Theodore E. Harris, *The Theory of Branching Processes*. Springer, Berlin, 1963. DM 36 xiv + 230 pp.; also Prentice-Hall, Englewood Cliffs, New Jersey, \$9.00. Vol. 119 of Die Grundlehren der Mathematischen Wissenschaften.

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This is a meticulously written, well organised, account of an important branch of probability theory to which the author has been one of the major contributors. It is a book intended primarily for mathematicians, and I am inclined to agree with the author's hope that most of Chapters I, II, V, can be mastered by anyone with a knowledge of W. Feller's *Probability Theory* and Parzen's *Modern Probability Theory*. The remainder of the book, however, demands some knowledge of measure-theoretic probability.

As stated by the author a branching process may be regarded as "a mathematical representation of the development of a population whose members reproduce and die, subject to the laws of chance. The objects may be of different types, depending on their age, energy, position, or other factors. However, they must not interfere with one another". It is the last condition which unifies this book. Chapter I is devoted to a detailed examination of the fundamental Galton-Watson process $\{Z_n\}$, under the assumption that EZ_1 is finite, and that a particular object has a non-zero probability of producing two or more offspring. The problem of extinction is treated rigorously and thoroughly but I think the section on stationary measures would have been improved if some motivation such as is contained on pp. 114–115 of the author's own paper "Stationary Measures for Markov Processes" (*Third Berkeley Symposium*) had preceded the mathematical analysis.

It has also been pointed out (J. Lamperti, private communication) that p. 22 of this chapter contains a slight error. In 10.4 the result actually proved is that under the conditions of Theorem 10.1, the conditional characteristic function of $2Zn/nf''(1)$, given $Z_N \neq 0$ approaches a limit $l(n)$ as $N \to \infty$. Letting n approach ∞, $l(n)$ can be shown to approach $(1 - it)^{-2}$ which is the characteristic function of the distribution having the density ue^{-u} for $u > 0$.

A straightforward generalisation of the Galton-Watson model to cover the case where the offspring of an object may be of various types is considered in Chapter II. Once again the author deals only with the discrete parameter process and further assumes that the expected number of offspring of type i produced by an object of type j is finite for all i, j.

Chapter III, possibly the most difficult in the book, formulates rigorously the general branching process in which objects are described by continuous variables. The opening pages contain an elegant and readable account of some results about random point distributions, and their "moment generating functionals". A moment generating functional as defined here is just the generating functional of Bochner's *Harmonic Analysis and the Theory of Probability*. The general branching process is defined in terms of these point distributions but a more extensive use of mathematical symbols would, I think, make for greater clarity.

In Chapter IV, the theory of general branching processes as developed in Chapter III, is applied to the problem of the transport and multiplication of neutrons in matter. Here, as everywhere else in the book, the explanation of the physical problem to the uninitiated is refreshingly lucid. I have been asked by the author to mention that the reference to Mullikin on p. 87 of this chapter is not as stated, but is to Mullikin's "Neutron Branching Processes" (*J. Math. Anal. Appl.*, **3** (1961) pp. 507–525).

The author defines a Markov branching process to be a continuous parametered Markov chain with the non-negative integers as states and with transition probabilities $\{P_{ik}(\tau, t)\}$ satisfying a particular form of the forward equation. He shows rigorously that there exists a unique non-negative, absolutely continuous solution $\{P_{ik}(\tau, t)\}$ satisfying $\sum_k P_{ik} \leqq 1$, and this satisfies the basic branching process relation

$$P_{ik} = \sum_{r_1+\cdots+r_i=k} P_{1r_1}P_{1r_2} \cdots P_{1r_i} .$$

Chapter V is devoted to these processes, though most of the work assumes homogeneity in time. Such processes correspond to family trees in which an object has exponential life length density be^{-bt}.

The author studies age dependent branching process in Chapter VI by means of "family histories". I found the definition on p. 123 to be vague; for example, I failed at first to see any reason behind the ordering $(0, 1, 2, 11, 3, 21, 12, 111, \cdots)$ and I think it would have been better if Definition 2.2. had been framed in more general terms rather than by means of an example. This is a slight point and the author is to be congratulated on making readable a chapter which has very difficult notational problems. The proof of Theorem 17.3 in the Springer edition of this book contains an error, the correction of which strengthens the theorem slightly.

On p. 144, second line following (17.6), change (b) to (a) and insert the following after $(0, \infty)$: "and $e^{-\alpha t}(1 - G(t))$ is exhibited as the difference of two monotone functions which, from the condition of the theorem, are integrable on $(0, \infty)$". The last sentence of the italics at the top of p. 153 should be deleted. On p. 161 of the appendix on results concerning the renewal equation, the part of Lemma 2 preceding (5) should read as follows *Lemma 2:* Suppose that G is not a lattice distribution and that α exists. Suppose also that the following condition holds: (a) $f(t)e^{-\alpha t}$ is the difference of two bounded non-increasing, functions, each of which is integrable on $(0, \infty)$, and $\int_0^\infty te^{-\alpha t}\, dG(t) < \infty$. Then $K(t) \cdots$.

Also the last sentence of the proof of Lemma 2 should be deleted and replaced by "Smith's $Q(t)$ should be assumed bounded". These corrections enable the condition $\int_0^\infty t^2e^{-\alpha t}\, dG(t) < \infty$ in the statement of Theorem 17.3. to be weakened to $\int_0^\infty te^{-\alpha t}\, dG(t) < \infty$. These corrections were realised by the author after the Springer edition had been published and have been corrected in the Prentice-Hall edition.

In Chapter VII the author shows the use of branching processes in the theory of cosmic rays. He emphasises that no attempt is made to describe the many different lines of attack that have been proposed for electron photon cascades, and merely tries to formulate probabilistically two of the more important models. This he does clearly, and wisely makes no attempt (other than by references) to justify these approximations. This chapter will I think awaken an interest in physical problems among many of the non-physical mathematicians who read it. Admittedly, little is known about the estimation of parameters of a branching process. Nevertheless, the importance of the estimation problem for stochastic process in general is such that proofs of some of the few statements relating to this topic and an extension of the two estimation sections giving further details of the contents of the quoted references would be welcome.

To sum up, Dr. Harris' book is an up to date, well written account of a subject in which modern references are widely scattered. For a book of this nature the number of minor misprints is exceedingly small, and I think many will find it a boon for reference purposes and as a guide to the use of various techniques. It should prove to be an admirable book for graduate study, for although it contains no examples the author has in many cases left proofs, with or without hints, to the reader. A systematic study of this book should improve a student's mathematical technique enormously.

The price of the book is $9.00, but the Prentice-Hall edition has a special academic price of $6.75, and in it most of the minor misprints and the proof of Theorem 17.3 are corrected.