Electronic Journal of Statistics
Vol. 8 (2014) 998–1028
ISSN: 1935-7524
DOI: 10.1214/14-EJS918

Estimation of the shift parameter in regression models with unknown distribution of the observations

Philippe Fraysse

Université Bordeaux 1, Institut de Mathématiques de Bordeaux, UMR CNRS 5251, 351 cours de la libération, 33405 Talence cedex, France
e-mail: philippe.fraysse@math.u-bordeaux1.fr

Abstract: This paper is devoted to the estimation of the shift parameter in a semiparametric regression model when the distribution of the observation times is unknown. Hence, we propose to use a stochastic algorithm which takes into account the estimation of the distribution of the observation times. We establish the almost sure convergence of our estimator and the asymptotic normality. The main result of the paper is that, with little assumptions on the regularity of the regression function, the asymptotic variance obtained is the same as when the distribution is known. In that sense, we improve the recent work of Bercu and Fraysse [1].

MSC 2010 subject classifications: Primary 62G05; secondary 62G20.
Keywords and phrases: estimation of the shift parameter, asymptotic properties.

Received December 2013.

Contents

1. Introduction

We propose to study the problem of the estimation of the shift parameter θ in the semi parametric regression model defined, for all $n \geq 0$, by

$$Y_n = f(X_n - \theta) + \varepsilon_n, \tag{1.1}$$

where (X_n) and (Y_n) are observed and where (X_n) and (ε_n) are two independent sequences of independent and identically distributed random variables. Model (1.1) belongs to the family of shape invariant models introduced by Lawton *et al.* [9]. One can find studies of that kind of models in the papers of Dalalyan *et al.* [5], of Gamboa *et al.* [7] or Vimond [15], whereas Castillo and Loubès [4] and Trigano *et al.* [13] were interested in such a model when the parameter θ is random. Recent advances on the subject have also provided by Bigot and Charlier [2] and Bigot and Gendre [3].

Contrary to all the papers quoted previously, we are dealing with random observation times (X_n) and we assume that their distribution is unknown. Our goal is the estimation of θ in that case. More precisely, we propose to generalize the work of Bercu and Fraysse [1] when the distribution of (X_n) is assumed to be known. Hence, the motivation of the paper is mainly theoretical. It consists in comparing the estimation of θ when g is known and when g is unknown. In particular, we want to know if we keep the same asymptotic properties for the estimation of θ when g is unknown than when g is known. For that, we implement a stochastic algorithm in order to estimate the unknown parameter θ without any preliminary evaluation of the regression function f. When the distribution of (X_n) is known, Bercu and Fraysse propose to use the algorithm similar to that of Robbins-Monro [12], defined, for all $n \geq 0$, by

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n + \gamma_n T_{n+1} \tag{1.2}$$

where (γ_n) is a positive sequence of real numbers decreasing towards zero and (T_n) is a sequence of random variables such that $\mathbb{E}[T_{n+1}|\mathcal{F}_n] = \phi(\widehat{\theta}_n)$ where $\mathcal{F}_n$ stands for the σ-algebra of the events occurring up to time n. References on algorithm (1.2) can be found in [1]. Nevertheless, the expression of T_{n+1} depends on the distribution of (X_n). To overcome this problem, we propose to replace the algorithm given by (1.2) by the one defined, for all $n \geq 0$, by

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \gamma_n \widehat{T}_{n+1}, \tag{1.3}$$

where $\widehat{T}_{n+1}$ depends only on an estimator of the distribution of (X_n) which will be explicited in the sequel. In particular, we no longer have $\mathbb{E}[\widehat{T}_{n+1}|\mathcal{F}_n] = \phi(\widehat{\theta}_n)$. Algorithms of the form (1.3) have been studied by Pelletier [10], [11] where the author establishes convergence results under the hypothesis that $(\widehat{T}_{n+1} - T_{n+1})^2 = o_{\mathbb{P}}(\gamma_n)$. Nevertheless, in our situation, such an hypothesis is not verified and we can not apply this kind of convergence results. However, we get to answer to our issue: effectively, we keep the almost sure convergence of our estimator of θ, but we also keep the asymptotic normality of our estimator with the same asymptotic variance as the one obtained in [1] when g is known. Hence, from a theoretical point of view, we really improve the results of [1].

The paper is organized as follows. Section 2 is devoted to the explanation of the estimation procedure of θ. We establish the almost sure convergence of $\widehat{\theta}_n$ as well as the asymptotic normality under some little assumptions on the regularity of f. In particular, we establish that the asymptotic variance is the same as the one obtained in the paper [1], that is to say the estimation of the distribution of (X_n) does not disturb the asymptotic behaviour of $\widehat{\theta}_n$. The proofs of the results are given is Section 3.

2. Estimation procedure and main results

We focus our attention on the estimation of the shift parameter θ in the semi-parametric regression model given by (1.1). We assume that (ε_n) is a sequence of independent and identically distributed random variables with zero mean and unknown positive variance σ^2. Moreover, we add the two several hypothesis similar to that of [1].

($\mathcal{H}_1$) The observation times (X_n) are independent and identically distributed with unknown probability density function g, positive on its support $[-1/2, 1/2]$. In addition, g is continuous, twice differentiable with bounded derivatives. We denote by $C_g > 0$ the minimum of g on $[-1/2, 1/2]$.

($\mathcal{H}_2$) The shape function f is symmetric, bounded, periodic with period 1.

When $|\theta| < 1/4$ and when the density g is known, Bercu and Fraysse [1] propose to use the algorithm defined, for all $n \geq 0$, by

$$\widetilde{\theta}_{n+1} = \pi_C\Big(\widetilde{\theta}_n + \mathrm{sign}(f_1)\gamma_{n+1}T_{n+1}\Big), \tag{2.1}$$

where the initial value $\widetilde{\theta}_0 \in C$ and the random variable T_{n+1} is defined by

$$T_{n+1} = \frac{\sin(2\pi(X_{n+1} - \widetilde{\theta}_n))}{g(X_{n+1})} Y_{n+1}.$$

We recall that π_C is the projection on the compact set $C = [-1/4; 1/4]$ defined, for all $x \in \mathbb{R}$, by

$$\pi_C(x) = \begin{cases} x & \text{if } \ |x| \leq 1/4, \\ 1/4 & \text{if } \ x \geq 1/4, \\ -1/4 & \text{if } \ x \leq -1/4. \end{cases}$$

Moreover, we denote by the first Fourier coefficient of f

$$f_1 = \int_{-1/2}^{1/2} \cos(2\pi x) f(x)\, dx,$$

and we define the function ϕ, for all $x \in \mathbb{R}$, by

$$\phi(x) = \sin\left(2\pi\left(\theta - x\right)\right) f_1. \tag{2.2}$$

Finally, (γ_n) is a decreasing sequence of positive real numbers satisfying

$$\sum_{n=1}^{\infty} \gamma_n = +\infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \gamma_n^2 < +\infty. \tag{2.3}$$

When the density g is unknown, it is not possible to use algorithm (2.1). The idea is to replace g in the expression of (2.1) by an estimator of g. More precisely, we study the algorithm defined, for all $n \geq 0$, by

$$\widehat{\theta}_{n+1} = \pi_C\left(\widehat{\theta}_n + \operatorname{sign}(f_1)\gamma_{n+1}\widehat{T}_{n+1}\right), \tag{2.4}$$

with

$$\widehat{T}_{n+1} = \frac{\sin(2\pi(X_{n+1} - \widehat{\theta}_n))}{\widehat{g}_n(X_{n+1})} Y_{n+1}. \tag{2.5}$$

where $\widehat{g}_n$ is the recursive Parzen-Rosenblatt kernel estimator of g (see [14], [16] and [17] for references) defined, for all $x \in [-1/2; 1/2]$ and for all $n \geq 0$, by

$$\widehat{g}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_i} K\left(\frac{X_i - x}{h_i}\right). \tag{2.6}$$

and where the kernel K is a symmetric density of probability function, positive, with compact support and with

$$\int_{-\infty}^{+\infty} K^2(x)dx = \mu^2 < +\infty \qquad \text{and} \qquad \frac{1}{2}\int_{-\infty}^{+\infty} x^2 K(x)dx = \nu^2 < +\infty.$$

All the results which follow are based on the following lemma.

Lemma 2.1. *If $h_n = n^{-\alpha}$ with $0 < \alpha < 1/2$, then, for all β such that $(1 + \alpha)/2 < \beta < 1$,*

$$\sup_{|x| \leq 1/2} |\widehat{g}_n(x) - g(x)| = \mathcal{O}\left(n^{-2\alpha} + n^{\beta - 1}\right) \qquad \text{a.s.}$$

Proof. The proof is given in Section 4. □

In the sequel, for sake of clarity, we choose $h_n = n^{-\alpha}$ with $0 < \alpha < 1$ and $\gamma_n = 1/n$. Our first result concerns the almost sure convergence of the estimator $\widehat{\theta}_n$.

Theorem 2.1. *Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold and that $|\theta| < 1/4$. Then, if K is a Lipschitz function, for all $0 < \alpha < 1/2$, $\widehat{\theta}_n$ converges almost surely to θ.*

In addition, the number of times that the random variable $\widehat{\theta}_n + sign(f_1)\gamma_{n+1}\widehat{T}_{n+1}$ goes outside of C is almost surely finite.

Proof. The proof is given in Section 4. □

Remark 2.1. At first sight, the estimation procedure needs the knowledge of the sign of f_1. However, it is possible to do without. Indeed, denote by $(\widehat{\theta}_n^+)$ the sequence defined, for all $n \geq 1$, by

$$\widehat{\theta}_{n+1}^+ = \pi_C\Big(\widehat{\theta}_n^+ + \gamma_{n+1}\widehat{T}_{n+1}^+\Big),$$

where

$$\widehat{T}_{n+1}^+ = \frac{\sin(2\pi(X_{n+1} - \widehat{\theta}_n^+))}{\widehat{g}_n(X_{n+1})}Y_{n+1},$$

and by $(\widehat{\theta}_n^-)$ the sequence defined, for all $n \geq 1$, by

$$\widehat{\theta}_{n+1}^- = \pi_C\Big(\widehat{\theta}_n^- - \gamma_{n+1}\widehat{T}_{n+1}^-\Big),$$

where

$$\widehat{T}_{n+1}^- = \frac{\sin(2\pi(X_{n+1} - \widehat{\theta}_n^-))}{\widehat{g}_n(X_{n+1})}Y_{n+1}.$$

Then, two events are possible. More precisely, one have

$$\lim_{n\to+\infty}\widehat{\theta}_n^+ = \theta \qquad \text{and} \qquad \lim_{n\to+\infty}|\widehat{\theta}_n^-| = 1/4 \quad \text{a.s.}$$

or

$$\lim_{n\to+\infty}\widehat{\theta}_n^- = \theta \qquad \text{and} \qquad \lim_{n\to+\infty}|\widehat{\theta}_n^+| = 1/4 \quad \text{a.s.}$$

Hence, for n large enough, the value $\widehat{\theta}_n$ which is considered is the value given by $\min(|\widehat{\theta}_n^+|, |\widehat{\theta}_n^-|)$. Nevertheless, for the sake of clarity, we shall do as if the sign of f_1 is known.

Before establishing the asymptotic normality of $\widehat{\theta}_n$, we need the following lemma on the mean square error of $\widehat{\theta}_n$.

Lemma 2.2. *Let $m \geq 0$ and $\varepsilon > 0$ such that $C_g > \varepsilon$ and define*

$$\tau_m = \inf\left\{n \geq m \ : \ |\widehat{\theta}_n - \theta| \geq \varepsilon \quad \text{or} \quad \inf_{x\in[-1/2;1/2]}\widehat{g}_n(x) \leq C_g - \varepsilon\right\}. \tag{2.7}$$

Suppose that $4\pi|f_1| > 1$, then, for $0 < \alpha < 1/2$,

$$\mathbb{E}\left[\left(\widehat{\theta}_n - \theta\right)^2 \mathrm{I}_{\{\tau_m=+\infty\}}\right] = \mathcal{O}\left(n^{-2\alpha} + n^{\frac{\alpha-1}{2}}\right).$$

Proof. The proof is given in Section 4. □

In order to establish the asymptotic normality of $\widehat{\theta}_n$, it is necessary to introduce a second auxiliary function φ defined, for all $t \in \mathbb{R}$, by

$$\begin{aligned}\varphi(t) &= \mathbb{E}\Big[\frac{\sin^2(2\pi(X-t))}{g^2(X)}(f^2(X-\theta)+\sigma^2)\Big], \qquad (2.8)\\ &= \int_{-1/2}^{1/2}\frac{\sin^2(2\pi(x-t))}{g(x)}(f^2(x-\theta)+\sigma^2)\,dx.\end{aligned}$$

As soon as $4\pi|f_1| > 1$, denote

$$\xi^2(\theta) = \frac{\varphi(\theta)}{4\pi|f_1| - 1}. \tag{2.9}$$

Moreover, we need to add the following hypothesis on the regularity of f.

