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LIMIT POINTS OF $\{n^{-1/\alpha}S_n\}$

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Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed (i.i.d.) positive random variables in the domain of attraction of a completely asymmetric stable law with characteristic exponent $\alpha \in (0, 1)$, i.e. their common distribution function G is given by

$$P(X_1 > x) = 1 - G(x) = x^{-\alpha}L(x),$$

where L is a slowly varying function at infinity. In this paper we study the set of limit points of $\{n^{-1/\alpha}(X_1 + \cdots + X_n) : n = 1, 2, \cdots\}$. The sets of limit points that are possible are $\{0\}$, $\{\infty\}$, $[0, \infty]$ and $[b, \infty]$ for some positive number b.

In Section 2 we consider the case where L is non-decreasing and in Section 3 the case where L is non-increasing. In both sections we give the conditions in terms of L for each of the limit sets.

1. Introduction. Let X_1, X_2, $\cdots$ be i.i.d. random variables with common stable distribution function $F(\cdot; \alpha, 1)$, with $0 < \alpha < 1$, and characteristic function ϕ given by

$$\log\phi(t) = -|t|^{\alpha}\{1 - i\,\mathrm{sign}(t)\tan(\pi\alpha/2)\}. \tag{1.1}$$

This standardization for the characteristic function of stable distributions is used in the monograph [7]. In this monograph one can find the values of the several constants that appear in this paper.

Put $S_n = X_1 + \cdots + X_n$, for $n = 1, 2, \cdots$. Then $n^{-1/\alpha}S_n$ has, for all n, the same distribution as X_1. The set of limit points of $\{n^{-1/\alpha}S_n(2\log_2 n)^{(1-\alpha)/\alpha}\}$ is given by the interval $[\{2B(\alpha)\}^{(1-\alpha)/\alpha}, \infty]$, where $B(\alpha)$ is a (known) constant given in formula (2.1.7) of [7]. This assertion easily follows from the results obtained in [10] or in Chapter 9 of [7].

In the case where X_1, X_2, $\cdots$ are i.i.d. positive random variables in the domain of attraction of a completely asymmetric stable distribution with characteristic function given by (1.1), their common distribution function satisfies

$$P(X_1 > x) = 1 - G(x) = x^{-\alpha}L(x), \tag{1.2}$$

where L is a slowly varying function at infinity and $0 < \alpha < 1$. Put $S_n = X_1 + \cdots + X_n$. Now we can prove the existence of a sequence h_n such that

$$\liminf_{n\to\infty} h_n^{-1/\alpha}S_n = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \quad \text{a.s.} \tag{1.3}$$

As in the case where the random variables have a stable distribution, we can prove that the set of limit points of $\{h_n^{-1/\alpha}S_n\}$ is given by the interval $[\{2B(\alpha)\}^{(1-\alpha)/\alpha}, \infty]$. In [5] Fristedt and Pruitt define a sequence of normalizing constants h_n such that (1.3) holds, by using the inverse of the logarithm of the Laplace transform of X_1. In [8] we define an(other) normalizing sequence h_n by making use of the function L given in (1.2). There we also prove the asymptotic equivalence of both sequences of normalizing constants.

It is proved by Kesten [6], Theorem 1, that the set of accumulation points of $\{n^{-1/\alpha}S_n\}$ is w.p. 1 equal to a fixed (non-random) closed set. In the same paper Kesten formulates the following problem. Find the structure of the set of the limit points $B(L, \alpha)$ of the sequence $\{n^{-1/\alpha}S_n\}$. Here follows a summary of the results obtained in Erickson and Kesten [1] and Erickson [2].

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EXAMPLE 1. (See [2] Theorem 5, page 804.)
$L(x) = (2 \log_2 x)^{(1-\alpha)}$ for x sufficiently large. Then $B(L, \alpha) = [b, \infty]$ for some (unknown) $b \in (0, \infty)$. It follows from Theorem 2 of [8] that $b = \{2B(\alpha)\}^{(1-\alpha)/\alpha}$.

EXAMPLE 2. (See [2] Example 1, page 817.)
$L(x) = (2 \log_2 x)^{(1-\alpha)} \exp\{(1-\alpha)(\log_3 x)(1-\cos(\log_4 x))\}$ for x sufficiently large. Then $\liminf_{n\to\infty} n^{-1/\alpha} S_n = b$ a.s. for some (unknown) $b \in (0, \infty)$. It follows from the results obtained in this paper that in this case $B(L, \alpha) = [\{2B(\alpha)\}^{(1-\alpha)/\alpha}, \infty]$.

EXAMPLE 3. (See [2] Theorem 8 Part 1, page 818.)
Suppose $\lim_{x\to\infty} L(x) = 0$ and $\int^\infty x^{-1} L(x)\, dx = \infty$ then $B(L, \alpha) = [0, \infty]$.

EXAMPLE 4. (See [2] Theorem 8 Part 2, page 818.)
Suppose $\lim_{x\to\infty} L(x) = \infty$, L satisfies, for $x \to \infty$, $L(x^\theta) \sim L(x)$ uniformly for $\theta \in [p^{-1}, p]$ for some $p > 1$ and

$$\int^\infty x^{-1} L(x) \exp\{-k(L(x))^{1/(1-\alpha)}\}\, dx = \infty$$

for every $k > 0$. Then $B(L, \alpha) = [0, \infty]$.

EXAMPLE 5. (Compare with [1] Example 5, page 578.)
$L(x) = (\log_2 x)^\beta$ with $0 < \beta < 1 - \alpha$, for x sufficiently large. Then $B(L, \alpha) = [0, \infty]$.

In this paper we distinguish two cases. In Section 2 we make the assumption that L is non-decreasing and in Section 3 that L is non-increasing. In both cases we shall prove an integral test that decides which is the set of limit points of $\{n^{-1/\alpha} S_n\}$. The general case, without the assumption that either L is monotone or L is slowly varying at infinity, is more complicated and therefore not considered in this paper. Some of the approximations of the probabilities that we shall use in this paper can also be derived in the general case. See, for example, Mijnheer [8].

2. L non-decreasing.

Consider the integral I defined by

$$I(L, k) = \int^\infty x^{-1} \{L(x)\}^{1/(2(1-\alpha))} \exp\{-k(L(x))^{1/(1-\alpha)}\}\, dx. \tag{2.1}$$

Note that the integrand is different from the one in Example 4 in the introduction. Define the functions ψ_k, $k > 0$, by

$$\frac{1}{2} \psi_k^2(x) = k\{L(x)\}^{1/(1-\alpha)}. \tag{2.2}$$

Then I can be rewritten as the well-known integral

$$J(\psi_k) = (2k)^{-1/2} \int^\infty x^{-1} \psi_k(x) \exp\left\{-\frac{1}{2}\psi_k^2(x)\right\} dx. \tag{2.3}$$

This integral occurs in the generalized law of the iterated logarithm. See, for example, Chapters 4, 5 and 6 in [7].

