tag{3.8}$$

$$\lim_{N\to\infty} z_n{}^N = z_n \qquad \textit{for all} \quad n\,, \tag{3.9}$$

$$\lim_{N\to\infty} d(F_n{}^N, F_n) = 0 \qquad \textit{for all} \quad n\,. \tag{3.10}$$

The proof of this theorem is divided into several lemmas. The first of these is:

LEMMA 3.11. *Under the conditions of Theorem* 3.7, $d(v_1{}^N, v_1) \to 0$ *and* $z_1{}^N \to z_1$.

PROOF. To show that $d(v_1{}^N, v_1) \to 0$, we fix N sufficiently large so $d(F_1{}^N, F_1) < \varepsilon$. Then for any given θ there is an $f_1 \in F_1$ such that $f_1(\theta) \leqq v_1(\theta) + \varepsilon$ and there is an $f_1{}^N \in F_1{}^N$ such that $f_1{}^N(\theta) \leqq f_1(\theta) + \varepsilon$. Thus $v_1{}^N(\theta) \leqq v_1(\theta) + 2\varepsilon$. Similarly, $v_1(\theta) \leqq v_1{}^N(\theta) + 2\varepsilon$ and it follows that $d(v_1{}^N, v_1) \to 0$. To show $z_1{}^N \to z_1$, we observe that $z_1 \leqq d(v_1, v_1{}^N) + z_1{}^N + \varepsilon \leqq z_1{}^N + 2\varepsilon$ for N sufficiently large. Similarly $z_1{}^N \leqq z_1 + 2\varepsilon$, and therefore $z_1{}^N \to z_1$.

LEMMA 3.12. *Under the conditions of Theorem* 3.7, $d(F_2{}^N, F_2) \to 0$.

PROOF. When $z_1 = 0$, we have $F_2 = \{v_1\}$. Thus for arbitrary $f_2{}^N \in F_2{}^N$ we have $d(v_1, f_2{}^N) \leqq d(v_1, v_1{}^N) + z_1{}^N + \varepsilon_1 z_1{}^N$, and on taking limits we find $d(F_2{}^N, F_2) \to 0$.

If $z_1 > 0$, let $\varepsilon > 0$ be given, and assume that $(\varepsilon/2) \leqq z_1 + \varepsilon_1 z_1$. Let $a = 9\varepsilon\varepsilon_1 z_1/(20)(z_1 + \varepsilon_1 z_1)$. Fix N sufficiently large so that

$$d(F_1, F_1{}^N) \leqq a/9$$
$$d(v_1, v_1{}^N) \leqq a/9$$
$$|z_1 - z_1{}^N| \leqq a/18\,, \qquad \varepsilon_1|z_1 - z_1{}^N| \leqq a/18$$
$$z_1/2 < z_1{}^N < 3z_1/2\,.$$

Part A. We show that $\sup_{f\in F_2} \inf_{f^N \in f_2{}^N} d(f, f^N) \leqq \varepsilon$. Let f be a fixed but arbitrary element of F_2.

Case I. If $d(v_1, f) \leqq z_1 + \varepsilon_1 z_1 - a/3$, we choose $f^N \in F_1{}^N$ such that $d(f, f^N) \leqq a/9 < \varepsilon/2$, and we see that $f^N \in F_2{}^N$.

Case II. If $d(v_1, f) > z_1 + \varepsilon_1 z_1 - a/3$, we choose an $f_0 \in F_2$ with $d(v_1, f_0) < z_1 + \varepsilon_1 z_1/10$. If $d(f, f_0) \leqq \varepsilon/2$, we choose (as in Case I) $f_0{}^N \in F_2{}^N$ with $d(f_0, f_0{}^N) < \varepsilon/2$, and therefore $d(f, f_0{}^N) < \varepsilon$.

When $d(f, f_0) > \varepsilon/2$, define $g = pf_0 + (1 - p)f$ where $p = \varepsilon/2d(f, f_0)$. Then by the triangle inequality for norms, and since $d(f, f_0) \leqq z_1 + \varepsilon_1 z_1$, we have $d(v_1, g) \leqq z_1 + \varepsilon_1 z_1 - a/3$. Thus, as in Case I, we can find a $g^N \in F_2{}^N$ with $d(g, g^N) < \varepsilon/2$, and so $d(f, g^N) < \varepsilon$.