$(\mathcal{H}_3)$ The shape function f is twice differentiable with bounded derivatives.

Theorem 2.2. *Assume that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *hold and that* $|\theta| < 1/4$. *Moreover, suppose that* (ε_n) *has a finite moment of order* > 2 *and that* $4\pi|f_1| > 1$. *Then, if* K *is a Lipschitz function, we have the asymptotic normality, for* $1/4 < \alpha < 1/2$,

$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \xi^2(\theta)). \tag{2.10}$$

Proof. The proof is given in Section 4. □

Remark 2.2. We could expect that asymptotic variance obtained in Theorem 2.2 be larger than asymptotic variance obtained when the density g is supposed to be known. Nevertheless, comparing the result obtained here and the one obtained in Theorem 2.2 of [1], we point out that the asymptotic variances are exactly the same. Hence, with a little assumption on the regularity of f, estimation of the density g does not disturb the estimation of θ. In that sense, the estimation procedure of θ by algorithm (2.4) improves the one proposed in [1].

Remark 2.3. In the particular case where $4\pi|f_1| = 1$, it is may be also possible to show [6] that

$$\sqrt{\frac{n}{\log(n)}}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \varphi(\theta)).$$

Asymptotic results are also available when $0 < 4\pi|f_1| < 1$. However, we have chosen to focus our attention on the more attractive case $4\pi|f_1| > 1$.

3. Numerical illustrations

3.1. Numerical comparison when g is known with f smooth

In this section, we propose to compare the numerical asymptotic behavior of the estimator $\widetilde{\theta}_n$ defined by (2.1) and the one of the new estimator $\widehat{\theta}_n$ given by (2.4). For that, we consider data X_n and Y_n generating according to the model given, for all $n \geq 1$, by

$$Y_n = f(X_n - \theta) + \varepsilon_n$$

where $\theta = 0.05$ and (ε_n) is a sequence of independent random variables of law $\mathcal{N}(0, 1)$. In this part, we are going to compare the numerical performances of the two estimators with a function f very smooth and given by

$$\forall x \in \mathbb{R}, \quad f(x) = \sum_{k=1}^{5} \cos(2k\pi x),$$

with $f_1 = 1/2 > 0$. Moreover, the sequence (X_n) is a sequence of independent random variables of law $\mathcal{N}(0,1)$ truncated on the interval $[-1/2;1/2]$ (generated according to an accept-reject algorithm). Hence, the density g of the sequence (X_n) is given by

$$\forall x \in \mathbb{R}, \quad g(x) = \frac{e^{-x^2/2}}{\int_{-1/2}^{1/2} e^{-t^2} dt} \mathbf{1}_{\{x\in[-1/2;1/2]\}}.$$

Thus, the sequence $\widetilde{\theta}_n$ defined by (2.1) can be computed with $\widetilde{\theta}_n \in [-1/4;1/4]$ and for all $n \geq 0$,

$$\begin{cases} \widetilde{\theta}_{n+1} = \pi_{[-1/4;1/4]}\left(\widetilde{\theta}_n + \frac{1}{n}T_{n+1}\right) \\ T_{n+1} = \frac{\sin(2\pi(X_{n+1}-\widetilde{\theta}_n))}{g(X_{n+1})}. \end{cases}$$

In order to compute the sequence $\widehat{\theta}_n$, one have to estimate the density g. For that, we make the choice of taking two different kernels, the gaussian one K_G and the uniform one K_U which are defined, for all $x \in \mathbb{R}$, by

$$K_G(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad \text{and} \qquad K_U(x) = \frac{1}{2}\mathbf{1}_{\{-1/2\leq x\leq 1/2\}}.$$

We also choose the bandwidth $h_n = n^{-1/5}$ where $\alpha = 1/5$ is known to be the optimal choice in kernel estimation. Hence, the sequence $\widehat{\theta}_n^G$ defined by (2.4) with the kernel K_G and the sequence $\widehat{\theta}_n^U$ defined by (2.4) with the kernel K_U can be computed with $\widehat{\theta}_n^G \in [-1/4;1/4]$ and $\widehat{\theta}_n^U \in [-1/4;1/4]$ and for all $n \geq 0$,

$$\begin{cases} \widehat{\theta}_{n+1}^G = \pi_{[-1/4;1/4]}\left(\widehat{\theta}_n^G + \frac{1}{n}\widehat{T}_{n+1}^G\right), \\ \widehat{T}_{n+1}^G = \dfrac{\sin(2\pi(X_{n+1}-\widehat{\theta}_n^G))}{\widehat{g}_n^G(X_{n+1})}, \\ \widehat{g}_n^G(X_{n+1}) = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\frac{1}{h_k}K_G\left(\frac{X_k - X_{n+1}}{h_k}\right), \end{cases}$$

and

$$\begin{cases} \widehat{\theta}_{n+1}^U = \pi_{[-1/4;1/4]}\left(\widehat{\theta}_n^U + \frac{1}{n}\widehat{T}_{n+1}^U\right), \\ \widehat{T}_{n+1}^U = \dfrac{\sin(2\pi(X_{n+1}-\widehat{\theta}_n^U))}{\widehat{g}_n^U(X_{n+1})}, \\ \widehat{g}_n^U(X_{n+1}) = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\frac{1}{h_k}K_U\left(\frac{X_k - X_{n+1}}{h_k}\right). \end{cases}$$

We compute these three algorithms until $n = 500$. The almost sure convergence of the three estimators are represented in Figure 1.

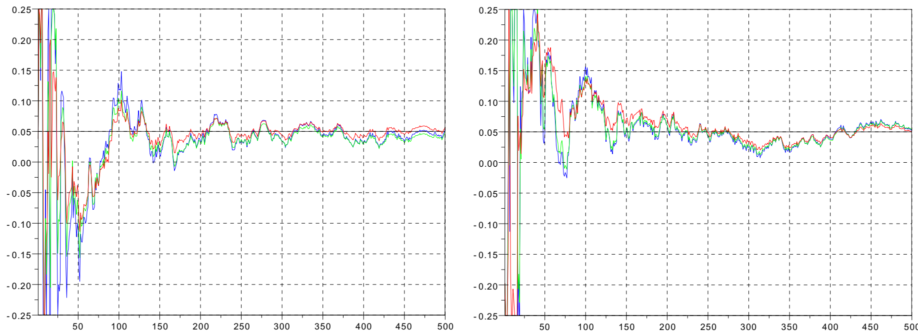

FIG 1. *Almost sure convergence of* $\widehat{\theta}^G_{500}$ *(green),* $\widehat{\theta}^U_{500}$ *(blue) and* $\widetilde{\theta}_{500}$ *(red).*

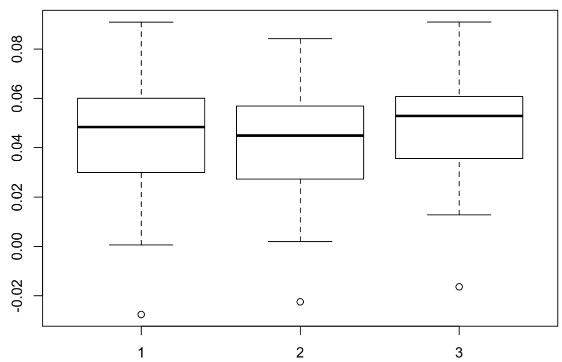

FIG 2. *Boxplots of the* 100 *values obtained for* $\widehat{\theta}^G_{500}$ *(1 on the Figure),* $\widehat{\theta}^U_{500}$ *(2) and* $\widetilde{\theta}_{500}$ *(3).*

We can point out that, if we look at Figure 1, *a priori* the numerical performances of the three estimators are similar. Nevertheless, this comparison is not so convincing. Then, in order to be more precise and to compare the robustness of these three estimates, we compute 100 times these three algorithms until $n = 500$. We have plotted in Figure 2 the boxplots of the 100 values obtained for each estimator. Moreover, the mean of the 100 values for each estimator is 0.0493 for $\widetilde{\theta}$, 0.0448 for $\widehat{\theta}^G$ and 0.0430 for $\widehat{\theta}^U$. Hence, from a practical point of view, it is clear that $\widetilde{\theta}_{500}$ is a better estimate than the two others : it has a mean and a median closer to θ than the two others, and the distance between its first and its third quartile is smaller than for the two others estimates. In addition, we can also say that, on this data set, the numerical performances of $\widehat{\theta}^G$ are a little better than the ones of $\widehat{\theta}^U$. Hence, it is not surprising that the choice of the kernel plays a role in the performance of the estimation.

To conclude, with a very smooth regression function, the numerical performances of each estimate are good and logically, the performances of the estimator using the density g are better. We have also observed that the estimator computed with the gaussian kernel performs a little better than the one computed with the uniform kernel.

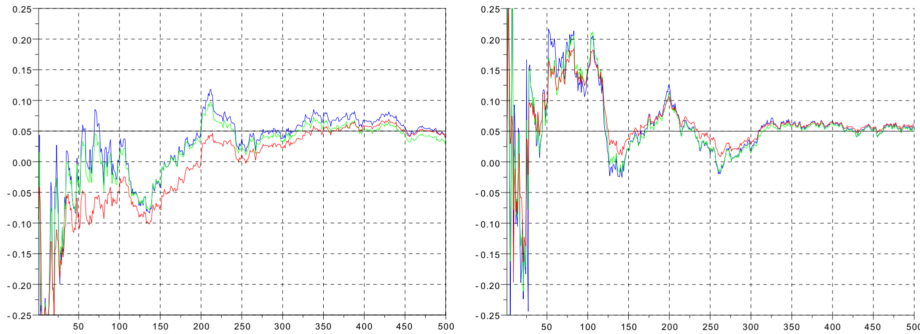

FIG 3. *Almost sure convergence of* $\widehat{\theta}^G_{500}$ *(green),* $\widehat{\theta}^U_{500}$ *(blue) and* $\widetilde{\theta}_{500}$ *(red).*

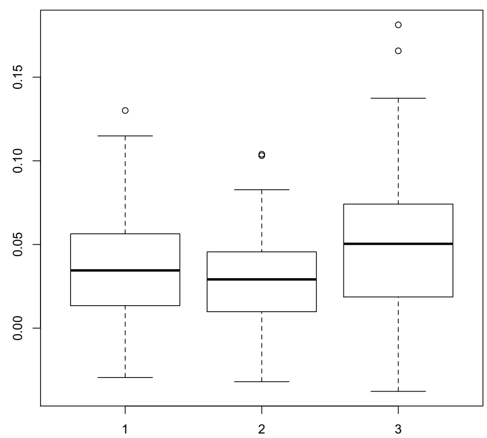

FIG 4. *Boxplots of the* 100 *values obtained for* $\widehat{\theta}^G_{500}$ *(1),* $\widehat{\theta}^U_{500}$ *(2) and* $\widetilde{\theta}_{500}$ *(3).*

3.2. Numerical comparison when g is known with f non smooth

In this part, we propose to compare the numerical asymptotic behavior of the estimator $\widetilde{\theta}_n$ defined by (2.1) and the new estimator proposed in the paper $\widehat{\theta}_n$ given by (2.4) with a function f which is not continuous, with period 1 and defined, for $x \in [-1/2; 1/2[$, by

$$f(x) = \frac{1}{2}\mathbf{1}_{-1/2 \leq x < -1/4} + \mathbf{1}_{-1/4 \leq x < 1/4} + \frac{1}{2}\mathbf{1}_{1/4 \leq x < 1/2},$$

with $f_1 < 0$. We keep every other functions and values of the last part. Only the regression function f is changed. As in the first part, we have plotted in Figure 3 the almost sure convergence of each estimate, and the boxplots of the 100 values obtained for each estimator. In that case, the mean of the 100 values is 0.0492 for $\widetilde{\theta}$, 0.0342 for $\widehat{\theta}^G$ and 0.0286 for $\widehat{\theta}^U$.