THEOREM 2.1. *Let $X_1, X_2, \cdots$ be a sequence of* i.i.d. *positive random variables with common distribution function G given by* (1.2). *Let $B(L, \alpha)$ be the* (*non-random*) *set of limit points of $\{n^{-1/\alpha} S_n\}$. Suppose L is non-decreasing. Then*

a. $B(L, \alpha) = \{\infty\}$ *if $I(L, k) < \infty$ for all $k > 0$;*
b. $B(L, \alpha) = [0, \infty]$ *if $I(L, k) = \infty$ for all $k > 0$;*
c. $B(L, \alpha) = [b, \infty]$ *with $b \in (0, \infty)$ if there exists a number k_0 such that $I(L, k) = \infty$ for*

all $k < k_0$ and $I(L, k) < \infty$ for all $k > k_0$. The constants b and k_0 in Part c *satisfy the following relation*

$$k_0 = B(\alpha)\{\Gamma(1-\alpha)\cos(\pi\alpha/2)\}^{1/(1-\alpha)} b^{-\alpha/(1-\alpha)}, \tag{2.4}$$

where $B(\alpha)$ is defined in (2.17) *of* [7].

REMARK. Define the function

$$f: \mathbb{R}^+ \to \mathbb{R}^+$$

by

$$f(x) = B(\alpha)\{\Gamma(1-\alpha)\cos(\pi\alpha/2)\}^{1/(1-\alpha)} x^{-\alpha/(1-\alpha)}. \tag{2.5}$$

As usual we define $f(0) = \infty$ and $f(\infty) = 0$. Then we can summarize the assertions of Theorem 2.1 as follows.

If $I(L, k) < \infty$ for all $k > k_0$ then $P(S_n \le b_1 n^{1/\alpha}$ i.o.$) = 0$ for $0 < b_1 < b \le \infty$. If $I(L, k) = \infty$ for all $k < k_0$ then $P(S_n n^{-1/\alpha} \in (b_1 - \varepsilon, b_1 + \varepsilon)$ i.o.$) = 1$ for all $0 \le b \le b_1 \le \infty$ and all $\varepsilon > 0$.

In the proof of Theorem 2.1 we need some parts of the proof of the generalized law of the iterated logarithm and some approximations for the probabilities on events of the type $\{S_n \le bn^{1/\alpha}\}$. These results will be stated and proved below. Let $\{X(t): 0 \le t < \infty\}$ be a completely asymmetric stable process with characteristic exponent $\alpha \in (0, 1)$. The characteristic function of $X(1)$ is given by (1.1). For $t > 0$, $X(t)$ has the same distribution as $t^{1/\alpha}X(1)$. The sample paths of these completely asymmetric stable processes with $\alpha \in (0, 1)$ are non-decreasing pure jump functions. The expansion of the distribution function near the origin (see, for example, Part IV of Theorem 2.1.7 in [7]) is given by

$$P(X(1) \le x) \sim (2/\alpha)^{1/2} P(U \ge (2B(\alpha))^{1/2} x^{-\alpha/(2(1-\alpha))}) \quad \text{for } x \downarrow 0, \tag{2.6}$$

where U is the standard normal random variable and $B(\alpha)$ the constant given by (2.1.7) of [7]. From the characteristic function given in (1.1), we can deduce the following expansion for the distribution function $F(\cdot; \alpha, 1)$ of a stable random variable

$$F(x; \alpha, 1) = 1 - \textstyle\sum_{n=1}^{\infty} A_n x^{-\alpha n}. \tag{2.7}$$

The values of the constants A_n can be found in Theorem 2.1.6 of [7]. Another standard reference, Feller (1971) Vol. II, gives a different standarization for the characteristic function of a stable random variable and therefore obtains other constants. But the values of A_n are only important in the relation between b and k_0 as given in (2.4). From the expansion of the density $p(\cdot; \alpha, 1)$, we easily obtain, for $x \to \infty$,

$$p(x; \alpha, 1) = \alpha A_1 x^{-\alpha-1} + \mathcal{O}(x^{-2\alpha-1}) \tag{2.8}$$

and

$$1 - F(x; \alpha, 1) = A_1 x^{-\alpha} + \mathcal{O}(x^{-2\alpha}). \tag{2.9}$$

The following assertion is proved by Feller in [3]. In fact, he proved a more general result because he did not assume that the function L in (1.2) is slowly varying at infinity. For any sequence c_n such that $n^{-1}c_n$ increases, we have $P(S_n > c_n$ i.o.$) = 0$ or 1 according as $\sum P(X_n > c_n) < \infty$ or $= \infty$. If we take $c_n = n^{(1+\varepsilon)/\alpha}$ for some $\varepsilon > 0$ then $\sum P(X_n > c_n) < \infty$ and, w.p. 1, we have, for all sufficiently large n, $X_n \le c_n$ and $S_n \le c_n$. Obviously the last n such that $S_n > c_n$ is a random variable, depending on the sample path. The result above suggests a truncation at c_n. The next lemma gives a second truncation.

LEMMA 2.2 *Let $X_1, X_2, \cdots$ be a sequence of positive* i.i.d. *random variables with common distribution function G given by* (1.2). *Let ℓ be a positive slowly varying function* *at infinity and take* $b_n = n^{1/\alpha}\ell(n)$, $n = 1, 2, \cdots$. *Suppose that* b_n *satisfies, for* $n \to \infty$,

(2.10) $$n^{-1}b_n^\alpha(\log_2 n)(L(b_n))^{-1} \to 0,$$

(2.11) $$n^{-1}b_n^\alpha(\log_2 n)^{\alpha/(2-\alpha)}(L(b_n))^{\alpha/(2-\alpha)} \to 0,$$

(2.12) $$n^{-1}b_n^\alpha(L(b_n))^{\alpha/(1-\alpha)} \to 0.$$

Then

$$n^{-1/\alpha}\{\textstyle\sum_{j=1}^n X_j I(X_j \le b_j)\} \to 0 \quad \text{a.s.}$$

PROOF. Put $\tilde{X}_j = X_j I(X_j \le b_j)$, for $j = 1, 2, \cdots$, and $s_n^2 = \sum_{j=1}^n E\tilde{X}_j^2$ for $n = 1, 2, \cdots$. From the theory of regular variation (cf. Feller [4] Vol. II, Section VIII-9), it follows that, for $n \to \infty$,

$$s_n^2 \sim k_1 \textstyle\sum_{j=1}^n b_j^{2-\alpha}L(b_j) \sim k_2 n\, b_n^{2-\alpha}L(b_n),$$

where k_1, k_2 are independent of n. Condition (2.10) implies

$$\tilde{X}_n = o(s_n(\log_2 s_n^2)^{-1/2}) \quad \text{a.s.} \quad \text{for } n \to \infty.$$