Part B. To show that $\sup_{f^N \in F_2^N} \inf_{f \in F_2} d(f^N, f) \leqq \varepsilon$, we proceed as in Part A, using in addition the fact that $z_1/2 < z_1^N < 3z_1/2$. Parts A and B together give $d(F_2, F_2^N) \leqq \varepsilon$, and Lemma 3.12 is complete.

PROOF OF THEOREM 3.7. Lemmas 3.11 and 3.12 show that $d(v_1^N, v_1) \to 0$, $z_1^N \to z_1$, and $d(F_2, F_2^N) \to 0$. From the proofs of these lemmas it is apparent that by assuming $\lim_{N\to\infty} d(F_n^N, F_n) = 0$ we can show that $\lim_{N\to\infty} d(v_n^N, v_n) = 0$, $\lim_{N\to\infty} z_n^N = z_n$, and $\lim_{N\to\infty} d(F_{n+1}^N, F_{n+1}) = 0$. Therefore, by induction, we have proved Theorem 3.7.

Using Theorem 3.7 we now prove that the iterated minimax regret criterion satisfies the continuity property.

THEOREM 3.13. *Let $Q = (\Theta, F)$, $Q^N = (\Theta, F^N)$ for $N = 1, 2, \cdots$ all satisfy Assumptions* 1.1. *Assume that $d(F, F^N) \to 0$ and that F is closed. Assume $f^N \in$ IMR (Q^N) for all N and that $\{f^N\}$ is a sequence such that $d(f^N, f) \to 0$ for some f. Then $f \in$ IMR (Q).*

PROOF. Note that we assume only that $f \in B$ (defined in Section 1).

Case I. $z_1 < \infty$. For fixed n, by Theorem 3.7 we can choose the sequence $\{g_n^N\} \in F_n$ such that $d(g_n^N, f^N) \to 0$ as $N \to \infty$. Thus we have $d(g_n^N, f) \to 0$ as $N \to \infty$, and we have $f \in F_n$. Therefore $f \in F_n$ for all n, i.e. $f \in$ IMR (Q).

Case II. If $z_1 = \infty$. In this case Lemma 3.11 gives $z_1^N = \infty$ for N sufficiently large. As in Case I, we can show that $f \in F_1$. Since $F_1 =$ IMR (Q), we have shown that $f \in$ IMR (Q). Theorem 3.13 is complete.

So far we have proved that the iterated minimax regret criterion satisfies six of the basic properties with only minimal assumptions on the decision problem. To prove that the two remaining properties (admissibility and same complete class) are satisfied, we assume that F is compact. Note that if F is convex and compact, then F satisfies Assumptions 1.1, z_1 is finite, and F is weak intrinsically compact. We now show that IMR (Q) is a single point which is admissible.

LEMMA 3.14. *If F is compact and convex, then $d(v_n, v) \to 0$, $z_n \to z$, and any $f \in$ IMR (Q) has $d(v, f) = z$.* (Recall that v and z refer to IMR (Q); see Section 2.)

PROOF. Assume that $\limsup d(v_n, v) > 2\varepsilon > 0$. Then there is a subsequence $\{v_{n(k)}\}$ of $\{v_n\}$ and a sequence $\{\theta_{n(k)}\}$ such that $v(\theta_{n(k)}) - v_{n(k)}(\theta_{n(k)}) > \varepsilon$ for all k. Since $F_{n(k)}$ is compact for every k, for every k we can find an $f_{n(k)} \in F_{n(k)}$ such that $f_{n(k)}(\theta_{n(k)}) = v_{n(k)}(\theta_{n(k)})$. Thus

$$v(\theta_{n(k)}) - f_{n(k)}(\theta_{n(k)}) > \varepsilon \qquad \text{for all} \quad k\,. \tag{3.15}$$

Since $f_{n(k)} \in F$ for all k, the compactness of F implies there is a convergent subsequence of $\{f_{n(k)}\}$, say $\{f_{n(k(m))}\}$ and $d(f_{n(k(m))}, f_0) \to 0$ for some $f_0 \in F$. Since $f_{n(k(m))} \in F_n$ for $n(k(m)) \geqq n$, we see that $f_0 \in F_n$ for all n, and so $f_0 \in$ IMR (Q).

Thus for m sufficiently large $v(\theta) \leqq f_0(\theta) \leqq f_{n(k(m))}(\theta) + (\varepsilon/2)$ for all θ. This contradicts (3.15) and therefore $d(v_n, v) \to 0$. It follows that $z_n \leqq z + \varepsilon$ for n sufficiently large, and since $z \leqq z_n$ we have $z_n \to z$. Since $\varepsilon_n \to 0$, the triangle inequality then gives $d(v, f) = z$ for any $f \in \mathrm{IMR}(Q)$.