Here, comparing Figure 2 and Figure 4, it is clear that the non smooth function f damaged the estimation. However, the numerical performances of the estimate $\widetilde{\theta}_{500}$ is again better than the ones of the two others : the mean and the median are closer to θ than the ones of the two others. Nevertheless, one can observe in Figure 4 than the distance between the first and the third quartile for $\widetilde{\theta}_{500}$ is larger than when the density f were smooth and is also larger than for the two other estimates. One can also point out that the estimates $\widehat{\theta}^{G}_{500}$ and $\widehat{\theta}^{U}_{500}$ underestimate θ.

To conclude, with a regression function which is not continuous, the numerical performances of each estimate are quite good but damaged by the non smooth function.

4. Proofs

4.1. Proof of Lemma 2.1.

For all $x \in [-1/2; 1/2]$, we have

$$\begin{aligned}\widehat{g}_n(x) - g(x) &= \widehat{g}_n(x) - \mathbb{E}\left[\widehat{g}_n(x)\right] + \mathbb{E}\left[\widehat{g}_n(x)\right] - g(x)\\ &= \frac{M_n(x)}{n} + \frac{R_n(x)}{n}\end{aligned} \tag{4.1}$$

where, for all $n \geq 1$ and for all $x \in [-1/2; 1/2]$,

$$M_n(x) = \sum_{k=1}^{n} K_{h_k}\left(X_k - x\right) - \mathbb{E}\left[K_{h_k}\left(X_k - x\right)\right], \tag{4.2}$$

and

$$R_n(x) = \sum_{k=1}^{n} \mathbb{E}\left[K_{h_k}\left(X_k - x\right)\right] - g(x), \tag{4.3}$$

with, for $x \in [-1/2; 1/2]$,

$$K_{h_k}(x) = \frac{1}{h_k} K\left(\frac{x}{h_k}\right).$$

Firstly, $(M_n(x))$ is a square integrable martingale whose increasing process is given, for all $n \geq 1$ and $x \in [-1/2; 1/2]$, by

$$\begin{aligned}\langle M(x)\rangle_n &= \sum_{k=1}^{n} \mathbb{E}\left[K_{h_k}\left(X_k - x\right)^2\right] - \mathbb{E}\left[K_{h_k}\left(X_k - x\right)\right]^2\\ &\leq \sum_{k=1}^{n} \mathbb{E}\left[K_{h_k}\left(X_k - x\right)^2\right].\end{aligned}$$

However, for all $1 \leq k \leq n$,

$$\mathbb{E}\left[K_{h_k}\left(X_k - x\right)^2\right] = \frac{1}{h_k}\int_{\mathbb{R}} K^2\left(y\right) g(x + h_k y) dy.$$

Hence, as g is bounded, we deduce, for all $x \in [-1/2; 1/2]$, that

$$\langle M(x) \rangle_n \leq ||g||_\infty \, \mu^2 \sum_{k=1}^{n} \frac{1}{h_k} \leq ||g||_\infty \, \mu^2 \frac{n}{h_n} \qquad \text{a.s.} \tag{4.4}$$

Moreover, denote, for all $n \geq 1$ and for all $x \in [-1/2; 1/2]$, $\Delta M_n(x) = M_n(x) - M_{n-1}(x)$. Then, we have

$$|\Delta M_n(x)| \leq \frac{2}{h_n} \, ||K||_\infty. \tag{4.5}$$

In particular, with the choice $h_n = 1/n^\alpha$, we infer from (4.4) and from (4.5) that there exists two constants a and b such that

$$\langle M(0) \rangle_n \leq a n^{1+\alpha} \qquad \text{and} \qquad |\Delta M_n(0)| \leq b n^\alpha. \tag{4.6}$$

Moreover, as the kernel K is bounded and Lipschitz, for all $\delta \in \,]0; 1[$, there exists a constant C_δ such that, for all $x, y \in \mathbb{R}$,

$$|K(x) - K(y)| \leq C_\delta \, |x - y|^\delta. \tag{4.7}$$

Thus, for all $x, y \in [-1/2; 1/2]$, one obtain that

$$|\Delta M_n(x) - \Delta M_n(y)| \leq 2 C_\delta \, |x - y|^\delta \, n^{\alpha(1+\delta)}.$$

In addition, for all $x, y \in [-1/2; 1/2]$,

$$\begin{aligned} \langle M(x) - M(y) \rangle_n &\leq \sum_{k=1}^{n} \mathbb{E} \left[\left(K_{h_k}\left(X_k - x\right) \right) - K_{h_k}^2 \left(X_k - y\right) \right) \right)^2 \right], \\ &\leq \sum_{k=1}^{n} k^{2\alpha} \int_{\mathbb{R}} \left(K\left(k^\alpha \left(u - x\right)\right) - K\left(k^\alpha \left(u - y\right)\right) \right)^2 g(u) du. \end{aligned}$$

With the change of variables $t = k^\alpha \, (u - x)$, one then obtain that

$$\langle M(x) - M(y) \rangle_n \leq ||g||_\infty \sum_{k=1}^{n} k^\alpha \int_{\mathbb{R}} \left(K\left(t\right) - K\left(t + k^\alpha (x - y)\right) \right)^2 dt. \tag{4.8}$$

Consequently, we deduce from (4.7) that, for all $1 \leq k \leq n$,

$$\int_{\mathbb{R}} \left(K\left(t\right) - K\left(t + k^\alpha (x - y)\right) \right)^2 dt \leq 2 C_{2\delta} \, |x - y|^{2\delta} \, k^{2\alpha\delta}.$$

Hence, it follows from (4.8) that, for all $x, \ y \in [-1/2; 1/2]$,

$$\langle M(x) - M(y) \rangle_n \leq 2 C_{2\delta} \, |x - y|^{2\delta} \, n^{1+\alpha+2\alpha\delta}.$$

As δ ca be choosen as small as we want, the four conditions of Theorem 6.4.34 page 220 of [6] are satisfactory with $s_{n-1}^2 = n^{1+\alpha}$, that is to say the martingale $(M_n(x))$ checks, for all $(1 + \alpha)/2 < \beta < 1$,

$$\sup_{|x| \leq 1/2} |M_n(x)| = o(n^\beta) \qquad \text{a.s.} \tag{4.9}$$

Finally, it keeps to control the term $R_n(x)$ defined by (4.3). However, for all $x \in [-1/2; 1/2]$ and for all $1 \leq k \leq n$,

$$\begin{aligned}|\mathbb{E}\left[K_{h_k}\left(X_k - x\right)\right] - g(x)| &\leq \int_{\mathbb{R}} K(y)\left|g(x + h_k y)dy - g(x)\right| \\ &\leq h_k^2 \left|\left|g''\right|\right|_\infty \nu^2.\end{aligned}$$

Hence, for $0 < \alpha < 1/2$, one obtain that

$$\sup_{|x|\leq 1/2} \frac{|R_n(x)|}{n} = \mathcal{O}\left(\frac{1}{n}\sum_{k=1}^{n} h_k^2\right) = \mathcal{O}\left(h_n^2\right) \qquad \text{a.s.} \tag{4.10}$$

The conjunction of (4.1), (4.9) and (4.10) leads to, for $0 < \alpha < 1/2$ and for all $(1+\alpha)/2 < \beta < 1$,

$$\sup_{|x|\leq 1/2} |\widehat{g}_n(x) - g(x)| = \mathcal{O}\left(n^{-2\alpha} + n^{\beta-1}\right) \qquad \text{a.s.} \tag{4.11}$$

which concludes the proof. □

4.2. Proof of Theorem 2.1.

Without loss of generality, we suppose that $f_1 > 0$. Denote by $\mathcal{F}_n$ the sigma-algebra $\mathcal{F}_n = \sigma\left(X_0, Y_0, \ldots, X_n, Y_n\right)$. We compute the two first conditional moments of $\widehat{T}_{n+1}$ in order to find suitable upper bounds. On the one hand, for all $n \geq 0$,

$$\mathbb{E}\left[\widehat{T}_{n+1}|\mathcal{F}_n\right] = \mathbb{E}\left[T_{n+1}|\mathcal{F}_n\right] + \mathbb{E}\left[\left(\widehat{T}_{n+1} - T_{n+1}\right)|\mathcal{F}_n\right]. \tag{4.12}$$

where

$$T_{n+1} = \frac{\sin\left(2\pi(X_{n+1} - \widehat{\theta}_n)\right)}{g(X_{n+1})} Y_{n+1}. \tag{4.13}$$

Thanks to (5.2) of [1], we have

$$\mathbb{E}\left[T_{n+1}|\mathcal{F}_n\right] = \phi\left(\widehat{\theta}_n\right) \qquad \text{a.s.,}$$

where the function ϕ is defined by (2.2). Hence, we deduce from (4.12) that

$$\mathbb{E}\left[\widehat{T}_{n+1}|\mathcal{F}_n\right] = \phi\left(\widehat{\theta}_n\right) + \mathbb{E}\left[\left(\widehat{T}_{n+1} - T_{n+1}\right)|\mathcal{F}_n\right] \qquad \text{a.s.} \tag{4.14}$$

On the other hand, for all $n \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\widehat{T}_{n+1}^2|\mathcal{F}_n\right] &= \mathbb{E}\left[\left(\widehat{T}_{n+1} - T_{n+1} + T_{n+1}\right)^2|\mathcal{F}_n\right] \\ &\leq 2\mathbb{E}\left[T_{n+1}^2|\mathcal{F}_n\right] + 2\mathbb{E}\left[\left(\widehat{T}_{n+1} - T_{n+1}\right)^2|\mathcal{F}_n\right].\end{aligned} \tag{4.15}$$

Thanks (5.4) of [1], there exists a constant $M > 0$ such that

$$\sup_{n\geq 0} \mathbb{E}\left[T_{n+1}^2|\mathcal{F}_n\right] \leq M \quad \text{a.s.}$$

Hence, it follows from (4.15) that, for all $n \geq 0$,

$$\mathbb{E}\left[\widehat{T}_{n+1}^2|\mathcal{F}_n\right] \leq 2M + 2\mathbb{E}\left[\left(\widehat{T}_{n+1} - T_{n+1}\right)^2|\mathcal{F}_n\right]. \tag{4.16}$$

Moreover, for all $n \geq 0$, denote $V_n = \left(\widehat{\theta}_n - \theta\right)^2$. Using the fact that π_K is Lipschitz with constant 1, we have, for all $n \geq 0$,

$$\mathbb{E}\left[V_{n+1}|\mathcal{F}_n\right] \leq V_n + \gamma_{n+1}^2\mathbb{E}\left[\widehat{T}_{n+1}^2|\mathcal{F}_n\right] + 2\gamma_{n+1}\left(\widehat{\theta}_n - \theta\right)\mathbb{E}\left[\widehat{T}_{n+1}|\mathcal{F}_n\right]. \tag{4.17}$$

Hence, it follows from (4.14), (4.16) and the previous inequality (4.17), that

$$\mathbb{E}\left[V_{n+1}|\mathcal{F}_n\right] \leq V_n+2\gamma_{n+1}^2\left(M + P_n\right)+2\gamma_{n+1}\left(\widehat{\theta}_n - \theta\right)\phi\left(\widehat{\theta}_n\right)+2\gamma_{n+1}Q_n, \tag{4.18}$$

where

$$P_n = \mathbb{E}\left[\left(\widehat{T}_{n+1} - T_{n+1}\right)^2|\mathcal{F}_n\right] \tag{4.19}$$

and

$$Q_n = \mathbb{E}\left[\left|\widehat{T}_{n+1} - T_{n+1}\right||\mathcal{F}_n\right]. \tag{4.20}$$