Kolmogorov's law of the iterated logarithm (see, for example, [9] Chapter X) implies

(2.13) $$\limsup_{n\to\infty}\{\textstyle\sum_{j=1}^n (\tilde{X}_j - E\tilde{X}_j)\}(2s_n^2 \log_2 s_n^2)^{-1/2} = 1 \quad \text{a.s.}$$

Consequences of (2.11) and (2.12) are

(2.14) $$\{s_n^2(\log_2 s_n^2)\}^{1/2}n^{-1/\alpha} \to 0$$

and

(2.15) $$n^{-1/\alpha}\textstyle\sum_{j=1}^n E\tilde{X}_j \to 0$$

for $n \to \infty$. Then (2.13), (2.14) and (2.15) imply the assertion stated in the lemma. □

Take $c_n = n^{(1+\varepsilon)/\alpha}$, $\varepsilon > 0$, $n = 1, 2, \cdots$, and let b_n satisfy the conditions as mentioned in Lemma 2.3. Then it is obvious that only the truncated random variables $X_n I(b_n < X_n < c_n)$ contribute to the non-zero limit points.

We shall define new random variables which have the same set of limit points. The following lemma will be useful in the approximation of the distribution function of these new random variables.

LEMMA 2.3. *Let t be a continuous and non-decreasing real function. Define*

(2.16) $$C_n = \{(x, y): x \ge t(y), y \in (b_n, c_n)\} \cup \{(x, y): x \ge t(b_n), 0 \le y \le b_n\} \cup \{(x, y): x \ge t(c_n), c_n \le y\},$$

where b_n *and* c_n *are chosen as before. Define*

(2.17) $$T_n(\omega) = \inf\{t: X(t, \omega) \in C_n\}.$$

Then for $y \in [b_n, c_n]$ *we have*

(2.18) $$P[X(T_n) \le y] \le P[X(t(b_n)) \le b_n] + \int_{b_n}^y f_{X(t(z))}(z)\, dz,$$

where $f_{X(t(z))}$ *is the density of the random variable* $X(t(z))$.

PROOF. Let $b_n = y_1 < y_2 < \cdots < y_m = y$ be a division of $[b_n, y]$ which becomes dense for $m \to \infty$. Because of the monotonicity of $X(\cdot, \omega)$ it follows that

$$P[X(T_n) \le y] = P[X(t(b_n)) \le b_n \quad \text{or} \quad X(t(v)) \le v \quad \text{for some} \quad v \in [b_n, y]]$$

$$\le P[X(t(b_n)) \le b_n] + \textstyle\sum_{j=2}^{m} P[y_{j-1} \le X(t(y_{j-1})) \le y_j]$$

$$= P[X(t(b_n)) \le b_n] + \textstyle\sum_{j=2}^{m} \displaystyle\int_{y_{j-1}}^{y_j} f_{X(t(y_{j-1}))}(z)\, dz.$$

The inequality on the right-hand side in Part a follows if $m \to \infty$. □

Take $\Delta > 0$ (fixed). Define the sequence of (non-random) stopping times $T_{1,n}$, $n = 1, 2, \cdots$, by

$$T_{1,n}(\omega) = (1-\Delta)A_1^{-1}L(b_n)$$

for all ω. Then we have, for $b_n \le y$,

$$1 - P(X(T_{1,n}) \le y) = 1 - P(X(1) \le (1-\Delta)^{-1/\alpha}A_1^{1/\alpha}L^{-1/\alpha}(b_n)y)$$

$$\sim (1-\Delta)L(b_n)y^{-\alpha} \quad \text{for} \quad n \to \infty.$$

Because of the monotonicity of L, we see that there exists a number $n_1(\Delta)$ such that for $n \ge n_1(\Delta)$ we have

$$P(X(T_{1,n}) \le y) \ge 1 - L(y)y^{-\alpha} = G(y) \tag{2.19}$$

for $y \ge b_n$.

The standard representation of slowly varying functions is given by (see Feller [4] Vol. II page 282)

$$L(y) = c(y)\exp\left\{\int_1^y x^{-1}\epsilon(x)\, dx\right\}, \tag{2.20}$$

where $c(x) \to c \in (0, \infty)$ and $\epsilon(x) \to 0$ for $x \to \infty$.

Let us first suppose that the function c is constant, i.e. we can write $L(y) = c \exp \{\int_1^y x^{-1}\epsilon(x)\, dx\}$. Because L is also non-decreasing, we obtain that L is a continuous differentiable function and $\epsilon(x) \ge 0$ for all x. Define the functions t_n, $n = 1, 2, \cdots$, by

$$\begin{aligned} t_n(y) &= (1+\Delta)A_1^{-1}L(b_n) \quad \text{for} \quad 0 \le y \le b_n \\ &= (1+\Delta)A_1^{-1}L(y) \quad \text{for} \quad b_n \le y \le c_n \\ &= (1+\Delta)A_1^{-1}L(c_n) \quad \text{for} \quad y > c_n. \end{aligned} \tag{2.21}$$

For every function t_n we define a set C_r as in Lemma 2.3 and a stopping time $T_{2,n}$ as in (2.17). From Lemma 2.3 we have, for $y \in [b_n, c_n]$,

$$P[X(T_{2,n}) \le y] \le P[X(t_n(b_n)) \le b_n] + \int_{b_n}^{y} f_{X(t_n(z))}(z)\, dz. \tag{2.22}$$

Because $\{X(t) : 0 \le t < \infty\}$ is a stable process, we have

$$P[X(t_n(b_n)) \le b_n] = P[X(1) \le t_n^{-1/\alpha}(b_n)b_n].$$

From the expansion given in (2.9), it follows that the last probability is equal to

$$F(t_n^{-1/\alpha}(b_n)b_n;\, \alpha, 1) = 1 - (1+\Delta)L(b_n)b_n^{-\alpha} + \mathcal{O}(L^2(b_n)b_n^{-2\alpha}) \tag{2.23}$$

for $n \to \infty$.

From the scaling property and the expansion given in (2.8), we obtain

$$\int_{b_n}^{y} f_{X(t_n(z))}(z)\,dz = (1+\Delta)\{b_n^{-\alpha}L(b_n) - y^{-\alpha}L(y)\} + R_n, \tag{2.24}$$

where $R_n = \mathcal{O}(b_n^{-2\alpha-1}L^2(b_n)y)$ for $n \to \infty$.

The results in (2.23) and (2.24) together with our choice $b_n = n^{1/\alpha}\ell(n)$, where ℓ is a slowly varying function at infinity, give, for $y \in [b_n, c_n]$,

$$P[X(T_{2,n}) \le y] = 1 - (1+\Delta)y^{-\alpha}L(y) + R_n^*, \tag{2.25}$$

where $R_n^* = \mathcal{O}(n^{-2+\delta})$ for $n \to \infty$ and some $\delta > 0$.