THEOREM 3.16. *If F is convex and compact, then $z = 0$, i.e.* IMR(Q) *is a single point, and moreover* IMR(Q) *is admissible.*

PROOF. Assume $z > 0$ and let $0 < \varepsilon < z$.

Let $\theta_1 \in \Theta$, and choose $f_1 \in \mathrm{IMR}(Q)$ such that $f_1(\theta_1) = v(\theta_1)$. Define $A_1 = \{\theta \in \Theta : f_1(\theta) \geqq v(\theta) + z - \varepsilon\}$, and therefore A_1 is not empty.

We proceed inductively by assuming A_n is not empty. Pick $\theta_{n+1} \in A_n$ and $f_{n+1} \in \mathrm{IMR}(Q)$ such that $f_{n+1}(\theta_{n+1}) = v(\theta_{n+1})$. Define $A_{n+1} = \{\theta \in A_n : f_{n+1}(\theta) \geqq v(\theta) + z - \varepsilon\}$. Thus A_{n+1} is not empty, for if it were we would have $\sum_{k=1}^{n+1} (f_k(\theta)/(n+1)) \leqq v(\theta) + z - (\varepsilon/(n+1))$ for all θ which contradicts Lemma 3.14.

By the above method we have selected a sequence $\{\theta_n\}$ in Θ and a sequence $\{f_n\}$ in IMR(Q) such that for $m < n$, $f_m(\theta_n) \geqq v(\theta_n) + z - \varepsilon$ and $f_n(\theta_n) = v(\theta_n)$. Thus $d(f_m, f_n) \geqq z - \varepsilon$ for $m \neq n$, contradicting the compactness of IMR(Q). Therefore z must be zero. To show the admissibility part, let $f_0 \in \mathrm{IMR}(Q)$ and assume that f_0 is inadmissible. Then by definition, there is an $f_1 \in F$ which is better than f_0. By induction, f_1 is also in IMR(Q). Since this contradicts $z = 0$, f_0 must be admissible. End of Theorem 3.16.

We now prove that the iterated minimax regret criterion satisfies Property 8, the same complete class property. We first show that if $Q = (\Theta, F)$ has F convex and compact, then the set A of admissible rules is a complete class. We then show that if $Q' = (\Theta, F')$ has F' convex and compact with admissible rules A' equal to A, then IMR$(Q) =$ IMR(Q'). By applying both of these facts, we show that if C is complete in both Q and Q' where $C \subset F$ and $C \subset F'$, then IMR$(Q) =$ IMR(Q'), i.e. IMR satisfies Property 8.

THEOREM 3.17. *If $Q = (\Theta, F)$ has F convex and compact, and if A is the set of admissible rules in F, then A is a complete class.*

PROOF. The method of proof is to use the definition of complete class 1.4. Let $g \in F$ with $g \notin A$. Define $T_g = \{f \in B : f(\theta) \leqq g(\theta)$ for all $\theta\}$. The set $T_g \cap F$ is convex and compact and is not empty. Thus $Q' = (\Theta, T_g \cap F)$ has IMR$(Q') = g_0$, say, with g_0 admissible in Q'. If there were a $g_1 \in F$ better than g_0, then $g_1 \in T_g \cap F$, contradicting g_0 admissible in $Q' = (\Theta, T_g \cap F)$. Therefore, g_0 is admissible in Q, and it follows that A is a complete class.

THEOREM 3.18. *Let $Q = (\Theta, F)$ have F convex and compact with A the set of admissible rules. Similarly, let $Q' = (\Theta, F')$ have F' convex and compact with A' the admissible rules. Assume $A = A'$. Then* IMR$(Q) =$ IMR(Q').

PROOF. This theorem follows from Theorem 3.16 and the following lemma.

DEFINITION 3.19. Using the notation of Theorem 3.18, define A_n (A_n') to be the admissible rules in F_n (F_n'), for $n = 1, 2, \cdots$. (A rule $f \in F_n$ is in A_n if there are no better rules in F_n.)

LEMMA 3.20. *Under the hypothesis of Theorem* 3.18, $A_n = A \cap F_n$ ($A_n' = A' \cap F_n'$) *and* $A_n = A_n'$ *for all* n.