However, for all $n \geq 0$,

$$\begin{aligned}\widehat{T}_{n+1} - T_{n+1} &= \sin\left(2\pi(X_{n+1} - \widehat{\theta}_n)\right)Y_{n+1}\left(\frac{1}{\widehat{g}_n(X_{n+1})} - \frac{1}{g(X_{n+1})}\right)\\ &= \frac{\sin\left(2\pi(X_{n+1} - \widehat{\theta}_n)\right)Y_{n+1}}{g(X_{n+1})\widehat{g}_n(X_{n+1})}\left(g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right).\end{aligned}$$

Since g does not vanish on its support, f is bounded and ε_{n+1} is independent of $\mathcal{F}_n$ with finite moment of order 2, we immediately deduce the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}P_n &= \mathbb{E}\left[\frac{\sin^2\left(2\pi(X_{n+1} - \widehat{\theta}_n)\right)Y_{n+1}^2}{g^2(X_{n+1})\widehat{g}_n^2(X_{n+1})}\left(g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right)^2\Big|\mathcal{F}_n\right]\\ &\leq C_1\mathbb{E}\left[\frac{\left(g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right)^2}{\widehat{g}_n^2(X_{n+1})}\Big|\mathcal{F}_n\right],\end{aligned} \tag{4.21}$$

and

$$Q_n \leq \mathbb{E}\left[\frac{\left|\sin\left(2\pi(X_{n+1} - \widehat{\theta}_n)\right)Y_{n+1}\right|}{g(X_{n+1})\widehat{g}_n(X_{n+1})}\left|g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right|\Big|\mathcal{F}_n\right],$$

$$\leq C_2 \mathbb{E}\left[\frac{\left|g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right|}{\widehat{g}_n(X_{n+1})}\Big|\mathcal{F}_n\right]. \tag{4.22}$$

On the one hand,

$$\mathbb{E}\left[\frac{\left(g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right)^2}{\widehat{g}_n^2(X_{n+1})}\Big|\mathcal{F}_n\right] = \int_{-1/2}^{1/2} \left(g(x) - \widehat{g}_n(x)\right)^2 \frac{g(x)}{\widehat{g}_n^2(x)} dx, \tag{4.23}$$

and on the other hand,

$$\mathbb{E}\left[\frac{|g(X_{n+1}) - \widehat{g}_n(X_{n+1})|}{\widehat{g}_n(X_{n+1})}\Big|\mathcal{F}_n\right] = \int_{-1/2}^{1/2} |g(x) - \widehat{g}_n(x)| \frac{g(x)}{\widehat{g}_n(x)} dx. \tag{4.24}$$

Then, it follows from Lemma 2.1 and the two previous calculations (4.23) and (4.24) that, for all $(1+\alpha)/2 < \beta < 1$,

$$\mathbb{E}\left[\frac{\left(g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right)^2}{\widehat{g}_n^2(X_{n+1})}\Big|\mathcal{F}_n\right] = \mathcal{O}\left(\left(n^{-4\alpha} + n^{2(\beta-1)}\right) \int_{-1/2}^{1/2} \frac{g(x)}{\widehat{g}_n^2(x)} dx\right) \quad \text{a.s.,} \tag{4.25}$$

and

$$\mathbb{E}\left[\frac{|g(X_{n+1}) - \widehat{g}_n(X_{n+1})|}{\widehat{g}_n(X_{n+1})}\Big|\mathcal{F}_n\right] = \mathcal{O}\left(\left(n^{-2\alpha} + n^{\beta-1}\right) \int_{-1/2}^{1/2} \frac{g(x)}{\widehat{g}_n(x)} dx\right) \quad \text{a.s.} \tag{4.26}$$

We immediately deduce from (4.21) and (4.22) that

$$P_n = \mathcal{O}\left(\left(n^{-4\alpha} + n^{2(\beta-1)}\right) \int_{-1/2}^{1/2} \frac{g(x)}{\widehat{g}_n^2(x)} dx\right) \quad \text{a.s.} \tag{4.27}$$

and

$$Q_n = \mathcal{O}\left(\left(n^{-2\alpha} + n^{\beta-1}\right) \int_{-1/2}^{1/2} \frac{g(x)}{\widehat{g}_n(x)} dx\right) \quad \text{a.s.} \tag{4.28}$$

Moreover, for all $x \in [-1/2; 1/2]$,

$$\lim_{n \to +\infty} \widehat{g}_n(x) = g(x) \quad \text{a.s.}$$

Since g and $\widehat{g}_n$ are defined on the compact set $[-1/2; 1/2]$, it follows from (4.27) and (4.28) that

$$P_n = \mathcal{O}\left(n^{-4\alpha} + n^{2(\beta-1)}\right) \quad \text{a.s.} \tag{4.29}$$

and

$$Q_n = \mathcal{O}\left(n^{-2\alpha} + n^{\beta-1}\right) \quad \text{a.s.} \tag{4.30}$$

Finally, as $\gamma_n = 1/n$, we infer from the two previous equations that, for all $0 < \alpha < 1$,

$$\sum_{n=0}^{+\infty} \gamma_{n+1}^2 P_n < +\infty \quad \text{a.s.} \tag{4.31}$$

and

$$\sum_{n=0}^{+\infty} \gamma_{n+1} Q_n < +\infty \qquad \text{a.s.} \tag{4.32}$$

To conclude, we deduce from (4.31) and (4.32) together with (4.18) and the Robbins-Siegmund Theorem (see [6] page 18), that the sequence (V_n) converges almost surely to a finite random variable and

$$\sum_{n=1}^{+\infty} \gamma_{n+1} \left(\widehat{\theta}_n - \theta\right) \phi\left(\widehat{\theta}_n\right) < +\infty \qquad \text{a.s.}$$

Following exactly the same lines as proof of Theorem 2.1 of [1] from equation (5.6), we deduce that $(\widehat{\theta}_n)$ converges almost surely to θ and that the number of times that the random variable $\widehat{\theta}_n + \gamma_{n+1}\widehat{T}_{n+1}$ goes outside of C is almost surely finite. $\square$

4.3. Proof of Lemma 2.2

Denote by (W_n) the sequence defined, for all $n \geq 0$, by

$$W_n = \frac{V_n}{\gamma_n}.$$

Then, one deduce from (4.18) and from the choice of step $\gamma_n = 1/n$ that

$$\mathbb{E}\left[W_{n+1}|\mathcal{F}_n\right] \leq W_n\left(1+\gamma_n\right) + 2\gamma_{n+1}\left(M+P_n\right) + 2\left(\widehat{\theta}_n - \theta\right)\phi\left(\widehat{\theta}_n\right) + 2Q_n. \tag{4.33}$$

Moreover, since (4.29), there exists a constant $L > 0$ such that

$$\mathbb{E}\left[W_{n+1}|\mathcal{F}_n\right] \leq W_n\left(1+\gamma_n\right) + \gamma_{n+1}L + 2\left(\widehat{\theta}_n - \theta\right)\phi\left(\widehat{\theta}_n\right) + 2Q_n. \tag{4.34}$$

In addition, we have for all $x \in \mathbb{R}$, $\phi(x) = 2\pi f_1(\theta - x) + f_1(\theta - x)v(x)$ where

$$v(x) = \frac{\sin(2\pi(\theta - x)) - 2\pi(\theta - x)}{(\theta - x)}.$$

By the continuity of the function v, one can find $0 < \varepsilon < 1/2$ such that, if $|x - \theta| < \varepsilon$,

$$\frac{q}{2f_1} < v(x) < 0. \tag{4.35}$$

Hence, it follows from (4.34) that for all $n \geq 1$,

$$\mathbb{E}[W_{n+1}|\mathcal{F}_n] \leq W_n + 2\gamma_n W_n(q - f_1 v(\widehat{\theta}_n)) + \gamma_n L + 2Q_n \tag{4.36}$$

with $2q = 1 - 4\pi f_1$ which means that $q < 0$. Then, it follows from (2.7) and (4.35) that

$$0 < -f_1 v(\widehat{\theta}_n)\mathrm{I}_{\{\tau_m > n\}} < -\left(\frac{q}{2}\right)\mathrm{I}_{\{\tau_m > n\}}. \tag{4.37}$$

Hence, we deduce from the conjunction of (4.36) and (4.37) that,

$$\begin{aligned}
&\mathbb{E}[W_{n+1}\mathrm{I}_{\{\tau_m>n\}}|\mathcal{F}_n] \\
&\quad\le W_n\mathrm{I}_{\{\tau_m>n\}} + 2\gamma_n W_n\mathrm{I}_{\{\tau_m>n\}}\left(q-\frac{q}{2}\right) + \gamma_n L + 2Q_n\mathrm{I}_{\{\tau_m>n\}} \\
&\quad\le W_n\mathrm{I}_{\{\tau_m>n\}}(1+q\gamma_n) + \gamma_n L + 2Q_n\mathrm{I}_{\{\tau_m>n\}}. \qquad (4.38)
\end{aligned}$$

Since $\{\tau_m > n+1\} \subset \{\tau_m > n\}$, we obtain by taking the expectation on both sides of (4.38) that for all $n \ge m$,

$$\mathbb{E}[W_{n+1}\mathrm{I}_{\{\tau_m>n+1\}}] \le (1+q\gamma_n)\mathbb{E}[W_n\mathrm{I}_{\{\tau_m>n\}}] + \gamma_n L + 2\mathbb{E}\left[Q_n\mathrm{I}_{\{\tau_m>n\}}\right]. \qquad (4.39)$$

From now on, denote $\alpha_n = \mathbb{E}[W_n\mathrm{I}_{\{\tau_m>n\}}]$. We infer from (4.39) that for all $n \ge m$,

$$\alpha_{n+1} \le \beta_n\alpha_m + L\beta_n\sum_{k=m}^{n}\frac{\gamma_k}{\beta_k} + 2\beta_n\sum_{k=m}^{n}\frac{1}{\beta_k}\mathbb{E}\left[Q_k\mathrm{I}_{\{\tau_m>k\}}\right] \qquad (4.40)$$

where

$$\beta_n = \prod_{k=m}^{n}(1+q\gamma_k).$$

As $\gamma_n = 1/n$, it follows from straightforward calculations that $\beta_n = O(n^q)$ and

$$\sum_{k=1}^{n}\frac{\gamma_k}{\beta_k} = O(n^{-q}).$$

Moreover, one have from (4.22) and (4.24) that there exists $C_2 > 0$ such that,

$$Q_n\mathrm{I}_{\{\tau_m>n\}} \le C_2\int_{-1/2}^{1/2}|g(x)-\widehat{g}_n(x)|\,\frac{g(x)}{\widehat{g}_n(x)}\mathrm{I}_{\{\tau_m>n\}}dx. \qquad (4.41)$$

However, for all $x \in [-1/2;1/2]$, we have

$$\frac{g(x)}{\widehat{g}_n(x)}\mathrm{I}_{\{\tau_m>n\}} \le \frac{g(x)}{C_g-\varepsilon}.$$

Then, taking expectation on both sides of (4.41), it follows from the previous inequality that

$$\begin{aligned}
\mathbb{E}\left[Q_n\mathrm{I}_{\{\tau_m>n\}}\right] &\le C_2\int_{-1/2}^{1/2}\mathbb{E}\left[|g(x)-\widehat{g}_n(x)|\right]\frac{g(x)}{C_g-\varepsilon}dx, \\
&\le \frac{C_2}{C_g-\varepsilon}\sup_{x\in[-1/2;1/2]}\mathbb{E}\left[|g(x)-\widehat{g}_n(x)|\right]. \qquad (4.42)
\end{aligned}$$

The quantity $\mathbb{E}\left[|g(x)-\widehat{g}_n(x)|\right]$ corresponds to the mean error of the recursive Parzen-Rosenblatt estimator. Hence, it is well-known that for $0 < \alpha < 1/2$,

$$\sup_{x\in[-1/2;1/2]}\mathbb{E}\left[|g(x)-\widehat{g}_n(x)|\right] = \mathcal{O}\left(n^{-2\alpha} + n^{\frac{\alpha-1}{2}}\right).$$

Then, one deduce from (4.42) that, for $0 < \alpha < 1/2$,

$$\mathbb{E}\left[Q_n \mathrm{I}_{\{\tau_m > n\}}\right] = \mathcal{O}\left(n^{-2\alpha} + n^{\frac{\alpha-1}{2}}\right), \tag{4.43}$$

which implies that

$$\beta_n \sum_{k=m}^{n} \frac{1}{\beta_k} \mathbb{E}\left[Q_k \mathrm{I}_{\{\tau_m > k\}}\right] = \mathcal{O}\left(n^{-2\alpha+1} + n^{\frac{\alpha+1}{2}}\right).$$