Thus there exists a number $n_2(\Delta)$ such that for all $n \ge n_2(\Delta)$ we have for $y \in [b_n, c_n]$,

$$P[X(T_{2,n}) \le y] \le 1 - y^{-\alpha}L(y) = G(y). \tag{2.26}$$

In the general case, the function c in the representation (2.20) is not constant. Then we define slowly varying functions $\tilde{L}_n$ such that

$$\tilde{L}_n(y) = \tilde{c}_n(y)\exp\left\{\int_1^y x^{-1}\tilde{\varepsilon}_n(x)\,dx\right\},$$

where the functions $\tilde{c}_n$ are constant, $\tilde{\varepsilon}_n$ is non-negative, and $L(y) \le \tilde{L}_n(y)$ for $y \in [b_n, c_n]$ and n sufficiently large. We define the stopping times $T_{2,n}$ by making use of the functions $\tilde{L}_n$. This yields (2.26) in the case that we only make the assumption that L is non-decreasing. Now we can summarize these results.

LEMMA 2.4. *Let the distribution function G be given by* (1.2). *Let $\{X(t): 0 \le t < \infty\}$ be a completely asymmetric stable process with characteristic exponent $\alpha \in (0, 1)$. Suppose $b_n = n^{1/\alpha}\ell(n)$ and satisfies* (2.10), (2.11) *and* (2.12). *Take $\varepsilon > 0$ and $c_n = n^{(1+\varepsilon)/\alpha}$. Then there exist two sequences of stopping times $\{T_{1,n}\}$ and $\{T_{2,n}\}$ such that, for sufficiently large n and $y \in [b_n, c_n]$, we have*

$$P(X(T_{2,n}) \le y) \le G(y) \le P(X(T_{1,n}) \le y). \tag{2.27}$$

The stopping times satisfy, for $\Delta > 0$ and for $n \to \infty$,

$$\textstyle\sum_{k=1}^n T_{1,k} \sim (1-\Delta)A_1^{-1}nL(b_n) \tag{2.28}$$

and

$$(1+\Delta)^{-1}A_1 n^{-1}L^{-1}(b_n)\{T_{2,1} + \cdots + T_{2,n}\} \to 1 \quad \text{a.s.} \tag{2.29}$$

PROOF. Assertion (2.27) is a combination of (2.19), (2.26) and the remarks after that formula. From the theory of slowly varying functions (see for example Feller [4]), it follows that

$$\textstyle\sum_{k=1}^n L(b_k) \sim nL(b_n) \quad \text{for} \quad n \to \infty.$$

This implies (2.28). From the definition of $T_{2,n}$, we have, for some constant c,

$$|\,ET_{2,n} - (1+\Delta)A_1^{-1}L(b_n)| \le cL(c_n)b_n^{-\alpha}L(b_n).$$

This yields

$$(1+\Delta)^{-1}A_1 n^{-1}L^{-1}(b_n)\{E(T_{2,1} + \cdots + T_{2,n})\} \to 1 \quad \text{for} \quad n \to \infty. \tag{2.30}$$

There exists some constant k such that

$$\textstyle\sum_{j=1}^\infty j^{-2}L^{-2}(b_j)\mathrm{Var}(T_{2,j}) < k\sum_{j=1}^\infty j^{-2}L^{-2}(b_j)L^2(c_j) < \infty.$$

Convergence of the last series and (2.30) imply (2.29).

From Lemma 2.2 and the assertion before that lemma, it follows that $\{n^{-1/\alpha}S_n\}$ and $\{n^{-1/\alpha}\sum_{k=1}^n X_k I(b_k < X_k < c_k)\}$ have the same set of limit points. We write F instead of $F(\cdot;\alpha, 1)$. Define the sequences of real numbers t_n and s_n, $n = 1, 2, \cdots$, by

$$t_n = b_n^\alpha\{F^{-1}G(b_n)\}^{-\alpha} \tag{2.31}$$

and

$$s_n = c_n^\alpha\{F^{-1}G(c_n)\}^{-\alpha}. \tag{2.32}$$

Then

$$P(X(t_n) \le b_n) = G(b_n)$$

and

$$P(X(s_n) \le c_n) = G(c_n).$$

Define the sequence of random variables X_n^*, $n = 1, 2, \cdots$, with distribution function F_n^* given by

$$\begin{aligned} F_n^*(x) &= P(X(t_n) \le x) && \text{for} \quad 0 \le x \le b_n \\ &= G(x) && \text{for} \quad b_n \le x \le c_n \\ &= P(X(s_n) \le x) && \text{for} \quad x \ge c_n. \end{aligned} \tag{2.33}$$

Then it easily follows that $\{n^{-1/\alpha}S_n\}$ and $\{n^{-1/\alpha}S_n^*\}$, where

$$S_n^* = X_1^* + \cdots + X_n^*, \tag{2.34}$$

have the same set of limit points.

In the next lemma we give some bounds for probabilities on events in which the random variables S_n^* occur.

LEMMA 2.5. *Let S_n^* be defined by* (2.34). *Let U be a standard normal random variable. Suppose* $\lim_{x\to\infty}L(x) = \infty$. *Take some $b > 0$. Let the functions ψ_k and f be defined by* (2.2) *and* (2.5).

a. *There exist positive constants b', k' and k_1 with*

$$0 < b < b' < \infty \text{ and } k' = f(b')$$

such that

$$P(S_n^* \le bn^{1/\alpha}) \le k_1 P(U \ge \psi_{k'}(b_n)).$$

b. *For $\delta > 0$ there exist positive constants b'', k'' and k_2 with $0 < b < b'' < b(1 + \delta) < \infty$ and $k'' = f(b'')$ such that*

$$P(bn^{1/\alpha}(1 - \delta) \le S_n^* \le bn^{1/\alpha}(1 + \delta)) \ge k_2 P(U \ge \psi_{k''}(b_n)) + r_n,$$

where $r_n = o(n^{-3/2})$ for $n \to \infty$.

PROOF. a. The expansion of the tail of F^{-1} and the definition of G as given in (1.2) give that the numbers t_n and s_n as defined in (2.31) and (2.32) satisfy, for $n \to \infty$,

$$t_n \sim A_1^{-1}L(b_n) \tag{2.35}$$

$$\text{and} \quad s_n \sim A_1^{-1}L(c_n). \tag{2.36}$$

From the theory of slowly varying functions, we obtain

$$\textstyle\sum_{k=1}^n t_k \sim A_1^{-1}nL(b_n) \quad \text{for} \quad n \to \infty.$$

Thus for n sufficiently large we have

$$\textstyle\sum_{k=1}^n T_{1,k} < \sum_{k=1}^n t_k.$$

It follows from Lemma 2.4 and the definition of F_n^* that there exists a number $n_1 = n_1(\Delta)$ such that for $n \geq n_1$ we have

$$P(X(T_{1,n}) \leq y) \geq F_n^*(y) \quad \text{for all} \quad y \in (0, \infty).$$

Let $c > 1$. Then for $n \geq n_2(\Delta, c)$ we have

$$P(S_n^* \leq y) \leq cP(X(\textstyle\sum_{k=1}^n T_{1,k}) \leq y)$$

for all positive y. From (2.28), it follows that for n sufficiently large we have

$$P(S_n^* \leq bn^{1/\alpha}) \leq c_1 P(X(1) \leq b' A_1^{1/\alpha} L^{-1/\alpha}(b_n)).$$

From $\lim_{x\to\infty} L(x) = \infty$, the expansion (2.6) and the definitions in (2.2) and (2.5), we obtain

$$P(S_n^* \leq bn^{1/\alpha}) \leq k_1 P(U \geq \psi_{k'}(b_n)).$$

b. From Lemma 2.4 and the definition of F_n^*, it follows that there exists a number $n_3 = n_3(\Delta)$ such that for all $n \geq n_3$ we have

$$P(X(T_{2,n}) \leq y) \leq F_n^*(y) \quad \text{for all positive } y. \tag{2.37}$$

Remember that the stopping times $T_{2,n}$, $n = 1, 2, \cdots$, are random stopping times. From Chebyshev's inequality we obtain, for $\varepsilon > 0$,

$$\begin{aligned} r_n &= P(|\textstyle\sum_{k=1}^n (T_{2,k} - E\, T_{2,k})| > \varepsilon n L(b_n)) \\ &\leq \varepsilon^{-2} n^{-2} L^{-2}(b_n) \operatorname{Var}(\textstyle\sum_{k=1}^n T_{2,k}) \\ &\leq c\, \varepsilon^{-2} n^{-2} L^{-2}(b_n) \textstyle\sum_{k=1}^n t_k^2(c_k) L(b_k) b_k^{-\alpha} \end{aligned}$$

for some positive constant c and $t_k(\cdot)$ is used to define $T_{2,k}$. From properties of slowly varying functions and the choice $b_k = k^{1/\alpha}\ell(k)$, it follows that, for $n \to \infty$

$$r_n = o(n^{-3/2}) \tag{2.38}$$

and

$$\textstyle\sum_{k=1}^n E\, T_{2,k} \sim (1 + \Delta) A_1^{-1} n L(\prime_n). \tag{2.39}$$

We can write

$$P(n^{1/\alpha} b(1 - \delta) \leq S_n^* \leq n^{1/\alpha} b(1 + \delta)) = P(S_n^* \leq n^{1/\alpha} b(1 + \delta)) - P(S_n^* \leq n^{1/\alpha} b(1 - \delta)).$$

The last probability on the right-hand side has an upper bound as given in Part a of this lemma.

From (2.37), it follows that for any positive constant $c > 1$ there exists a number $n_4 = n_4(\Delta, c)$ such that, for all $n \geq n_4$, we have

$$\begin{aligned} P(S_n^* \leq n^{1/\alpha} b(1 + \delta)) &\geq c\, P(X(\textstyle\sum_{k=1}^n T_{2,k}) \leq n^{1/\alpha} b(1 + \delta)) \\ &= c\, P(X(\textstyle\sum_{k=1}^n T_{2,k}) \leq n^{1/\alpha} b(1 + \delta) \wedge |\textstyle\sum_{k=1}^n (T_{2,k} - ET_{2,k})| \leq \varepsilon n L(b_n)) \\ &\quad + c\, P(X(\textstyle\sum_{k=1}^n T_{2,k}) \leq n^{1/\alpha} b(1 + \delta) \wedge |\textstyle\sum_{k=1}^n (T_{2,k} - ET_{2,k})| > \varepsilon n L(b_n)) \\ &= P_1 + P_2. \end{aligned}$$

From the scaling property of stable processes and (2.39), we obtain

$$P_1 \geq P[X(1) \leq (1 + \Delta + \varepsilon)^{-1/\alpha} A_1^{1/\alpha} b(1 + \delta) \{L(b_n)\}^{-1/\alpha}].$$

From the expansion given in (2.6), $\lim_{x\to\infty} L(x) = \infty$, (2.2) and (2.5) it follows that

$$P_1 \geq c_1\, P[U \geq \psi_{k''}(b_n)].$$

Then the assertion stated in Part b easily follows. □

Let $\{n_j\}$ be a sequence of integers, such that

$$b_{n_j}/b_{n_{j-1}} \sim \exp\{\psi_k^{-2}(b_{n_j})\} \quad \text{for } j \to \infty, \tag{2.40}$$

where ψ_k is defined by (2.2).

LEMMA 2.6. *Let $\{b_n\}$ satisfy the conditions as given in Lemma 2.2. Let n_j be defined such that* (2.40) *holds and let the integral $J(\psi_k)$ be defined by* (2.3). *Let U be the standard normal random variable. Suppose $\lim_{x\to\infty}\psi_k(x)=\infty$. Then $J(\psi_k)<\infty$ iff*

$$\sum_{j=1}^{\infty} P(U \geq \psi_k(b_{n_j})) < \infty.$$

PROOF. A similar assertion is proved in the proof of the generalized law of the iterated logarithm. We shall write β_j instead of b_{n_j}. Then we have, for some constants c_1 and c_2,

$$c_1 \sum_j \psi_k^{-1}(\beta_{j+1}) \exp\left(-\frac{1}{2}\psi_k^2(\beta_{j+1})\right) \leq \sum_j \psi_k(\beta_{j+1}) \exp\left(-\frac{1}{2}\psi_k^2(\beta_{j+1})\right) \log(\beta_{j+1}/\beta_j)$$
$$\leq J(\psi_k) \leq \sum_j \psi_k(\beta_j) \exp\left(-\frac{1}{2}\psi_k^2(\beta_j)\right) \log(\beta_{j+1}/\beta_j) \leq c_2 \sum_j \psi_k^{-1}(\beta_j) \exp\left(-\frac{1}{2}\psi_k^2(\beta_j)\right).$$

Now the assertion of the lemma easily follows from the following well-known asymptotic expansion of the tail of the standard normal distribution function

$$P(U \geq x) \sim \frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2} \quad \text{for } x \to \infty. \qquad \square$$

Now we can prove the main result of this section.