PROOF. We use induction. By hypothesis, the lemma is true for $n = 1$. Assume the lemma is true for n. By applying Theorem 3.17 to (Θ, F_n), we see that $v_n(\theta) = \inf_{f \in A_n} f(\theta)$. The analogous equation holds for the prime quantities, so the induction assumption $A_n = A_n'$ gives $v_n = v_n'$. A similar argument gives $z_n = \inf_{f \in A_n} d(v_n, f)$, from whence we have $z_n = z_n'$. It is straightforward to verify that $A_{n+1} = A_n \cap F_{n+1}$, i.e. $A_{n+1} = A \cap F_{n+1}$. It follows that $A_{n+1} = A'_{n+1}$, and Lemma 3.20 is complete.

We now prove that IMR satisfies the same complete class property.

THEOREM 3.21. *Let* $Q = (\Theta, F)$, $Q' = (\Theta, F')$ *have each of* F *and* F' *convex and compact. Let* C *be a subset of both* F *and* F', *with* C *a complete class in both* Q *and* Q'. *Then* IMR$(Q') =$ IMR(Q).

PROOF. By Theorem 3.17, A (the set of admissible rules in F) is a complete class. Thus A is minimal complete and therefore $A \subset C$. Similarly, A' (the set of admissible rules in F') is a complete class and $A' \subset C$. It is straightforward to verify that $A = A'$. Applying Theorem 3.18, we have IMR$(Q) =$ IMR(Q').

4. Further results and examples. In Section 3 we gave sufficient conditions on the decision problem so that the iterated minimax regret criterion satisfied all eight basic properties. We recall that weak intrinsic compactness was used to insure that IMR(Q) was nonempty, closure of F was needed to show the continuity property, and compactness was assumed to prove the admissibility and same complete class properties. All other properties were proved for the general decision problem. In this section we discuss the weakening of the sufficient conditions. We also discuss the convergence of v_n to v and z_n to z.

Efron (1965) has several examples where $Q = (\Theta, F)$ has F closed and bounded, but where IMR(Q) is empty. Of course, for these examples, F is not weak intrinsically compact. Thus, closed and bounded does not insure a nonempty optimal set, while weak intrinsic compactness does insure a nonempty optimal set.

When proving the continuity property, we used the assumption that F was closed. Obviously if F is not closed, IMR does not have to satisfy the continuity property.

We now investigate the compactness assumption used to show that IMR satisfied the admissibility and same complete class properties. We give examples which are weak intrinsically compact, closed, and bounded, but for which IMR does not satisfy the admissibility and same complete class properties. We remark that these same examples show that Theorem 2 of Efron is incorrect. Efron's Theorem 2 says that IMR satisfies the admissibility and same complete class properties for the general decision problem.

We now give an example which is weak intrinsically compact, closed, and bounded, but for which IMR (Q) contains an inadmissible rule.

EXAMPLE 4.1. Let Θ be the positive integers. Define $f_0(\theta) = 1$ for all θ,

$$f_n(\theta) = 1 \quad \text{if} \quad \theta \neq n$$
$$= 0 \quad \text{if} \quad \theta = n \qquad \text{for } n = 1, 2, \cdots .$$

(See the matrix below.) Define $F = \{f : f = \sum_{n=0}^{\infty} p_n f_n, p_n \geqq 0, \sum_{n=0}^{\infty} p_n = 1\}$. Then F is convex and pointwise bounded below. So the decision problem $Q = (\Theta, F)$ satisfies Assumptions 1.1. We also have the following:

F is closed and weak intrinsically compact;
for any applicable sequence $\{\varepsilon_n\}$, IMR $(Q) = F$; and
$f_0 \in$ IMR (Q) but f_0 is not admissible.

PROOF OF EXAMPLE 4.1. We write the matrix defining the sequence $\{f_n\}$.

		f_0	f_1	f_2	$f_3 \cdots$
	1	1	0	1	$1 \cdots$
	2	1	1	0	1
Θ	3	1	1	1	0
	4	1	1	1	1
	$\vdots$	$\vdots$			

From the definition of F, it is seen that F is both closed and weak intrinsically compact. From the matrix above, we also see that $v_1(\theta) \equiv 0$, $z_1 = 1$, and any $f \in F$ has $d(v_1, f) = 1$. Thus for any applicable sequence $\{\varepsilon_n\}$ we have IMR $\{\varepsilon_n\}(Q) = F$. Therefore f_0, which is inadmissible, is in IMR $\{\varepsilon_n\}(Q)$. Example 4.1 is complete.