Thus, (4.40) together the previous equation implies that

$$\alpha_n = \mathcal{O}\left(n^{-2\alpha+1} + n^{\frac{\alpha+1}{2}}\right).$$

Hence, for all $m \geq 0$,

$$\mathbb{E}[W_n \mathrm{I}_{\{\tau_m = +\infty\}}] = \mathcal{O}\left(n^{-2\alpha+1} + n^{\frac{\alpha+1}{2}}\right), \tag{4.44}$$

that is to say, for $0 < \alpha < 1/2$,

$$\mathbb{E}\left[\left(\widehat{\theta}_n - \theta\right)^2 \mathrm{I}_{\{\tau_m = +\infty\}}\right] = \mathcal{O}\left(n^{-2\alpha} + n^{\frac{\alpha-1}{2}}\right).$$

□

4.4. *Proof of Theorem 2.2*

Without loss of generality, we suppose that $f_1 > 0$. We have the decomposition, for all $n \geq 0$,

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \gamma_{n+1}\left(\widehat{T}_{n+1} - \phi\left(\widehat{\theta}_n\right)\right) + \gamma_{n+1}\phi\left(\widehat{\theta}_n\right) + d_{n+1}, \tag{4.45}$$

where

$$d_{n+1} = \pi_K\left(\widehat{\theta}_n + \gamma_{n+1}\widehat{T}_{n+1}\right) - \left(\widehat{\theta}_n + \gamma_{n+1}\widehat{T}_{n+1}\right). \tag{4.46}$$

Moreover, as ϕ is two times differentiable, there exists $0 < \xi_n < 1$ such that

$$\phi\left(\widehat{\theta}_n\right) = \left(\widehat{\theta}_n - \theta\right)\phi'(\theta) + \frac{\left(\widehat{\theta}_n - \theta\right)^2}{2}\phi''\left(\theta + \xi_n\left(\widehat{\theta}_n - \theta\right)\right). \tag{4.47}$$

Then, it follows from (4.45) and (4.47) that, for all $n \geq 0$,

$$\widehat{\theta}_{n+1} - \theta = \alpha_n\left(\widehat{\theta}_n - \theta\right) + \gamma_{n+1}\left(\widehat{T}_{n+1} - \phi\left(\widehat{\theta}_n\right)\right) + \gamma_{n+1} r_n + d_{n+1}, \tag{4.48}$$

where

$$\alpha_n = 1 + \gamma_{n+1}\phi'(\theta), \tag{4.49}$$

and

$$r_n = \frac{\left(\widehat{\theta}_n - \theta\right)^2}{2} \phi''\left(\theta + \xi_n \left(\widehat{\theta}_n - \theta\right)\right). \tag{4.50}$$

In addition, for all $n \geq 0$,

$$\widehat{T}_{n+1} = T_{n+1} + A_{n+1} + B_{n+1} + C_{n+1}, \tag{4.51}$$

where T_{n+1} is given by (4.13) and

$$A_{n+1} = \sin\left(2\pi(X_{n+1} - \theta)\right) Y_{n+1} \left(\frac{g(X_{n+1}) - \widehat{g}_n(X_{n+1})}{g^2(X_{n+1})}\right), \tag{4.52}$$

$$B_{n+1} = \frac{(\sin(2\pi(X_{n+1} - \widehat{\theta}_n)) - \sin(2\pi(X_{n+1} - \theta)))(g(X_{n+1}) - \widehat{g}_n(X_{n+1}))}{g^2(X_{n+1})} Y_{n+1}, \tag{4.53}$$

and

$$C_{n+1} = \sin\left(2\pi(X_{n+1} - \widehat{\theta}_n)\right) Y_{n+1} \left(\frac{\left(g(X_{n+1}) - \widehat{g}_n(X_{n+1})\right)^2}{g^2(X_{n+1})\widehat{g}_n(X_{n+1})}\right). \tag{4.54}$$

Then, denoting $D_{n+1} = A_{n+1} + B_{n+1} + C_{n+1}$, it follows from (4.51), that for all $n \geq 0$,

$$\widehat{T}_{n+1} = T_{n+1} + \mathbb{E}\left[A_{n+1}|\mathcal{F}_n\right] + D_{n+1} - \mathbb{E}\left[D_{n+1}|\mathcal{F}_n\right] + \mathbb{E}\left[B_{n+1}|\mathcal{F}_n\right] + \mathbb{E}\left[C_{n+1}|\mathcal{F}_n\right]. \tag{4.55}$$

Finally, we deduce from (4.48) that, for all $n \geq 0$,

$$\begin{aligned}
\widehat{\theta}_{n+1} - \theta = & \left(\alpha_n + \gamma_{n+1} \frac{\mathbb{E}\left[B_{n+1}|\mathcal{F}_n\right]}{\widehat{\theta}_n - \theta} + \gamma_{n+1} \frac{r_n}{\widehat{\theta}_n - \theta}\right) \left(\widehat{\theta}_n - \theta\right) \\
& + \gamma_{n+1} \left(T_{n+1} + \mathbb{E}\left[A_{n+1}|\mathcal{F}_n\right] - \phi\left(\widehat{\theta}_n\right)\right) \\
& + \gamma_{n+1} \left(D_{n+1} - \mathbb{E}\left[D_{n+1}|\mathcal{F}_n\right]\right) + \gamma_{n+1} \mathbb{E}\left[C_{n+1}|\mathcal{F}_n\right] + d_{n+1}.
\end{aligned} \tag{4.56}$$

An immediate recurrence in the previous equality leads to, for all $n \geq 1$,

$$\begin{aligned}
\sqrt{n}\left(\widehat{\theta}_n - \theta\right) = & \, n^{1/2}\beta_{n-1}\left(\widehat{\theta}_1 - \theta\right) + n^{1/2}\beta_{n-1}S_{n-1} + n^{1/2}\beta_{n-1}R^1_{n-1} \\
& + n^{1/2}\beta_{n-1}R^2_{n-1} + n^{1/2}\beta_{n-1}R^3_{n-1},
\end{aligned} \tag{4.57}$$

where

$$\beta_{n-1} = \prod_{k=1}^{n-1} \left(\alpha_k + \gamma_{k+1} \frac{\mathbb{E}\left[B_{k+1}|\mathcal{F}_k\right]}{\widehat{\theta}_k - \theta} + \gamma_{k+1} \frac{r_k}{\widehat{\theta}_k - \theta}\right), \tag{4.58}$$

$$S_{n-1} = \sum_{k=1}^{n-1} \frac{\gamma_{k+1}}{\beta_k} \left(T_{k+1} - \phi\left(\widehat{\theta}_k\right) + \mathbb{E}\left[A_{k+1}|\mathcal{F}_k\right]\right), \tag{4.59}$$

$$R^1_{n-1} = \sum_{k=1}^{n-1} \frac{\gamma_{k+1}}{\beta_k} \left(D_{k+1} - \mathbb{E}\left[D_{k+1} | \mathcal{F}_k \right] \right), \tag{4.60}$$

$$R^2_{n-1} = \sum_{k=1}^{n-1} \frac{\gamma_{k+1}}{\beta_k} \mathbb{E}\left[C_{k+1} | \mathcal{F}_k \right], \tag{4.61}$$

and

$$R^3_{n-1} = \sum_{k=1}^{n-1} \frac{1}{\beta_k} d_{k+1}. \tag{4.62}$$

4.4.1. Equivalent of β_{n-1}

We begin by finding a simple equivalent of the sequence β_{n-1} given by (4.58). Firstly, one have, for all $n \geq 0$,

$$\begin{aligned} &\mathbb{E}\left[B_{n+1} | \mathcal{F}_n \right] \\ &= \int_{-1/2}^{1/2} \frac{\left(\sin(2\pi(x - \widehat{\theta}_n)) - \sin(2\pi(x-\theta))\right)}{g(x)} f(x-\theta) \left(g(x) - \widehat{g}_n(x) \right) dx. \end{aligned} \tag{4.63}$$

Hence, as f is bounded and g does not vanish on $[-1/2; 1/2]$, we deduce that there exists a constant $C > 0$ such that, for all $n \geq 1$,

$$\begin{aligned} &\left| \frac{\mathbb{E}\left[B_{n+1} | \mathcal{F}_n \right]}{\widehat{\theta}_n - \theta} \right| \\ &\leq \int_{-1/2}^{1/2} \left| \frac{\left(\sin(2\pi(x - \widehat{\theta}_n)) - \sin(2\pi(x-\theta))\right)}{\widehat{\theta}_n - \theta} \frac{f(x-\theta)}{g(x)} \left(g(x) - \widehat{g}_n(x) \right) \right| dx \\ &\leq C \sup_{-1/2 \leq x \leq 1/2} |g(x) - \widehat{g}_n(x)|. \end{aligned}$$

In particular, thanks to Lemma 2.1 and with $\gamma_n = 1/n$,

$$\sum_{n=1}^{+\infty} \gamma_{n+1} \left| \frac{\mathbb{E}\left[B_{n+1} | \mathcal{F}_n \right]}{\widehat{\theta}_n - \theta} \right| < +\infty \qquad \text{a.s.} \tag{4.64}$$

Secondly, by definition of ϕ given by (2.2), one have for all $x \in \mathbb{R}$,

$$\phi''(x) = -4\pi^2 f_1 \sin\left(2\pi \left(\theta - x \right) \right).$$

Hence, one can find a constant $C > 0$ such that for all $n \geq 0$,

$$\left| \frac{\widehat{\theta}_n - \theta}{2} \phi'' \left(\theta + \xi_n \left(\widehat{\theta}_n - \theta \right) \right) \right| \leq C \left(\widehat{\theta}_n - \theta \right)^2.$$

Thus, for all $n \geq 1$,

$$\sum_{n=1}^{+\infty} \gamma_{n+1} \left| \frac{\widehat{\theta}_n - \theta}{2} \phi'' \left(\theta + \xi_n \left(\widehat{\theta}_n - \theta \right) \right) \right| \leq C \sum_{n=1}^{+\infty} \gamma_{n+1} \left(\widehat{\theta}_n - \theta \right)^2.$$

Consequently, with $\gamma_n = 1/n$, one deduce from Lemma 2.2 that, if $0 < \alpha < 1/2$, on $\{\tau_m = +\infty\}$, the sequence

$$\sum_{k=1}^{n} \gamma_{k+1} \left| \frac{\widehat{\theta}_k - \theta}{2} \phi'' \left(\theta + \xi_k \left(\widehat{\theta}_k - \theta \right) \right) \right|$$

converges a.s. Moreover, as $\cup_{m=1}^{+\infty} \{\tau_m = +\infty\}$ is a set of probability 1, it follows that, for $0 < \alpha < 1/2$,

$$\sum_{n=1}^{+\infty} \gamma_{n+1} \left| \frac{\widehat{\theta}_n - \theta}{2} \phi'' \left(\theta + \xi_n \left(\widehat{\theta}_n - \theta \right) \right) \right| < +\infty \qquad \text{a.s.} \tag{4.65}$$

Finally, one infer from (4.58) together with (4.64) and (4.65) that there exists a constant $c > 0$ such that

$$\beta_{n-1} \sim c \prod_{k=1}^{n-1} \alpha_k,$$

that is to say

$$\beta_{n-1} \sim c n^q, \tag{4.66}$$

where $q := \phi'(\theta) = -2\pi f_1 < -1/2$. Finally, for $0 < \alpha < 1/2$, we infer from (4.66) that (4.57) is equivalent to

$$\begin{aligned} \sqrt{n} \left(\widehat{\theta}_n - \theta \right) = {} & n^{1/2+q} \left(\widehat{\theta}_1 - \theta \right) + n^{1/2+q} S_{n-1} + n^{1/2+q} R_{n-1}^1 \\ & + n^{1/2+q} R_{n-1}^2 + n^{1/2+q} R_{n-1}^3, \end{aligned} \tag{4.67}$$

where in each sum S_{n-1}, R_{n-1}^1, R_{n-1}^2 and R_{n-1}^3, $\frac{\gamma_{k+1}}{\beta k}$ is replaced by k^{-1-q}.

Now, we are going to analyze the asymptotic behaviour of each term of (4.67). Firstly, as $q < -1/2$, we immediately have

$$n^{1/2+q} \left(\widehat{\theta}_1 - \theta \right) = o(1) \qquad \text{a.s.} \tag{4.68}$$

4.4.2. Asymptotic behaviour of (R_{n-1}^3)

The sequence (R_{n-1}^3) is almost surely finite since the number of times that the random variable $\widehat{\theta}_n + \text{sign}(f_1)\gamma_{n+1}\widehat{T}_{n+1}$ goes outside of C is almost surely finite (Theorem 2.1). Hence, as $q < -1/2$,

$$n^{1/2+q} R_{n-1}^3 = \mathcal{O}(n^{1/2+q}) = o(1) \qquad \text{a.s.} \tag{4.69}$$

4.4.3. Asymptotic behaviour of (R^1_{n-1})

The sequence (R^1_n) is a square integrable martingale whose increasing process is given, for all $n \geq 1$, by

$$\begin{aligned}\langle R^1\rangle_{n-1} &= \sum_{k=1}^{n-1} \frac{1}{k^{2+2q}} \left(\mathbb{E}\left[D^2_{k+1}|\mathcal{F}_k\right] - \mathbb{E}\left[D_{k+1}|\mathcal{F}_k\right]^2\right) \\ &\leq \sum_{k=1}^{n-1} \frac{1}{k^{2+2q}} \mathbb{E}\left[D^2_{k+1}|\mathcal{F}_k\right] \qquad \text{a.s.} \end{aligned} \tag{4.70}$$

In addition, as $D_{k+1} = A_{k+1} + B_{k+1} + C_{k+1}$, one obtain that, for all $1 \leq k \leq n-1$,

$$\mathbb{E}\left[D^2_{k+1}|\mathcal{F}_k\right] \leq 4\left(\mathbb{E}\left[A^2_{k+1}|\mathcal{F}_k\right] + \mathbb{E}\left[B^2_{k+1}|\mathcal{F}_k\right] + \mathbb{E}\left[C^2_{k+1}|\mathcal{F}_k\right]\right)$$

Hence, since f is bounded, g does not vanish on its support and (ε_n) has a moment of order 2, one immediately deduce from (4.52), (4.53) and (4.54) that

$$\mathbb{E}\left[D^2_{n+1}|\mathcal{F}_n\right] = \mathcal{O}\left(\sup_{-1/2 \leq x \leq 1/2} (g(x) - \widehat{g}_n(x))^2\right) \qquad \text{a.s.}$$

Then, it follows from Lemma 2.1 and (4.70) that, for all $n \geq 1$, a.s.

$$\langle R^1\rangle_{n-1} = \mathcal{O}\left(\sum_{k=1}^{n-1} \frac{1}{k^{2+2q}}(k^{-4\alpha} + k^{2\beta-2})\right) = \mathcal{O}\left(\sum_{k=1}^{n-1} \frac{1}{k^{2+2q+4\alpha}} + \frac{1}{k^{4+2q-2\beta}}\right)$$

If $2 + 2q + 4\alpha > 1$ and $4 + 2q - 2\beta > 1$, $\langle R^1\rangle_{n-1}$ converges a.s., and then R^1_{n-1} converges a.s.

If $2 + 2q + 4\alpha = 1$ and $4 + 2q - 2\beta = 1$, then $\langle R^1\rangle_{n-1} = \mathcal{O}(\log(n))$. We then deduce from the strong law of large numbers for martingales that $R^1_{n-1} = o(\log(n))$ a.s.

If $2 + 2q + 4\alpha < 1$ and $4 + 2q - 2\beta < 1$, then

$$\langle R^1\rangle_{n-1} = \mathcal{O}\left(\frac{1}{n^{1+2q+4\alpha}} + \frac{1}{n^{3+2q-2\beta}}\right) \qquad \text{a.s.}$$

Then, we infer from the strong law of large numbers for martingales given by Theorem 1.3.15 of [6] that for any $\gamma > 0$,

$$R^1_{n-1} = \mathcal{O}\left(\frac{\log(n)^{1/2+\gamma/2}}{n^{1/2+q+2\alpha}} + \frac{\log(n)^{1/2+\gamma/2}}{n^{3/2+q-\beta}}\right) \qquad \text{a.s.}$$

In the three cases, as $q < -1/2$, one can conclude that

$$n^{1/2+q} R^1_{n-1} = o(1) \qquad \text{a.s.} \tag{4.71}$$

4.4.4. Asymptotic behaviour of (R_{n-1}^2)

From the same way as previously, one deduce from (4.54) that

$$\mathbb{E}\left[C_{n+1}^2|\mathcal{F}_n\right] = \mathcal{O}\left(\sup_{-1/2\le x\le 1/2}\left(g(x)-\widehat{g}_n(x)\right)^2\right) \qquad \text{a.s.}$$

Hence, it follows from Lemma 2.1 that

$$\begin{aligned} R_{n-1}^2 &= \mathcal{O}\left(\sum_{k=1}^{n-1}\frac{1}{k^{1+q}}\sup_{-1/2\le x\le 1/2}|g(x)-\widehat{g}_k(x)|^2\right) \\ &= \mathcal{O}\left(\sum_{k=1}^{n-1}\frac{1}{k^{1+q+4\alpha}}+\frac{1}{k^{3+q-2\beta}}\right) \qquad \text{a.s.} \end{aligned}$$

Thus, if $1+q+4\alpha>1$ and $3+q-2\beta>1$, the sequence (R_{n-1}^2) converges a.s. whereas if $1+q+4\alpha=1$ and $3+q-2\beta=1$, one obtain that

$$R_{n-1}^2 = \mathcal{O}\left(\log(n)\right) \qquad \text{a.s.}$$

In the case where $1+q+4\alpha<1$ and $3+q-2\beta<1$, one deduce that

$$R_{n-1}^2 = \mathcal{O}\left(\frac{1}{n^{q+4\alpha}}+\frac{1}{n^{q+2-2\beta}}\right) \qquad \text{a.s.}$$

Finally, in the three cases, one obtain that, if $\alpha>1/8$ and $\beta<3/4$,

$$n^{1/2+q}R_{n-1}^2 = o(1) \qquad \text{a.s.} \tag{4.72}$$

The hypothesis $\beta<3/4$ implies to take $\alpha<1/2$. Hence, one obtain from (4.67) together with (4.68), (4.69), (4.71) and (4.72), that if $1/8<\alpha<1/2$,

$$\sqrt{n}\left(\widehat{\theta}_n-\theta\right) = n^{1/2+q}S_{n-1}+o(1) \qquad \text{a.s.} \tag{4.73}$$

where we recall that, for all $n\ge 1$,

$$S_{n-1} = \sum_{k=1}^{n-1}\frac{1}{k^{1+q}}\left(T_{k+1}-\phi\left(\widehat{\theta}_k\right)+\mathbb{E}\left[A_{k+1}|\mathcal{F}_k\right]\right). \tag{4.74}$$

One deduce from (2.2) and (4.13) the decomposition, for $n\ge 1$,

$$\sum_{k=1}^{n-1}\frac{1}{k^{1+q}}\left(T_{k+1}-\phi\left(\widehat{\theta}_k\right)\right) = M_{n-1}^1+M_{n-1}^2 \tag{4.75}$$

where

$$M_{n-1}^1 = \sum_{k=1}^{n-1}\frac{1}{k^{1+q}}\sin\left(2\pi(\theta-\widehat{\theta}_k)\right)\left(\frac{\cos\left(2\pi(X_{k+1}-\theta)\right)}{g(X_{k+1})}Y_{k+1}-f_1\right) \tag{4.76}$$

and

$$M_{n-1}^2 = \sum_{k=1}^{n-1}\frac{1}{k^{1+q}}\cos\left(2\pi(\theta-\widehat{\theta}_k)\right)\frac{\sin\left(2\pi(X_{k+1}-\theta)\right)}{g(X_{k+1})}Y_{k+1}. \tag{4.77}$$

4.4.5. Asymptotic behaviour of (M^1_{n-1})

The sequence (M^1_{n-1}) is a square integrable martingale whose increasing process is given, for all $n \geq 1$, by

$$\langle M^1 \rangle_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k^{2+2q}} \sin\left(2\pi(\theta - \widehat{\theta}_k)\right)^2 \mathbb{E}\left[\left(\frac{\cos(2\pi(X_{k+1}-\theta))}{g(X_{k+1})} Y_{k+1} - f_1\right)^2 |\mathcal{F}_k\right]. \tag{4.78}$$

Moreover, since f is bounded, g does not vanish on its support and (ε_k) has a moment of order 2, one immediately obtain from (4.78) that

$$\langle M^1 \rangle_{n-1} = \mathcal{O}\left(\sum_{k=1}^{n-1} \frac{1}{k^{2+2q}} \sin\left(2\pi(\theta - \widehat{\theta}_k)\right)^2\right) \qquad \text{a.s.}$$

In addition, since $(\widehat{\theta}_n)$ converges almost surely to θ and $\sum_{k=1}^{n-1} \frac{1}{k^{2+2q}} = \mathcal{O}\left(n^{-1-2q}\right)$, we finally deduce that

$$\langle M^1 \rangle_{n-1} = o(n^{-1-2q}) \qquad \text{a.s.}$$

In addition, since ε_n admits a moment of order > 2, the sequence (M^1_{n-1}) checks a Lyapunov condition. Consequently, one can conclude from the central limit theorem for martingales given e.g. by Corollary 2.1.10 of [6] that

$$n^{1/2+q} M^1_{n-1} = o_{\mathbb{P}}(1). \tag{4.79}$$

Hence, it follows from (4.75) and (4.79) that

$$n^{1/2+q} \sum_{k=1}^{n-1} \frac{1}{k^{1+q}} \left(T_{k+1} - \phi\left(\widehat{\theta}_k\right)\right) = M^2_{n-1} + o_{\mathbb{P}}(1), \tag{4.80}$$

where M^2_{n-1} is given by (4.77). Furthermore, since $(\widehat{\theta}_n)$ converges almost surely to θ, one immediately deduce that the asymptotic behaviour of M^2_n is the same as the one of the sequence M^3_{n-1}, given for all $n \geq 1$, by

$$M^3_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k^{1+q}} \frac{\sin\left(2\pi(X_{k+1}-\theta)\right)}{g(X_{k+1})} Y_{k+1}. \tag{4.81}$$

4.4.6. Asymptotic behaviour of (S_{n-1})

Finally, one obtain from (4.74) and (4.80) that, for $n \geq 1$,

$$n^{1/2+q} S_{n-1} = n^{1/2+q} M^3_{n-1} + n^{1/2+q} \sum_{k=1}^{n-1} \frac{1}{k^{1+q}} \mathbb{E}\left[A_{k+1}|\mathcal{F}_k\right] + o_{\mathbb{P}}(1), \tag{4.82}$$

where M^3_{n-1} is given by (4.81). In addition, (4.52) and the symmetry of f leads to, for all $n \geq 1$,

$$\mathbb{E}\left[A_{n+1}|\mathcal{F}_n\right] = \int_{-1/2}^{1/2} \sin\left(2\pi(x-\theta)\right) f(x-\theta) \left(\frac{g(x)-\widehat{g}_n(x)}{g(x)}\right) dx$$
$$= -\int_{-1/2}^{1/2} a(x)\widehat{g}_n(x)dx. \tag{4.83}$$

where for all $-1/2 \leq x \leq 1/2$,

$$a(x) = \frac{\sin\left(2\pi(x-\theta)\right) f(x-\theta)}{g(x)}. \tag{4.84}$$

Hence, we deduce from (2.6) and the change of variables $u = \frac{X_i - x}{h_i}$, that

$$\int_{-1/2}^{1/2} a(x)\widehat{g}_n(x)dx = \frac{1}{n}\sum_{i=1}^{n} \int_{-1/2}^{1/2} a(x)\frac{1}{h_i}K\left(\frac{X_i - x}{h_i}\right) dx$$
$$= \frac{1}{n}\sum_{i=1}^{n} \int_{\frac{-1/2-X_i}{h_i}}^{\frac{1/2-X_i}{h_i}} a(X_i + h_i u)K(u)du. \tag{4.85}$$

Moreover, since $X_n \in [-1/2; 1/2]$, one have

$$\frac{-1/2 - X_n}{h_n} \underset{n\to+\infty}{\longrightarrow} -\infty \quad \text{a.s.} \qquad \text{and} \qquad \frac{1/2 - X_n}{h_n} \underset{n\to+\infty}{\longrightarrow} +\infty \quad \text{a.s.} \tag{4.86}$$

From now, denote by $[-A; A]$ the support of K. Then, we have, for all $n \geq 1$,

$$\int_{\frac{-1/2-X_n}{h_n}}^{\frac{1/2-X_n}{h_n}} a(X_n + h_n u)K(u)du = \int_{-A}^{A} a(X_n + h_n u)K(u)du$$
$$+ \int_{\frac{-1/2-X_n}{h_n}}^{-A} a(X_n + h_n u)K(u)du$$
$$+ \int_{A}^{\frac{1/2-X_n}{h_n}} a(X_n + h_n u)K(u)du.$$

Then, we deduce from the previous equality and (4.85) and (4.86) that

$$\int_{-1/2}^{1/2} a(x)\widehat{g}_n(x)dx = \frac{1}{n}\sum_{i=1}^{n} \int_{-A}^{A} a(X_i + h_i u)K(u)du + \mathcal{O}\left(\frac{1}{n}\right) \qquad \text{a.s.} \tag{4.87}$$

Moreover, as f and g are two times differentiable, one can write a Taylor expansion of the function a given by (4.84). More precisely, there exists $0 < \xi_i < 1$ such that a.s.

$$a(X_i + h_i u) = a(X_i) + h_i u a'(X_i) + \frac{h_i^2}{2}u^2 a''(X_i + \xi_i h_i u).$$

Consequently, since K is a symmetric density and f and g have bounded derivates, one infer from the previous equality and (4.87) that, as $\alpha < 1/2$,

$$\begin{aligned}\int_{-1/2}^{1/2} a(x)\widehat{g}_n(x)dx &= \frac{1}{n}\sum_{i=1}^{n} a(X_i) + \mathcal{O}\left(\frac{1}{n}\sum_{i=1}^{n} h_i^2\right) + \mathcal{O}\left(\frac{1}{n}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} a(X_i) + \mathcal{O}\left(\frac{1}{n^{2\alpha}}\right) \qquad \text{a.s.} \end{aligned} \tag{4.88}$$

Finally, one deduce from (4.83) together with (4.84) and (4.88) that, as $\alpha < 1/2$,

$$\mathbb{E}\left[A_{n+1}|\mathcal{F}_n\right] = -\frac{1}{n}\sum_{i=1}^{n} \frac{\sin\left(2\pi(X_i-\theta)\right) f(X_i-\theta)}{g(X_i)} + \mathcal{O}\left(\frac{1}{n^{2\alpha}}\right) \qquad \text{a.s.} \tag{4.89}$$

Moreover, if $1+q+2\alpha < 1$, then for $\alpha > 1/4$,

$$n^{1/2+q}\sum_{k=1}^{n-1} \frac{1}{k^{1+q+2\alpha}} = o(1).$$

If $1+q+2\alpha \geq 1$, it is obvious that

$$n^{1/2+q}\sum_{k=1}^{n-1} \frac{1}{k^{1+q+2\alpha}} = o(1).$$

Hence, it follows from (4.89) and the two previous equality that a.s.,

$$\begin{aligned} &n^{1/2+q}\sum_{k=1}^{n-1} \frac{1}{k^{1+q}}\mathbb{E}\left[A_{k+1}|\mathcal{F}_k\right] \\ &= -n^{1/2+q}\sum_{k=1}^{n-1}\frac{1}{k^{2+q}}\sum_{i=1}^{k} \frac{\sin\left(2\pi(X_i-\theta)\right) f(X_i-\theta)}{g(X_i)} + o(1). \end{aligned} \tag{4.90}$$

Finally, the conjunction of (4.73) together with (4.82) and (4.90) let us to conclude that, for $1/4 < \alpha < 1/2$,

$$\begin{aligned} &\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \\ &= n^{1/2+q}M_{n-1}^3 + n^{1/2+q}\sum_{k=1}^{n-1}\frac{1}{k^{2+q}}\sum_{i=1}^{k}\frac{\sin\left(2\pi(X_i-\theta)\right) f(X_i-\theta)}{g(X_i)} + o_{\mathbb{P}}(1) \\ &= n^{1/2+q}\left(\sum_{k=1}^{n-1}\frac{1}{k^{1+q}}v(X_{k+1})Y_{k+1} - \sum_{k=1}^{n-1}\frac{1}{k^{2+q}}\sum_{i=1}^{k} v(X_i)f(X_i-\theta)\right) + o_{\mathbb{P}}(1). \end{aligned} \tag{4.91}$$

where for all $-1/2 \leq x \leq 1/2$,

$$v(x) = \frac{\sin\left(2\pi(x-\theta)\right)}{g(x)}. \tag{4.92}$$

From now, denote for all $1 \leq k \leq n-1$,

$$U_k = \begin{pmatrix} X_k \\ \varepsilon_k \end{pmatrix}. \tag{4.93}$$

4.4.7. Conclusion

In the following, we note U a random variable independent of the sequence (U_n) and with the same law as U_n. We obtain from (4.91) that, for $1/4 < \alpha < 1/2$,

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) = n^{1/2+q}\left(\sum_{k=1}^{n-1} \frac{1}{k^{1+q}}\psi_1(U_{k+1}) - \sum_{k=1}^{n-1} \frac{1}{k^{2+q}} \sum_{i=1}^{k} \psi_2(U_i)\right) + o_{\mathbb{P}}(1), \tag{4.94}$$

where the function ψ_1 and ψ_2 are such that, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\psi_1(x) = v(x_1)\left(f(x_1 - \theta) + x_2\right) \qquad \text{and} \qquad \psi_2(x) = v(x_1)f(x_1 - \theta). \tag{4.95}$$

In addition,

$$\sum_{k=1}^{n-1} \frac{1}{k^{2+q}} \sum_{i=1}^{k} \psi_2\left(U_i\right) = \sum_{i=1}^{n-1}\sum_{k=i}^{n-1} \frac{1}{k^{2+q}}\psi_2\left(U_i\right) = \sum_{i=1}^{n-1} s_i^{n-1}\psi_2\left(U_i\right) \tag{4.96}$$

where for $1 \leq i \leq n-1$,

$$s_i^{n-1} = \sum_{k=i}^{n-1} \frac{1}{k^{2+q}}. \tag{4.97}$$

Thus, it follows from (4.96) that for $n \geq 3$,

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{1}{k^{1+q}}\psi_1(U_{k+1}) - \sum_{k=1}^{n-1} \frac{1}{k^{2+q}} \sum_{i=1}^{k} \psi_2(U_i) \\ &\quad = \sum_{k=1}^{n-2}\left(\frac{1}{k^{1+q}}\psi_1(U_{k+1}) - s_{k+1}^{n-1}\psi_2(U_{k+1})\right) + \frac{1}{(n-1)^{1+q}}\psi_1(U_n) - s_1^{n-1}\psi_2(U_1). \end{aligned} \tag{4.98}$$

Moreover, since $\psi_1(U_n)$ and $\psi_2(U_n)$ are integrable, we have

$$n^{1/2+q}\left(\frac{1}{(n-1)^{1+q}}\psi_1(U_n) - s_1^{n-1}\psi_2(U_1)\right) = o_{\mathbb{P}}(1).$$

One finally deduce from (4.94) and (4.98) that for $1/4 < \alpha < 1/2$,

$$\begin{aligned} \sqrt{n}\left(\widehat{\theta}_n - \theta\right) &= n^{1/2+q}\sum_{k=1}^{n-2}\left(\frac{1}{k^{1+q}}\psi_1(U_{k+1}) - s_{k+1}^{n-1}\psi_2(U_{k+1})\right) + o_{\mathbb{P}}(1), \\ &= (1 \;\; 1)\, n^{1/2+q}\mathcal{M}_n + o_{\mathbb{P}}(1), \end{aligned} \tag{4.99}$$

where for $n \geq 3$,

$$\mathcal{M}_n = \sum_{k=1}^{n-2} \begin{pmatrix} a_k \psi_1 \left(U_{k+1}\right) \\ b_k \psi_2 \left(U_{k+1}\right), \end{pmatrix} \tag{4.100}$$

and for $1 \leq k \leq n-2$,

$$a_k = \frac{1}{k^{1+q}} \quad \text{and} \quad b_k = -s_{k+1}^{n-1}. \tag{4.101}$$

Moreover, as f is symmetric and (ε_n) is of mean 0, it is not hard to see that, for $1 \leq k \leq n-2$,

$$\mathbb{E}\left[\begin{pmatrix} a_k \psi_1 \left(U_{k+1}\right) \\ b_k \psi_2 \left(U_{k+1}\right) \end{pmatrix} \middle| \mathcal{F}_k\right] = \mathbb{E}\left[\begin{pmatrix} a_k \psi_1 \left(U\right) \\ b_k \psi_2 \left(U\right) \end{pmatrix}\right] = 0.$$

Consequently, the sequence $(\mathcal{M}_n)$ is a vectorial martingale whose increasing process is the matrix defined for $n \geq 3$, by

$$\langle \mathcal{M} \rangle_n = \sum_{k=1}^{n-2} \begin{pmatrix} a_k^2 \sigma_1^2 & a_k b_k \sigma_{1,2} \\ a_k b_k \sigma_{1,2} & b_k^2 \sigma_2^2 \end{pmatrix}. \tag{4.102}$$

where

$$\sigma_1^2 = \mathbb{E}\left[\psi_1(U)^2\right], \quad \sigma_2^2 = \mathbb{E}\left[\psi_2(U)^2\right] \quad \text{and} \quad \sigma_{1,2} = \mathbb{E}\left[\psi_1(U)\psi_2(U)\right].$$

Moreover, for $q < -1/2$,

$$n^{1+2q} \sum_{k=1}^{n-2} a_k^2 = n^{1+2q} \sum_{k=1}^{n-2} \frac{1}{k^{2+2q}} \underset{n \to +\infty}{\longrightarrow} \int_0^1 \frac{dx}{x^{2+2q}} = -\frac{1}{1+2q}. \tag{4.103}$$

In addition, for $n \geq 3$,

$$\sum_{k=1}^{n-2} a_k b_k = -\sum_{k=1}^{n-2} \frac{1}{k^{1+q}} s_{k+1}^{n-1} = -\sum_{k=1}^{n-2} \sum_{i=k+1}^{n-1} \frac{1}{k^{1+q}} \frac{1}{i^{2+q}} = -\sum_{i=2}^{n-1} \sum_{k=1}^{i-1} \frac{1}{k^{1+q}} \frac{1}{i^{2+q}}.$$

Hence, one deduce from the Toeplitz lemma that, as $q < -1/2$,

$$\begin{aligned} &n^{1+2q} \sum_{k=1}^{n-2} a_k b_k \\ &\quad = -\frac{1}{n} \sum_{i=2}^{n-1} \frac{n^{2+2q}}{i^{2+2q}} \frac{1}{i} \sum_{k=1}^{i-1} \frac{i^{1+q}}{k^{1+q}} \underset{n \to +\infty}{\longrightarrow} -\int_0^1 \frac{dx}{x^{1+q}} \int_0^1 \frac{dx}{x^{2+2q}} = -\frac{1}{1+2q} \frac{1}{q}. \end{aligned} \tag{4.104}$$

Finally,

$$\sum_{k=1}^{n-2} b_k^2 = \sum_{k=1}^{n-2} \left(s_{k+1}^{n-1}\right)^2 = \sum_{k=1}^{n-2} \left(\sum_{i=k+1}^{n-1} \frac{1}{i^{2+q}} \right)^2 = \sum_{k=1}^{n-2} \sum_{i=k+1}^{n-1} \sum_{j=k+1}^{n-1} \frac{1}{i^{2+q}} \frac{1}{j^{2+q}}.$$

Consequently,

$$\lim_{n\to+\infty} n^{1/2+q}\sum_{k=1}^{n-2} b_k^2 = \lim_{n\to+\infty} n^{1/2+q}\left(\sum_{i=2}^{n-1}\frac{1}{i^{2+q}}\sum_{j=2}^{i}\frac{1}{j^{1+q}} + \sum_{j=3}^{n-1}\frac{1}{j^{2+q}}\sum_{i=2}^{j-1}\frac{1}{i^{1+q}}\right).$$

Hence, it immediately follows from (4.104) that

$$n^{1+2q}\sum_{k=1}^{n-2} b_k^2 \underset{n\to+\infty}{\longrightarrow} 2\frac{1}{1+2q}\frac{1}{q}. \tag{4.105}$$

Hence, we infer from (4.102) together with (4.103), (4.104) and (4.105) that

$$n^{1+2q}\langle\mathcal{M}\rangle_n \underset{n\to+\infty}{\longrightarrow} -\frac{1}{q(1+2q)}\begin{pmatrix} q\sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & -2\sigma_2^2\end{pmatrix} \qquad \text{a.s.} \tag{4.106}$$

In order to apply the central limit theorem for vectorial martingales, it remains to check the Lindeberg condition, that is to say, for all $\varepsilon > 0$,

$$n^{1+2q}\sum_{k=1}^{n-2}\mathbb{E}\left[\left|\left|V_{k+1}\right|\right|^2 1_{||V_{k+1}||\geq\varepsilon n^{-1/2-q}}\middle|\mathcal{F}_k\right] \xrightarrow[n\to+\infty]{\mathbb{P}} 0$$

where $V_{k+1} = \begin{pmatrix} a_k\psi_1(U_{k+1}) \\ b_k\psi_2(U_{k+1})\end{pmatrix}$. However, for $\delta > 0$, one have

$$\begin{aligned}\sum_{k=1}^{n-2}\mathbb{E}\left[\left|\left|V_{k+1}\right|\right|^2 1_{||V_{k+1}||\geq\varepsilon n^{-1/2-q}}\middle|\mathcal{F}_k\right] &= \sum_{k=1}^{n-2}\mathbb{E}\left[\frac{\left|\left|V_{k+1}\right|\right|^{2+\delta}}{\left|\left|V_{k+1}\right|\right|}1_{||V_{k+1}||\geq\varepsilon n^{-1/2-q}}\middle|\mathcal{F}_k\right] \\ &\leq \frac{1}{\varepsilon}n^{1/2+q}\sum_{k=1}^{n-2}\mathbb{E}\left[\left|\left|V_{k+1}\right|\right|^{2+\delta}\middle|\mathcal{F}_k\right]\end{aligned} \tag{4.107}$$

Moreover, for $1 \leq k \leq n-2$,

$$\begin{aligned}\mathbb{E}\left[\left|\left|V_{k+1}\right|\right|^{2+\delta}\middle|\mathcal{F}_k\right] &= \mathbb{E}\left[\left(a_k^2\psi_1(U_{k+1})^2 + b_k^2\psi_2(U_{k+1})^2\right)^{1+\delta/2}\middle|\mathcal{F}_k\right] \\ &\leq 2^{\delta/2}\left(a_k^{2+\delta}\mathbb{E}\left[\psi_1(U)^{2+\delta}\right] + b_k^{2+\delta}\mathbb{E}\left[\psi_2(U)^{2+\delta}\right]\right).\end{aligned} \tag{4.108}$$

Hence, as (ε_n) has a moment of order > 2, one immediately deduce from (4.107) and (4.108) that there exists $\kappa_\varepsilon > 0$ such that

$$n^{1+2q}\sum_{k=1}^{n-2}\mathbb{E}\left[\left|\left|V_{k+1}\right|\right|^2 1_{||V_{k+1}||\geq\varepsilon n^{-1/2-q}}\middle|\mathcal{F}_k\right] \leq \kappa_\varepsilon n^{3/2+3q}\sum_{k=1}^{n-2}\left(a_k^{2+\delta} + b_k^{2+\delta}\right). \tag{4.109}$$

However,

$$n^{3/2+3q}\sum_{k=1}^{n-2}\left(a_k^{2+\delta}+b_k^{2+\delta}\right)=n^{3/2+3q}\left(\sum_{k=1}^{n-2}\frac{1}{k^{(1+q)(2+\delta)}}+\sum_{k=1}^{n-2}\left(\sum_{i=k+1}^{n-1}\frac{1}{i^{2+q}}\right)^{2+\delta}\right). \tag{4.110}$$

If $q=-1$, then

$$\begin{aligned} &n^{3/2+3q}\left(\sum_{k=1}^{n-2}\frac{1}{k^{(1+q)(2+\delta)}}+\sum_{k=1}^{n-2}\left(\sum_{i=k+1}^{n-1}\frac{1}{i^{2+q}}\right)^{2+\delta}\right)\\ &\quad=\mathcal{O}\left(n^{3/2-3}(n+n\log(n)^{2+\delta})\right)=o(1). \end{aligned} \tag{4.111}$$

If $q<-1$ then,

$$\begin{aligned} n^{3/2+3q}\left(\sum_{k=1}^{n-2}\frac{1}{k^{(1+q)(2+\delta)}}+\sum_{k=1}^{n-2}\left(\sum_{i=k+1}^{n-1}\frac{1}{i^{2+q}}\right)^{2+\delta}\right)&=\mathcal{O}\left(\frac{n^{3/2+3q}}{n^{1+2q+q\delta+\delta}}\right)\\ &=\mathcal{O}\left(n^{1/2+q-\delta(q+1)}\right)=o(1). \end{aligned} \tag{4.112}$$

as soon as $\delta<\frac{1/2+q}{1+q}$. Moreover, in this case, $\frac{1/2+q}{1+q}>0$.

If $-1<q<-1/2$ there exists a constant $c>0$ such that

$$\sum_{i=k+1}^{n-1}\frac{1}{i^{2+q}}\leq\frac{c}{k^{1+q}},$$

which implies that

$$n^{3/2+3q}\left(\sum_{k=1}^{n-2}\frac{1}{k^{(1+q)(2+\delta)}}+\sum_{k=1}^{n-2}\left(\sum_{i=k+1}^{n-1}\frac{1}{i^{2+q}}\right)^{2+\delta}\right)=\mathcal{O}\left(\sum_{k=1}^{n-2}\frac{n^{3/2+3q}}{k^{(1+q)(2+\delta)}}\right).$$

In the case where $(1+q)(2+\delta)<1$ then,

$$n^{3/2+3q}\sum_{k=1}^{n-2}\frac{1}{k^{(1+q)(2+\delta)}}=\mathcal{O}\left(\frac{n^{3/2+3q}}{n^{1+2q+q\delta+\delta}}\right)=\mathcal{O}\left(n^{1/2+q-\delta(q+1)}\right)=o(1) \tag{4.113}$$

as soon as $\delta>\frac{1/2+q}{1+q}$, which is right because $\frac{1/2+q}{1+q}<0$. In the case where $(1+q)(2+\delta)\geq 1$, we clearly have

$$n^{3/2+3q}\sum_{k=1}^{n-2}\frac{1}{k^{(1+q)(2+\delta)}}=o(1). \tag{4.114}$$

Finally, one deduce from (4.107), (4.108) (4.109) and (4.110) together with (4.111), (4.112), (4.113) and (4.114) that

$$n^{1+2q}\sum_{k=1}^{n-2}\mathbb{E}\left[||V_{k+1}||^2\,1_{||V_{k+1}||\geq\varepsilon n^{-1/2-q}}\middle|\mathcal{F}_k\right]\underset{n\to+\infty}{\longrightarrow}0\qquad\text{a.s.} \tag{4.115}$$

that seems that the Lindeberg condition is satisfied. One can conclude from (4.106) and from (4.115) and the central limit theorem for martingales given e.g. in Corollary 2.1.10 page 46 of [6] that

$$n^{1/2+q}\mathcal{M}_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\Gamma\right), \tag{4.116}$$

where

$$\Gamma = -\frac{1}{q(1+2q)}\begin{pmatrix} q\sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & -2\sigma_2^2 \end{pmatrix}. \tag{4.117}$$

Moreover, as (ε_n) is of mean 0 and variance σ^2 and independent of (X_n), straightforward but tedious calculations lead to

$$\sigma_1^2 = \mathbb{E}\left[\psi_1(U)^2\right] = \int_{-1/2}^{1/2} \frac{\sin^2\left(2\pi(x-\theta)\right)}{g(x)}\left(f^2(x-\theta)+\sigma^2\right)dx,$$

$$\sigma_2^2 = \mathbb{E}\left[\psi_2(U)^2\right] = \int_{-1/2}^{1/2} \frac{\sin^2\left(2\pi(x-\theta)\right)}{g(x)} f^2(x-\theta)dx,$$

and

$$\sigma_{1,2} = \mathbb{E}\left[\psi_1(U)\psi_2(U)\right] = \sigma_2^2.$$

Finally, (4.99) together with (4.116) and the Slutsky theorem let us to conclude that, if $1/4 < \alpha < 1/2$,

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, -\frac{1}{1+2q}\sigma_1^2\right),$$

which ends the proof of Theorem 2.2. □

References

[1] Bercu, B. and Fraysse, P. (2012). A Robbins-Monro procedure for estimation in semiparametric regression models. *Ann. Statist.* **40**. MR2933662

[2] Bigot, J. and Charlier, B. (2011). On the consistency of Fréchet means in deformable models for curve and image analysis. *Electron. J. Stat.* **5** 1054–1089. doi:10.1214/11-EJS633 MR2836769

[3] Bigot, J. and Gadat, S. (2010). A deconvolution approach to estimation of a common shape in a shifted curves model. *Ann. Statist.* **38** 2422–2464. doi:10.1214/10-AOS800 MR2676894

[4] Castillo, I. and Loubes, J. M. (2009). Estimation of the distribution of random shifts deformation. *Math. Methods Statist. 18* **1** 21–42. MR2508947

[5] Dalalyan, A. S., Golubev, G. K. and Tsybakov, A. B. (2006). Penalized maximum likelihood and semiparametric second-order efficiency. *Ann. Statist. 34* **1** 169–201. MR2275239

[6] Duflo, M. (1997). *Random iterative models. Applications of Mathematics* **34**. Springer-Verlag, Berlin. MR1485774

[7] Gamboa, F., Loubes, J. M. and Maza, E. (2007). Semi-parametric estimation of shifts. *Electron. J. Stat.* **1** 616–640. MR2369028

[8] Hall, P. and Heyde, C. C. (1980). *Martingale limit theory and its application. Academic Press Inc.* MR0624435

[9] Lawton, W. H., Sylvestre, E. A. and Maggio, M. S. (1972). Self modeling nonlinear regression. *Technometrics* **14** 513–532.

[10] Pelletier, M. (1998a). On the almost sure asymptotic behaviour of stochastic algorithms. *Stochastic Process. Appl.* **78** 217–244. doi:10.1016/S0304-4149(98)00029-5 MR1654569

[11] Pelletier, M. (1998b). Weak convergence rates for stochastic approximation with application to multiple targets and simulated annealing. *Annals of Appli. Proba.* **8** 10–44. MR1620405

[12] Robbins, H. and Monro, S. (1951). A stochastic approximation method. *Ann. Math. Statistics* **22** 400–407. MR0042668

[13] Trigano, T., Isserles, U. and Ritov, Y. (2011). Semiparametric curve alignment and shift density estimation for biological data. *IEEE Trans. Signal Processing* **59** 1970–1984. MR2816476

[14] Tsybakov, A. B. (2004). *Introduction à l'estimation non-paramétrique. Mathématiques & Applications (Berlin)* **41**. Springer-Verlag, Berlin. MR2013911

[15] Vimond, M. (2010). Efficient estimation for a subclass of shape invariant models. *Ann. Statist.* **38** 1885–1912. MR2662362

[16] Wolverton, C. T. and Wagner, T. J. (1969). Asymptotically optimal discriminant functions for pattern classification. *IEEE Trans. Information Theory* **IT-15** 258–265. MR0275576

[17] Yamato, H. (1970/71). Sequential estimation of a continuous probability density function and mode. *Bull. Math. Statist.* **14** 1–12; correction, ibid. 15 (1972), 133. MR0381187