PROOF OF THEOREM 2.1. Suppose $\lim_{x\to\infty} L(x) < \infty$. Then $I(L, k) < \infty$ for all $k > 0$. The law of the iterated logarithm (see [5], [8] or Chapter 10 in [7]) implies

$$\liminf_{n\to\infty} n^{-1/\alpha} S_n (2\log_2 n)^{(1-\alpha)/\alpha} = c \in (0, \infty) \quad \text{a.s.} \tag{2.41}$$

This implies $\liminf_{n\to\infty} n^{-1/\alpha} S_n = 0$ a.s. As in Lemma 2.5 we obtain, for any $b > 0$ and $\varepsilon > 0$, that there exist constants b_1, k_1 and k_2, such that for all n

$$P[b(1-\varepsilon)n^{1/\alpha} \leq S_n \leq b(1+\varepsilon)n^{1/\alpha}] \geq k_1 P[b_1(1-\varepsilon) \leq X_1 \leq b_1(1+\varepsilon)] \geq k_2.$$

The constant k_2 is independent of n and depends only on b and ε. Then we easily obtain

$$P[b(1-\varepsilon)n^{1/\alpha} \leq S_n \leq b(1+\varepsilon)n^{1/\alpha} \text{ i.o.}] = 1$$

from Corollary 2, page 1178 of Kesten [6]. This implies $b \in B(L, \alpha)$. Since this holds for all $b > 0$, we have $B(L, \alpha) = [0, \infty]$. Thus the theorem is proved in the case $\lim_{x\to\infty} L(x) < \infty$. Therefore we shall, from now on, suppose that $\lim_{x\to\infty} L(x) = \infty$.

First we shall prove Part c. Let b satisfy (2.4). We shall show that

$$\liminf_{n\to\infty} n^{-1/\alpha} S_n \geq b \quad \text{a.s.} \tag{2.42}$$

As we have seen, it is sufficient to prove that

$$\liminf_{n\to\infty} n^{-1/\alpha} S_n^* \geq b \quad \text{a.s.,} \tag{2.43}$$

where S_n^* is defined by (2.34).

Take $\varepsilon > 0$, let A_n be the event $\{S_n^* \leq b(1-\varepsilon)n^{1/\alpha}\}$ and let B_j be the event $\{S_{n_j}^* \leq b(1-\varepsilon)n_{j+1}^{1/\alpha}\}$, where n_j is a subsequence of integers such that b_{n_j} satisfies (2.40). Obviously we have

$$\{A_n \text{ i.o.}\} \subset \{B_j \text{ i.o.}\}. \tag{2.44}$$

It follows from (2.40) and the choice $b_n = n^{1/\alpha}\ell(n)$, where ℓ is a slowly varying function, that $n_{j+1}/n_j \to 1$ for $j \to \infty$. Then we have

$$\begin{aligned} P(B_j) &= P(S_{n_j}^* \leq b(1-\varepsilon)n_{j+1}^{1/\alpha}) \\ &\leq P(S_{n_j}^* \leq b(1-\varepsilon_1)n_j^{1/\alpha}) \text{ for some } \varepsilon_1 \in (0, \varepsilon) \text{ and } j \text{ sufficiently large} \\ &\leq k_1 P(U \geq \psi_{k'}(b_{n_j})) \text{ by Lemma 2.5, Part a,} \end{aligned}$$

where k_1 is some positive constant and $k' > k_0$. Because $I(L, k) < \infty$ for all $k > k_0$, it follows from Lemma 2.6 that $\sum P(B_j) < \infty$. The Borel-Cantelli lemma gives $P(B_j \text{ i.o.}) = 0$. From (2.44) we obtain $P(A_n \text{ i.o.}) = 0$. This yields (2.43).

Next we shall show that every point in the interval $[b, \infty)$ is a limit point of $\{S_n n^{-1/\alpha}\}$. Take $b_1 > b$ and $\varepsilon > 0$. Define the events D_n, $n = 1, 2, \cdots$ by

$$b_1(1-\varepsilon)n^{1/\alpha} \le S_n \le b_1(1+\varepsilon)n^{1/\alpha}.$$

We remark that S_n is the sum of n i.i.d. random variables but S_n^* is the sum of n independent *non*-identically distributed random variables. We shall write, for $n = 1, 2, \cdots, S_n = U_n + V_n + W_n$, where

$$U_n = \textstyle\sum_{i=1}^n X_i I(0 \le X_i \le b_i), \tag{2.45}$$

$$V_n = \textstyle\sum_{i=1}^n X_i I(b_i < X_i < c_i) \tag{2.46}$$

and

$$W_n = \textstyle\sum_{i=1}^n X_i I(c_i \le X_i < \infty). \tag{2.47}$$

Similarly we can write $S_n^* = U_n^* + V_n^* + W_n^*$. From the definition of S_n^* it follows that, for all n, V_n and V_n^* have the same distribution. We showed in Lemma 2.2 that, w.p. 1.,

$$n^{-1/\alpha}U_n \to 0 \quad \text{and} \quad n^{-1/\alpha}U_n^* \to 0.$$

Lemma 3 from Chapter X of Petrov [9], page 296 implies together with (2.10) and (2.11) that, for $\varepsilon > 0$ and $\delta > 0$,

$$P[U_n \ge \varepsilon n^{1/\alpha}] \le (\log n)^{-1-\delta} \tag{2.48}$$

for n sufficiently large. We obtain the same upper bound for $P[U_n^* \ge \varepsilon n^{1/\alpha}]$.

Next we shall derive an upper bound for $P[W_n \ge \varepsilon n^{1/\alpha}]$. By Markov's inequality we have for $0 < \beta < \alpha$

$$P[W_n \ge \varepsilon n^{1/\alpha}] \le \varepsilon^{-\beta} n^{-\beta/\alpha} E W_n^\beta.$$

From the results in Chapter VIII in Feller [4] Vol. II and $c_n = n^{(1+\varepsilon)/\alpha}$, we obtain

$$E W_n^\beta \le E \textstyle\sum_{i=1}^n X_i^\beta I(c_i \le X_i < \infty) \le k\, n^{-\delta_2}$$

for some constant k and $\delta_2 > 0$. We obtain the same upper bound for $P[W_n^* \ge \varepsilon n^{1/\alpha}]$. Let

$$P_1 = P[0 \le U_n \le \varepsilon n^{1/\alpha}/4 \wedge (b-\varepsilon)n^{1/\alpha} \le V_n \le (b+\varepsilon/2)n^{1/\alpha} \wedge 0 \le W_n \le \varepsilon n^{1/\alpha}/4]$$

and P_1^* the probability on a similar event in which U_n^*, V_n^* and W_n^* occur. Then we easily obtain

$$P_1 \le P[(b-\varepsilon)n^{1/\alpha} \le S_n \le (b+\varepsilon)n^{1/\alpha}],$$

$$P[(b-\varepsilon)n^{1/\alpha} \le V_n \le (b+\varepsilon/2)n^{1/\alpha}] - P_1 = o((\log n)^{-1-\delta}) \quad \text{for } \delta > 0 \text{ and } n \to \infty,$$

$$P_1^* \ge P[(b-\varepsilon/2)n^{1/\alpha} \le S_n^* \le (b+\varepsilon/2)n^{1/\alpha}]$$

and

$$P[(b-\varepsilon)n^{1/\alpha} \le V_n^* \le (b+\varepsilon/2)n^{1/\alpha}] - P_1^* = o((\log n)^{-1-\delta}) \quad \text{for } \delta > 0 \text{ and } n \to \infty.$$

Now we have

$$\textstyle\sum_{n=1}^\infty \frac{1}{n} P[D_n] \ge \sum_{j=1}^\infty n_j^{-1} \sum_{n=n_{j-1}+1}^{n_j} P[D_n]. \tag{2.49}$$

We may restrict ourselves to the case where the slowly varying function ℓ is non-increasing. From (2.40) and $b_n = n^{1/\alpha}\ell(n)$, it follows that

$$\frac{n_j - n_{j-1}}{n_j} \ge c\, \psi_{k_0}^{-2}(b_{n_j}).$$

for some constant c.

From Lemma 2.5 Part b, it follows that the series (2.49) diverges according as

$$\sum_j \psi_{k''}^{-3}(b_{n_j}) \exp\left\{-\frac{1}{2}\psi_{k''}^2(b_{n_j})\right\} \tag{2.50}$$

diverges, where $k'' > k_0$. Take $k_1 \in (k_0, k'')$, then $I(L, k_1) = \infty$ and divergence of the series (2.50) easily follows. By Corollary 2 in Kesten [6] page 1178, we have $b_1 \in B(L, \alpha)$.

If we take a sequence ε_k with $\varepsilon_k \to 0$ for $k \to \infty$ and b_1 in a countable dense subset of $[b, \infty)$ then we easily obtain that, w.p. 1, $[b, \infty]$ is the set of limit points of $\{n^{-1/\alpha}S_n\}$. This completes the proof of Theorem 2.1 Part c.

Part b has already been proved in the case $\lim_{x\to\infty} L(x) < \infty$. In the case $\lim_{x\to\infty} L(x) = \infty$, Part b follows from the proof of the second assertion of Part c. Finally, Part a follows from the proof of the first assertion of Part c. □

We conclude this section with two examples.

EXAMPLE 1. (See also Examples 1 and 5 in Section 1.)
$L(x) = (2 \log_2 x)^\beta$ for x sufficiently large. Then

$$\begin{aligned} I(L, k) &< \infty \quad \text{for all} \quad \beta > 1 - \alpha \quad \text{or} \quad \beta = 1 - \alpha \quad \text{and} \quad k > 1/2 \\ &= \infty \quad \text{for all} \quad 0 < \beta < 1 - \alpha \quad \text{or} \quad \beta = 1 - \alpha \quad \text{and} \quad k \le 1/2. \\ B(L, \alpha) &= [0, \infty] \qquad \text{for} \quad 0 < \beta < 1 - \alpha \\ &= [b, \infty] \qquad \text{for some} \quad b > 0 \quad \text{if} \quad \beta = 1 - \alpha \\ &= \{\infty\} \qquad \text{for} \quad \beta > 1 - \alpha. \end{aligned}$$

For $\beta < 0$ see Example 1 in Section 3.

EXAMPLE 2. (Compare with a remark after Theorem 3.1 on page 1117 in Wichura [10].)

$$L(x) = \exp(\log x/\log_4 x) \quad \text{for } x \text{ sufficiently large.}$$

This slowly varying function does not satisfy the condition in Example 4 of Section 1. $I(L, k) < \infty$ for all $k > 0$ and therefore $B(L, \alpha) = \{\infty\}$.

3. L non-increasing. In this section we use results of stable processes which are related with the heavy tail of the completely asymmetric stable distribution. Now there are only two possible sets of limit points. Define the ingegral I_2 by

$$I_2(L) = \int^\infty x^{-1}L(x)\,dx. \tag{3.1}$$

Define the function ψ by

$$\psi(x) = \{L(x)\}^{-1/\alpha}; \tag{3.2}$$

then we can rewrite

$$I_2(L) = \int^\infty x^{-1}\psi^{-\alpha}(x)\,dx.$$

This integral occurs in the generalized law of the iterated logarithm for the heavy tails of random variables with a stable distribution. (See Chapter 8 in [7] for stable distributions or [3] for random variables in the domain of attraction of stable distributions.)

In this section we shall prove the following theorem.

THEOREM 3.1. *Let $X_1, X_2, \cdots$ be a sequence of positive* i.i.d. *random variables with common distribution function G, given by* (1.2). *Let $B(L, \alpha)$ be the (non-random) set of limit points of $\{n^{-1/\alpha}S_n\}$. Suppose L is non-increasing. Then*

a. $B(L, \alpha) = \{0\}$ if $I_2(L) < \infty$

b. $B(L, \alpha) = [0, \infty]$ if $I_2(L) = \infty$.

REMARK. The assertions of the theorem above are equivalent with the following two assertions.

For all $b \in (0, \infty)$, $P(S_n \geq bn^{1/\alpha}$ i.o.$) = 0$ if $I_2(L) < \infty$.

For all $b \in (0, \infty)$ and $\varepsilon > 0$, $P(S_n n^{-1/\alpha} \in (b - \varepsilon, b + \varepsilon)$ i.o.$) = 1$ if $I_2(L) = \infty$.

We define the sequences b_n and c_n as in Section 2. In the proof of Theorem 3.1 we make use of the following lemmas.

LEMMA 3.2. *Let t be a continuous and non-increasing real function. Define*

$$C_n = \{(x, y): x \geq t(y), y \in (b_n, c_n)\} \cup \{(x, y): x \geq t(b_n), 0 \leq y \leq b_n\} \cup \{(x, y): x \geq t(c_n), y \geq c_n\}. \tag{3.3}$$

Define $T_n(\omega)$ by (2.17). *Then for $y \in [b_n, c_n]$ we have*

$$P[X(t(b_n)) \leq b_n] + \int_{b_n}^{y} f_{X(t(z))}(z)\, dz \leq P[X(T_n) \leq y], \tag{3.4}$$

where $f_{X(t(z))}$ is the density of the random variable $X(t(z))$.

The proof of this lemma follows the same lines as the proof of Lemma 2.3. We omit the proof.

LEMMA 3.3. *Let G, $X(t)$, b_n and c_n be given as in Lemma 2.4. Then there exist two sequences of stopping times $\{T_{1,n}\}$ and $\{T_{2,n}\}$ such that, for sufficiently large n and $y \in [b_n, c_n]$, we have*

$$P(X(T_{2,n}) \leq y) \leq G(y) \leq P(X(T_{1,n}) \leq y) \tag{3.5}$$

and, for $n \to \infty$,

$$(1 - \Delta)A_1^{-1}n^{-1}L^{-1}(b_n)\{T_{1,1} + \cdots + T_{1,n}\} \to 1 \quad \text{a.s.} \tag{3.6}$$

and

$$\textstyle\sum_{k=1}^{n} T_{2,k} \sim (1 + \Delta)A_1^{-1}\, nL(b_n). \tag{3.7}$$

PROOF. The proof is essentially the same as the proof of Lemma 2.4. We only mention the points of difference. We define $T_{2,n}$, $n = 1, 2, \cdots$, by

$$T_{2,n}(\omega) = (1 + \Delta)A_1^{-1}L(b_n) \tag{3.8}$$

for all ω. Now the stopping times $T_{1,n}$, $n = 1, 2, \cdots$, are random. In the case where the function c in representation (2.20) is constant we define the function t_n by

$$\begin{aligned} t_n(y) &= (1 - \Delta)A_1^{-1}L(b_n) \quad \text{for} \quad 0 \leq y \leq b_n \\ &= (1 - \Delta)A_1^{-1}L(y) \quad \text{for} \quad b_n \leq y \leq c_n \\ &= (1 - \Delta)A_1^{-1}L(c_n) \quad \text{for} \quad y > c_n, \end{aligned} \tag{3.9}$$

where Δ is some positive constant. In the general case we proceed in a similar way as in the proof of Lemma 2.4. □

LEMMA 3.4. *Let S_n^* be defined by* (2.34). *Suppose* $\lim_{x\to\infty} L(x) = 0$. *Take* $b > 0$. *Let the function* ψ *be defined by* (3.2).

a. *There exists a positive constant* k_1 *such that*

$$P(S_n^* \geq bn^{1/\alpha}) \leq k_1 L(b_n) = k_1 \psi^{-\alpha}(b_n).$$

b. *For* $\delta > 0$ *there exists a positive constant* $k_2 = k_2(b, \delta)$ *such that*

$$P(bn^{1/\alpha}(1-\delta) \leq S_n^* \leq bn^{1/\alpha}(1+\delta)) \geq k_2 L(b_n) + r_n,$$

where $r_n = o(n^{-3/2})$ *for* $n \to \infty$.

PROOF. Part a. Similarly as in the proof of Lemma 2.5, we can show that

$$P[X(T_{2,n}) \leq y] \leq F_n^*(y) \qquad \text{for all} \quad y \in (0, \infty)$$

for n sufficiently large. The assertion easily follows from (3.7), the scaling property of stable processes and the expansion of the heavy tail of a stable distribution as given in (2.9).

The assertion in Part b can be proved following the same lines as the proof of Lemma 2.5 Part b. □

LEMMA 3.5. *Let* $\{b_n\}$ *satisfy* $b_n = n^{1/\alpha}\ell(n)$, $n = 1, 2, \cdots$, *where* ℓ *is a slowly varying function. Take* $m > 1$. *Let* n_k *denote the largest integer smaller than* m^k, $k = 1, 2, \cdots$. *Then*

$$I_2(L) < \infty \quad \text{iff} \quad \textstyle\sum_{k=1}^{\infty} L(b_{n_k}) < \infty.$$

PROOF. Since L is non-increasing we have, for some constants c_1 and c_2,

$$\alpha^{-1}c_1 \log m \textstyle\sum_{k=1}^{\infty} L(b_{n_k}) \leq \sum_{k=1}^{\infty} L(b_{n_k}) \displaystyle\int_{b_{n_{k-1}}}^{b_{n_k}} x^{-1}\,dx \leq \textstyle\sum_{k=1}^{\infty} \displaystyle\int_{b_{n_{k-1}}}^{b_{n_k}} x^{-1}L(x)\,dx$$

$$= I_2(L) \leq \textstyle\sum_{k=1}^{\infty} L(b_{n_{k-1}}) \displaystyle\int_{b_{n_{k-1}}}^{b_{n_k}} x^{-1}\,dx \leq \alpha^{-1}c_2 \log m \textstyle\sum_{k=1}^{\infty} L(b_{n_{k-1}}). \ \square$$

PROOF OF THEOREM 3.1. Suppose $\lim_{x\to\infty} L(x) > 0$. Then $I_2(L) = \infty$. Again (2.41) holds. As in the proof of Theorem 2.1 it follows that $B(L, \alpha) = [0, \infty]$. From now on we shall suppose $\lim_{x\to\infty} L(x) = 0$. Suppose $I_2(L) < \infty$. Define, for $n = 1, 2, \cdots$, the events

$$A_n : S_n^* > an^{1/\alpha},$$

where a is some positive real number. It follows that $P(A_n \text{ i.o.}) = 0$. This implies $\lim_{n\to\infty} S_n^* n^{-1/\alpha} = \lim\sup_{n\to\infty} S_n^* n^{-1/\alpha} = 0$ a.s. Because $\{S_n n^{-1/\alpha}\}$ and $\{S_n^* n^{-1/\alpha}\}$ have the same limit points, we have proved Part a of the theorem.

Suppose $I_2(L) = \infty$ and $\lim_{x\to\infty} L(x) = 0$. Take $b > 0$ and $\delta > 0$. Define, for $n = 1, 2, \cdots$, the events D_n by

$$D_n : b(1-\delta)n^{1/\alpha} \leq S_n \leq b(1+\delta)n^{1/\alpha}.$$

It easily follows from Lemma 3.5 and the monotony of L that

$$\textstyle\sum_{n=1}^{\infty} n^{-1}P(D_n) = \infty.$$

By Corollary 2 of [6] we have $b \in B(L, \alpha)$. Since this holds for all $b > 0$, we have $B(L, \alpha) = [0, \infty]$. □

EXAMPLE 1.

$$L(x) = (2\log_2 x)^{\beta}, \quad \beta < 0.$$

$$I_2(L) = \infty \qquad \text{for all} \quad \beta. \quad B(L, \alpha) = [0, \infty].$$

EXAMPLE 2.

$$L(x) = (\log x)^{\beta}, \qquad \beta < 0.$$

$$I_2(L) < \infty \quad \text{for} \quad \beta < -1$$
$$= \infty \quad \text{for} \quad -1 \le \beta < 0.$$

$$B(L, \alpha) = \{0\} \qquad \text{for} \quad \beta < -1$$
$$= [0, \infty] \quad \text{for} \quad -1 \le \beta < 0.$$

EXAMPLE 3.

$$L(x) = \exp(-\log x/\log_4 x) \qquad \text{for } x \text{ sufficiently large.}$$

$$I_2(L) < \infty \quad \text{and} \quad B(\mathscr{L}, \alpha) = \{0\}.$$

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