Our next example shows that weak intrinsic compactness is not sufficient to guarantee that IMR satisfies Property 8 (same complete class).

EXAMPLE 4.2. Let $Q = (\Theta, F)$ be as in Example 4.1. Let F' be the admissible rules in F, and define $Q' = (\Theta, F')$. Then F and F' have the same minimal complete class, but IMR $(Q) = F$ while IMR $(Q) = F'$. Thus IMR does not satisfy Property 8 for this example.

We now prove some remarks concerning the convergence of v_n to v and of z_n to z. We recall that if F is compact, then $d(v_n, v) \to 0$ and $z_n \to z = 0$. We will show that if F is weak intrinsically compact, then v_n converges to v pointwise, but we cannot conclude that $d(v_n, v) \to 0$ and we cannot conclude that $z_n \to z$. We then give an example which shows that in general v_n does not converge pointwise to v.

THEOREM 4.3. *Let $Q = (\Theta, F)$ have $z_1 > \infty$ and F weak intrinsically compact. Then v_n converges pointwise to v.*

PROOF. Fix $\theta \in \Theta$. Since $v_n(\theta)$ is monotone non-decreasing and bounded above, the sequence $\{v_n(\theta)\}$ converges to a limit, say $v_0(\theta)$, where $v_0(\theta) \leqq v(\theta)$. From

Theorem 3.2, F_n is weak intrinsically compact, and so there is $f_n \in F_n$ such that $f_n(\theta) = v_n(\theta)$. Then there is a subsequence $\{f_{n(k)}\}$ and an $f_0 \in \mathrm{IMR}(Q)$ such that $\liminf f_{n(k)}(\theta) \geqq f_0(\theta)$. Thus $v_0(\theta) \geqq v(\theta)$, and Theorem 4.3 is proved.

EXAMPLE 4.4. This example shows that weak intrinsic compactness of F does not imply that $d(v_n, v) \to 0$ nor that $z_n \to z$. Let the decision problem be $Q = (\Theta, F)$ where Θ is the positive integers, $\{f_n\}_{n=0}^{\infty}$ is defined by the matrix below, and F is the closed convex hull of $\{f_0, f_1, f_2, \cdots\}$. We let $\varepsilon_n = 10^{-n}$.

		f_0	f_1	f_2	$\cdots$
	1	1	0	1.01	$\cdots$
	2	1	1.1	0	
Θ	3	1	1.1	1.01	
	$\vdots$	$\vdots$			

Using induction, we see that F_n contains $\{f_0, f_{n-1}, f_n, f_{n+1}, \cdots\}$. It follows that $\mathrm{IMR}(Q) = \{f_0\}$. Thus $v_n(\theta) = 0$ for $\theta \geqq n$ while $v(\theta) = 1$ for all θ, and $z_n = 1$ for all n while $z = 0$. End of Example 4.4.

Theorem 4.3 gives sufficient conditions for the pointwise convergence of v_n to v. Our next example shows that in general we cannot conclude even the pointwise convergence of v_n to v.

EXAMPLE 4.5. Let Θ be the positive integers. Define $f_0(\theta) = 1$ for all θ,

$$\begin{aligned} f_n(\theta) &= 0.5 && \text{for} \quad \theta < n \\ &= 0 && \text{for} \quad \theta = n \\ &= 1 + 10^{-n} && \text{for} \quad \theta > n \qquad \text{for} \quad n = 1, 2, \cdots. \end{aligned}$$

The matrix is below.

		f_0	f_1	f_2	f_3	$\cdots$
	1	1	0	0.5	0.5	$\cdots$
	2	1	1.1	0	0.5	
Θ	3	1	1.1	1.01	0	
	4	1	1.1	1.01	1.001	
	$\vdots$	$\vdots$				

Let F be the closed convex hull of $\{f_0, f_1, \cdots\}$, $Q = (\Theta, F)$, and $\varepsilon_n = 10^{-n}$. Then, as in Example 4.4 we can show that for $n > 2$, F_n contains $\{f_0, f_{n-1}, f_n, \cdots\}$, and that $\mathrm{IMR}\{\varepsilon_n\}(Q) = \{f_0\}$. Thus we have $v_n(1) \leqq \frac{1}{2}$ while $v(1) = 1$, and therefore v_n does not converge pointwise to v.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA