Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 959586, 11 pages
http://dx.doi.org/10.1155/2014/959586

Research Article

Approximation by Genuine q-Bernstein-Durrmeyer Polynomials in Compact Disks in the Case $q > 1$

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Received 10 January 2014; Accepted 2 February 2014; Published 16 March 2014

Academic Editor: Sofiya Ostrovska

This paper deals with approximating properties of the newly defined q-generalization of the genuine Bernstein-Durrmeyer polynomials in the case $q > 1$, which are no longer positive linear operators on $C[0, 1]$. Quantitative estimates of the convergence, the Voronovskaja-type theorem, and saturation of convergence for complex genuine q-Bernstein-Durrmeyer polynomials attached to analytic functions in compact disks are given. In particular, it is proved that, for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the genuine q-Bernstein-Durrmeyer polynomials ($q > 1$) is of order q^{-n} versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials. We give explicit formulas of Voronovskaja type for the genuine q-Bernstein-Durrmeyer for $q > 1$. This paper represents an answer to the open problem initiated by Gal in (2013, page 115).

1. Introduction

In several recent papers, convergence properties of complex q-Bernstein polynomials, proposed by Phillips [1], attached to an analytic function f in closed disks, were intensively studied. Ostrovska [2, 3] and Wang and Wu [4, 5] have investigated convergence properies of $B_{n,q}$ in the case $q > 1$. In the case $q > 1$, the q-Bernstein polynomials are no longer positive operators; however, for a function analytic in a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$, $R > q$, it was proved in [2] that the rate of convergence of $\{B_{n,q}(f; z)\}$ to $f(z)$ has the order q^{-n} (versus $1/n$ for the classical Bernstein polynomials). Moreover, Ostrovska [3] obtained Voronovskaya-type theorem for monomials. If $q \geq 1$, then qualitative Voronovskaja-type theorem and saturation results for complex q-Bernstein polynomials were obtained by Wang and Wu [4]. Wu [5] studied saturation of convergence on the interval $[0, 1]$ for the q-Bernstein polynomials of a continuous function f for arbitrary fixed $q > 1$.

Genuine Bernstein-Durrmeyer operators were first considered by Chen [6] and Goodman and Sharma [7] around 1987. In recent years, the genuine Bernstein-Durrmeyer operators have been investigated intensively by a number of authors. Among the many papers written on the genuine Bernstein-Durrmeyer operators, we mention here only the ones by Gonska et al. [8], Parvanov and Popov [9], Sauer [10], Waldron [11], and the book of Păltănea [12].

On the other hand, Gal [13] obtained quantitative estimates of the convergence and of the Voronovskaja-type theorem in compact disks, for the complex genuine Bernstein-Durrmeyer polynomials attached to analytic functions. Besides, in other very recent papers, similar studies were done for complex Bernstein-Durrmeyer operators in Anastassiou and Gal [14], for complex Bernstein-Durrmeyer operators based on Jacobi weights in Gal [15], for complex genuine q-Bernstein-Durrmeyer operators ($0 < q < 1$) by Mahmudov [16], and for other kinds of complex Durrmeyer operators in Mahmudov [17] and Gal et al. [18]. It should be stressed out that study of q-Durrmeyer-type operators ($0 < q < 1$) in the real case was first initiated by Derriennic [19].

Also, for the case $q > 1$, exact quantitative estimates and quantitative Voronovskaja-type results for complex q-Lorentz polynomials, q-Stancu polynomials [20], q-Stancu-Faber polynomials, q-Bernstein-Faber polynomials, q-Kantorovich polynomials [21], q-Szász-Mirakjan operators [22] obtained by different researchers are collected in the recent book of Gal [23]. In this book the definition and study of complex q-Durrmeyer-kind operators for $q > 1$ presented an open problem. This paper presents a positive solution to this problem.

In this paper we define the genuine q-Bernstein-Durrmeyer polynomials for $q > 1$. Note that similar to the q-Bernstein operators the genuine q-Bernstein-Durrmeyer operators in the case $q > 1$ are not positive operators on $C[0,1]$. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than that for $0 < q < 1$. We present upper estimates in approximation and we prove the Voronovskaja-type convergence theorem in compact disks in $\mathbb{C}$, centered at origin, with quantitative estimate of this convergence. These results allow us to obtain the exact degrees of approximation by complex genuine q-Bernstein-Durrmeyer polynomials. Our results show that approximation properties of the complex genuine q-Bernstein-Durrmeyer polynomials are better than approximation properties of the complex Bernstein-Durrmeyer polynomials considered in [13].

2. Main Results

We begin with some notations and definitions of q-calculus; see, for example, [24, 25]. Let $q > 0$. For any $n \in \mathbb{N} \cup \{0\}$, the q-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [0]_q := 0; \tag{1}$$

and the q-factorial $[n]_q!$ is defined by

$$[n]_q! := [1]_q [2]_q \cdots [n]_q, \quad [0]_q! := 1. \tag{2}$$

For integers $0 \le k \le n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}. \tag{3}$$

For $q = 1$ we obviously get $[n]_q = n$, $[n]_q! = n!$, and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \binom{n}{k}$. Moreover

$$\begin{gathered} (1-z)_q^n := \prod_{s=0}^{n-1} (1 - q^s z), \\ p_{n,k}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (1-z)_q^{n-k}, \quad z \in \mathbb{C}. \end{gathered} \tag{4}$$

For fixed $q > 0$, $q \neq 1$, we denote the q-derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{5}$$

The q-analogue of integration in the interval $[0, A]$ (see [24]) is defined by

$$\int_0^A f(t)\, d_q t := A(1-q) \sum_{n=0}^{\infty} f(Aq^n) q^n, \quad 0 < q < 1. \tag{6}$$

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f \in H(\mathbb{D}_R)$ we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$ for all $z \in \mathbb{D}_R$. The norm $\|f\|_r := \max\{|f(z)| : |z| \le r\}$. We denote $e_m(z) = z^m$ for all $m \in \mathbb{N} \cup \{0\}$.

Definition 1. For $f : [0,1] \to \mathbb{C}$, the genuine q-Bernstein-Durrmeyer operator is defined as follows:

$$U_{n,q}(f;z) := \begin{cases} f(0)\, p_{n,0}(q;z) + f(1)\, p_{n,n}(q;z) \\ \quad + [n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;z) \\ \quad \times \int_0^1 p_{n-2,k-1}(q;qt) f(t)\, d_q t, & 0 < q < 1, \\ f(0)\, p_{n,0}(z) + f(1)\, p_{n,n}(z) \\ \quad + (n-1) \sum_{k=1}^{n-1} p_{n,k}(z) \\ \quad \times \int_0^1 p_{n-2,k-1}(t) f(t)\, dt, & q = 1, \\ f(0)\, p_{n,0}(q;z) + f(1)\, p_{n,n}(q;z) \\ \quad + [n-1]_{q^{-1}} \sum_{k=1}^{n-1} q^{k-1} p_{n,k}(q;z) \\ \quad \times \int_0^1 p_{n-2,k-1}\left(q^{-1}; q^{-1}t\right) f\left(q^{k-n}t\right) d_{q^{-1}} t, & q > 1, \end{cases} \tag{7}$$

where for $n = 1$ the sum is empty; that is, it is equal to 0.

$U_{n,q}(f;z)$ are linear operators reproducing linear functions and interpolating every function $f \in C[0,1]$ at 0 and 1. The genuine q-Bernstein-Durrmeyer operators are positive operators on $C[0,1]$ for $0 < q \le 1$, and they are not positive for $q > 1$. As a consequence, the cases $0 < q \le 1$ and $q > 1$ are not similar to each other regarding the convergence. For $q \to 1^-$ and $q \to 1^+$ we recapture the classical ($q = 1$) genuine Bernstein-Durrmeyer polynomials.

We start with the following quantitative estimates of the convergence for complex q-Bernstein-Durrmeyer polynomials attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 2. *Let $f \in H(\mathbb{D}_R)$, $1 \le r < R/q$, and $q > 1$. Then for all $|z| \le r$ one has*

$$\left| U_{n,q}(f;z) - f(z) \right| \le \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m|\, m(m-1)\, q^{m-2} r^{m-2}. \tag{8}$$

Theorem 2 says that, for functions analytic in $\mathbb{D}_R$, $R > q$, the rate of approximation by the genuine q-Bernstein-Durrmeyer polynomials ($q > 1$) is of order q^{-n} versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials; see [13].

The Voronovskaja theorem for the real case with a quantitative estimate is obtained by Gonska et al. [26] in the following form:

$$\left|U_n(f;x) - f(x) - \frac{x(1-x)}{n+1}f''(z)\right| \le \frac{x(1-x)}{n+1}\omega\left(f''\frac{2}{3\sqrt{n+3}}\right), \tag{9}$$

and, for all $n \in \mathbb{N}$, $0 \le x \le 1$. For the complex genuine q-Bernstein-Durrmeyer $(0 < q \le 1)$ a quantitative estimate is obtained by Gal [13] $(q = 1)$ and Mahmudov [16] $(0 < q < 1)$ in the following form:

$$\left|U_{n,q}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_q}f''(z)\right| \le \frac{M_{r,f}}{[n]_q^2}, \quad 0 < q \le 1, \tag{10}$$

and, for all $n \in \mathbb{N}$, $|z| \le r$.

To formulate and prove the Voronovskaja-type theorem with a quantitative estimate in the case $q > 1$ we introduce a function $L_q(f;z)$.

Let $R > q \ge 1$ and let $f \in H(\mathbb{D}_R)$. For $|z| < R/q^2$, we define

$$L_q(f;z) := \frac{(1-z)q\left(D_q f(z) - D_{q^{-1}}f(z)\right)}{q-1} \quad \text{for } q > 1. \tag{11}$$

And, for $0 < q \le 1$,

$$L_q(f;z) = L_1(f;z) := f''(z)z(1-z). \tag{12}$$

The next theorem gives Voronovskaja-type result in compact disks; for complex q-Bernstein-Durrmeyer polynomials attached to an analytic function in $\mathbb{D}_R$, $R > q^2 > 1$ and center 0 in terms of the function $L_q(f;z)$.

Theorem 3. *Let $f \in H(\mathbb{D}_R)$, $1 \le r < R/q^2$, and $q > 1$. The following Voronovskaja-type result holds:*

$$\left|U_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q}L_q(f;z)\right| \le \frac{4r^2(1+r)^2}{[n+1]_q^2}\sum_{m=3}^{\infty}|a_m|(m-1)^2(m-2)^2\left(q^2r\right)^{m-2}. \tag{13}$$

For all $n \in \mathbb{N}$, $|z| \le r$.

Now we are in position to prove that the order of approximation in Theorem 2 is exactly q^{-n} versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials; see [13].

Theorem 4. *Let $1 < q < R$, $1 \le r < R/q^2$, and $f \in H(\mathbb{D}_R)$. If f is not a polynomial of degree ≤1, the estimate,*

$$\left\|U_{n,q}(f) - f\right\|_r \ge \frac{1}{[n+1]_q}C_{r,q}(f), \quad n \in \mathbb{N}, \tag{14}$$

holds, where the constant $C_{r,q}(f)$ depends on f, q, and r but is independent of n.

From Theorem 3 we conclude that, for $q > 1$, $[n+1]_q\,(U_{n,q}(f;z) - f(z)) \to L_q(f;z)$ in $H(\mathbb{D}_{R/q^2})$ and therefore $L_q(f;z) \in H(\mathbb{D}_{R/q^2})$. Furthermore, we have the following saturation of convergence for the genuine q-Bernstein-Durrmeyer polynomials for fixed $q > 1$.

Theorem 5. *Let $1 < q < R$, $1 \le r < R/q^2$. If a function f is analytic in the disc $\mathbb{D}_{R/q^2}$, then $|U_{n,q}(f;z) - f(z)| = o(q^{-n})$ for infinite number of points having an accumulation point on $\mathbb{D}_{R/q^2}$ if and only if f is linear.*

The next theorem shows that $L_q(f;z)$, $q \ge 1$, is continuous in the parameter q for $f \in H(\mathbb{D}_R)$, $R > 1$.

Theorem 6. *Let $R > 1$ and $f \in H(\mathbb{D}_R)$. Then, for any r, $0 < r < R$,*

$$\lim_{q \to 1+} L_q(f;z) = L_1(f;z) \tag{15}$$

uniformly on $\mathbb{D}_R$.

3. Auxiliary Results

The q-analogue of beta function for $0 < q < 1$ (see [24]) is defined as

$$B_q(m,n) = \int_0^1 t^{m-1}(1-qt)_q^{n-1}d_q t, \quad m,n > 0,\ 0 < q < 1. \tag{16}$$

Since we consider the case $q > 1$, we need to use $B_{q^{-1}}(m,n)$ as follows:

$$B_{q^{-1}}(m,n) = \int_0^1 t^{m-1}\left(1 - q^{-1}t\right)_{q^{-1}}^{n-1}d_{q^{-1}}t, \quad m,n > 0,\ 0 < q^{-1} < 1. \tag{17}$$

Also, it is known that

$$B_{q^{-1}}(m,n) = \frac{[m-1]_{q^{-1}}![n-1]_{q^{-1}}!}{[m+n-1]_{q^{-1}}!}, \quad 0 < q^{-1} < 1. \tag{18}$$

For $m = 0, 1, \ldots$, we have

$$\begin{aligned}
&[n-1]_{q^{-1}}q^{k-1}\int_0^1 t^m p_{n-2,k-1}\left(q^{-1};q^{-1}t\right)d_{q^{-1}}t \\
&\quad = [n-1]_{q^{-1}}\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^{-1}} q^{m(k-n)} \\
&\qquad \times \int_0^1 t^{k+m-1}\left(1 - q^{-1}t\right)_{q^{-1}}^{n-k-1}d_{q^{-1}}t \\
&\quad = q^{m(k-n)}\frac{[n-1]_{q^{-1}}!}{[k-1]_{q^{-1}}![n-k-1]_{q^{-1}}!}B_{q^{-1}}(k+m,n-k)
\end{aligned}$$

$$= q^{m(k-n)} \frac{[n-1]_{q^{-1}}!}{[k-1]_{q^{-1}}![n-k-1]_{q^{-1}}!} \times \frac{[k+m-1]_{q^{-1}}![n-k-1]_{q^{-1}}!}{[k+m+n-k-1]_{q^{-1}}!}$$
$$= \frac{[n-1]_q![k+m-1]_q!}{[k-1]_q![n+m-1]_q!} = \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}. \tag{19}$$

Thus, we get the following formula for $U_{n,q}(e_m; z)$:

$$\begin{aligned} U_{n,q}(e_m; z) &= f(0)\, p_{n,0}(q; z) + f(1)\, p_{n,n}(q; z) \\ &\quad + [n-1]_{q^{-1}} \sum_{k=1}^{n-1} p_{n,k}(q; z) \\ &\quad \times \int_0^1 p_{n-2,k-1}\left(q^{-1}; q^{-1}t\right) f\left(q^{k-n}t\right) d_{q^{-1}}t \\ &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}. \end{aligned} \tag{20}$$

Note that, for $m = 0, 1, 2$, we have

$$U_{n,q}(e_0; z) = 1, \qquad U_{n,q}(e_1; z) = z,$$
$$U_{n,q}(e_2; z) = z^2 + \frac{(1+q)\, z(1-z)}{[n+1]}. \tag{21}$$

Lemma 7. *$U_{n,q}(e_m; z)$ is a polynomial of degree less than or equal to* $\min(m, n)$ *and*

$$U_{n,q}(e_m; z) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s B_{n,q}(e_s; z). \tag{22}$$

Proof. From (20) it follows that

$$\begin{aligned} U_{n,q}(e_m; z) &= \sum_{k=1}^{n} p_{n,k}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q} \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=1}^{n} [k]_q [k+1]_q \cdots [k+m-1]_q\, p_{n,k}(q; z). \end{aligned} \tag{23}$$

Now using

$$[k]_q[k+1]_q \cdots [k+m-1]_q = \prod_{s=0}^{m-1} \left(q^s [k]_q + [s]_q\right) = \sum_{s=1}^{m} S_q(m, s)\, [k]_q^s, \tag{24}$$

where $S_q(m, s) > 0$, $s = 1, 2, \ldots, m$, are the constants independent of k, we get

$$\begin{aligned} U_{n,q}(e_m; z) &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=0}^{n} \sum_{s=1}^{m} S_q(m, s)\, [k]_q^s p_{n,k}(q; z) \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s B_{n,q}(e_s; z). \end{aligned} \tag{25}$$

Since $B_{n,q}(e_s; z)$ is a polynomial of degree less than or equal to $\min(s, n)$ and $S_q(m, s) > 0$, $s = 1, 2, \ldots, m$, it follows that $U_{n,q}(e_m; z)$ is a polynomial of degree less than or equal to $\min(m, n)$. □

Lemma 8. *The numbers $S_q(m, s)$, $(m, s) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$, given by* (24)*, enjoy the following properties:*

$$S_q(0, 0) = 1, \quad S_q(m, 0) = 0, \quad m \in N,$$
$$S_q(m+1, s) = [m]_q S_q(m, s) + q^m S_q(m, s-1), \quad m \in N_0,\ s \in N, \tag{26}$$
$$S_q(m+1, m+1) = q^m S_q(m, m),$$
$$S_q(m, s) = 0 \quad \textit{for } s > m.$$

Also, the following lemma holds.

Lemma 9. *For all $m, n \in \mathbb{N}$ the identity,*

$$\frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s = 1, \tag{27}$$

holds.

Proof. It follows from end points interpolation property of $U_{n,q}(e_m; z)$ and $B_{n,q}(e_s; z)$. Indeed

$$\begin{aligned} 1 = U_{n,q}(e_m; 1) &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s B_{n,q}(e_s; 1) \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s. \end{aligned} \tag{28}$$

□

Lemma 9 implies that for all $m, n \in \mathbb{N}$ and $|z| \leq r$ we have

$$\begin{aligned} &\left|U_{n,q}(e_m; z)\right| \\ &\quad \leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s \left|B_{n,q}(e_s; z)\right| \\ &\quad \leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} S_q(m, s)\, [n]_q^s r^s \leq r^m. \end{aligned} \tag{29}$$

For our purpose first we need a recurrence formula for $U_{n,q}(e_m; z)$.

Lemma 10. *For all $m, n \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$ one has*

$$U_{n,q}(e_{m+1}; z) = \frac{q^m z(1-z)}{[n+m]_q} D_q U_{n,q}(e_m; z) + \frac{q^m [n] z + [m]_q}{[n+m]_q} U_{n,q}(e_m; z). \tag{30}$$

Proof. By simple calculation we obtain (see [27])

$$\begin{aligned} z(1-z) D_q(p_{n,k}(q;z)) &= ([k]_q - [n]_q z) p_{n,k}(q;z), \\ U_{n,q}(e_m; z) &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q;z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q} \\ &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q;z) I_{k,m}, \\ I_{k,m} &:= \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}. \end{aligned} \tag{31}$$

It follows that

$$\begin{aligned} & z(1-z) D_q U_{n,q}(e_m; z) \\ &= [n]_q z(1-z) z^{n-1} + \sum_{k=1}^{n-1} ([k]_q - [n]_q z) p_{n,k}(q;z) I_{k,m} \\ &= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q;z) I_{k,m} \\ &\quad - [n]_q z \sum_{k=1}^{n-1} p_{n,k}(q;z) I_{k,m} - [n]_q z^{n+1} \\ &= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q;z) I_{k,m} \\ &\quad - z[n]_q U_{n,q}(e_m; z) \\ &= [n]_q z^n + q^{-m} \sum_{k=1}^{n-1} p_{n,k}(q;z) (q^m [k]_q + [m]_q - [m]_q) I_{k,m} \\ &\quad - z[n]_q U_{n,q}(e_m; z) \\ &= [n]_q z^n + q^{-m} \sum_{k=1}^{n-1} p_{n,k}(q;z) (q^m [k]_q + [m]_q - [m]_q) I_{k,m} \\ &\quad - z[n]_q U_{n,q}(e_m; z) \\ &= q^{-m} (q^m [n]_q + [m]_q - [m]_q) z^n \\ &\quad + q^{-m} [n+m]_q \sum_{k=1}^{n-1} p_{n,k}(q;z) I_{k,m+1} \\ &\quad - q^{-m} [m]_q \sum_{k=1}^{n-1} p_{n,k}(q;z) I_{k,m} - z[n]_q U_{n,q}(e_m; z) \\ &= q^{-m} [n+m]_q U_{n,q}(e_{m+1}; z) - q^{-m} [m]_q U_{n,q}(e_m; z) \\ &\quad - z[n]_q U_{n,q}(e_m; z), \end{aligned} \tag{32}$$

which implies the recurrence in the statement. □

Let

$$\Theta_{n,m}(q;z) := U_{n,q}(e_m; z) - z^m - \frac{1}{[n+1]_q} \times \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z). \tag{33}$$

Using the recurrence formula (30) we prove two more recurrence formulas.

Lemma 11. *For all $m, n \in \mathbb{N}$ and $z \in \mathbb{C}$ one has*

$$\begin{aligned} & U_{n,q}(e_m; z) - z^m \\ &= \frac{q^{m-1} z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) \\ &\quad + \frac{q^{m-1} [n] z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1}, \end{aligned} \tag{34}$$

$$\begin{aligned} & \Theta_{n,m}(q;z) \\ &= \frac{q^{m-1} z(1-z)}{[n+m-1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{q^{m-1} [n] z + [m-1]_q}{[n+m-1]_q} \Theta_{n,m-1}(q;z) + R_{n,m}(q;z), \end{aligned} \tag{35}$$

where

$$\begin{aligned} & R_{n,m}(q;z) \\ &= \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \\ &\quad \times \left[(1 + q^{m-1}) + \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (z+1) \right] \\ &\quad \times z^{m-2} (1-z). \end{aligned} \tag{36}$$

Proof. From the recurrence formula in Lemma 10, for all $m \geq 2$, we get

$$\begin{aligned}
&U_{n,q}(e_m;z)-z^m\\
&=\frac{q^{m-1}z(1-z)}{[n+m-1]_q}D_qU_{n,q}(e_{m-1};z)\\
&\quad+\frac{q^{m-1}[n]z+[m-1]_q}{[n+m-1]_q}\left(U_{n,q}(e_{m-1};z)-z^{m-1}\right)\\
&\quad+\frac{q^{m-1}[n]z+[m-1]_q}{[n+m-1]_q}z^{m-1}-z^m\\
&=\frac{q^{m-1}z(1-z)}{[n+m-1]_q}D_qU_{n,q}(e_{m-1};z)\\
&\quad+\frac{q^{m-1}[n]z+[m-1]_q}{[n+m-1]_q}\left(U_{n,q}(e_{m-1};z)-z^{m-1}\right)\\
&\quad+\frac{[m-1]_q}{[n+m-1]_q}(1-z)z^{m-1},\\
&U_{n,q}(e_m;z)-z^m\\
&\quad-\frac{1}{[n+1]_q}\left(q\sum_{i=1}^{m-1}[i]_q+\sum_{i=1}^{m-1}[i]_{q^{-1}}\right)z^{m-1}(1-z)\\
&=\frac{q^{m-1}z(1-z)}{[n+m-1]_q}D_q\left(U_{n,q}(e_{m-1};z)-z^{m-1}\right)\\
&\quad+\frac{q^{m-1}[n]z+[m-1]_q}{[n+m-1]_q}\\
&\quad\times\Bigg(U_{n,q}(e_m;z)-z^{m-1}-\frac{1}{[n+1]_q}\\
&\qquad\times\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right)z^{m-2}(1-z)\Bigg)\\
&\quad+R_{n,m}(q;z),
\end{aligned}\tag{37}$$

where

$$\begin{aligned}
&R_{n,m}(q;z)\\
&=\frac{[m-1]_q}{[n+m-1]_q}(1-z)z^{m-1}\\
&\quad-\frac{1}{[n+1]_q}\left(q\sum_{i=1}^{m-1}[i]_q+\sum_{i=1}^{m-1}[i]_{q^{-1}}\right)z^{m-1}(1-z)\\
&\quad+\frac{q^{m-1}[m-1]_q}{[n+m-1]_q}(1-z)z^{m-1}\\
&\quad+\frac{q^{m-1}[n]z+[m-1]_q}{[n+m-1]_q}\frac{1}{[n+1]_q}\\
&\quad\times\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right)z^{m-2}(1-z)\\
&:=T_{n'm}(q)z^{m-1}(1-z)+\frac{[m-1]_q}{[n+m-1]_q[n+1]_q}\\
&\quad\times\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right)z^{m-2}(1-z).
\end{aligned}\tag{38}$$

Again by simple calculation we obtain

$$\begin{aligned}
&T_{n,m}(q)\\
&=\frac{[m-1]_q}{[n+m-1]_q}-\frac{1}{[n+1]_q}\left(q\sum_{i=1}^{m-1}[i]_q+\sum_{i=1}^{m-1}[i]_{q^{-1}}\right)\\
&\quad+\frac{q^{m-1}[m-1]_q}{[n+m-1]_q}+\frac{q^{m-1}[n]_q}{[n+m-1]_q}\frac{1}{[n+1]_q}\\
&\quad\times\left(q\sum_{i=1}^{m-1}[i]_q+\sum_{i=1}^{m-1}[i]_{q^{-1}}\right)\\
&\quad-\frac{q^{m-1}[n]_q}{[n+m-1]_q}\frac{1}{[n+1]_q}\left(q[m-1]_q+[m-1]_{q^{-1}}\right)\\
&=\Bigg(\frac{[m-1]_q}{[n+m-1]_q}+\frac{q^{m-1}[m-1]_q}{[n+m-1]_q}-\frac{q^{m-1}[n]_q}{[n+m-1]_q}\\
&\qquad\times\frac{1}{[n+1]_q}\left(q[m-1]_q+[m-1]_{q^{-1}}\right)\Bigg)\\
&\quad+\left(\frac{q^{m-1}[n]_q}{q^{m-1}[n]_q+[m-1]_q}-1\right)\frac{1}{[n+1]_q}\\
&\quad\times\left(q\sum_{i=1}^{m-1}[i]_q+\sum_{i=1}^{m-1}[i]_{q^{-1}}\right)\\
&:=T^1_{n,m}(q)+T^2_{n,m}(q),
\end{aligned}\tag{39}$$

where $T^1_{n,m}(q)$ and $T^2_{n,m}(q)$ can be simplified as follows:

$$\begin{aligned}
T^2_{n,m}(q)&=\left(1-\frac{q^{m-1}[n]_q}{[n+m-1]_q}\right)\frac{1}{[n+1]_q}\\
&\quad\times\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right)\\
&=\frac{[m-1]_q}{[n+m-1]_q[n+1]_q}\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right),
\end{aligned}$$

$$
\begin{aligned}
T^1_{n,m}(q) &= \frac{[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} \\
&\quad - \frac{q^{m-1}[n]_q}{[n+m-1]_q}\frac{1}{[n+1]_q} \\
&\quad \times \left(q[m-1]_q + [m-1]_{q^{-1}}\right) \\
&= [m-1]_q\left(\frac{1}{[n+m-1]_q} - \frac{q}{[n+1]_q}\frac{q^{m-1}[n]_q}{[n+m-1]_q}\right) \\
&\quad + [m-1]_q\left(\frac{q^{m-1}}{[n+m-1]_q} - \frac{1}{[n+1]_q}\frac{q[n]_q}{[n+m-1]_q}\right) \\
&= [m-1]_q\frac{[n+1]_q - q^m[n]_q}{[n+m-1]_q[n+1]_q} \\
&\quad + [m-1]_q\frac{q^{m-1}[n+1]_q - q[n]_q}{[n+m-1]_q[n+1]_q} \\
&= [m-1]_q\frac{\left(1+q^{m-1}\right)[n+1]_q - \left(1+q^{m-1}\right)q[n]_q}{[n+m-1]_q[n+1]_q} \\
&= \frac{[m-1]_q\left(1+q^{m-1}\right)}{[n+m-1]_q[n+1]_q}.
\end{aligned} \tag{40}
$$

□

Lemma 12. *Let $q > 1$ and $f \in H(\mathbb{D}_R)$. The function $L_q(f;z)$ has the following representation:*

$$
L_q(f;z) = \sum_{m=2}^{\infty} a_m\left(q\sum_{i=1}^{m-1}[i]_q + \sum_{i=1}^{m-1}[i]_{q^{-1}}\right)z^{m-1}(1-z), \quad z \in \mathbb{D}_R. \tag{41}
$$

Proof. Using the following identity:

$$
\begin{aligned}
&[m]_q - m \\
&= 1 + q + q^2 + \cdots + q^{m-1} - m \\
&= (1-1) + (q-1) + \left(q^2-1\right) + \cdots + \left(q^{m-1}-1\right) \\
&= (q-1)[1]_q + (q-1)[2]_q + \cdots + (q-1)[m-1]_q \\
&= (q-1)\left([1]_q + \cdots + [m-1]_q\right) = (q-1)\sum_{i=1}^{m-1}[i]_q,
\end{aligned} \tag{42}
$$

we get

$$
\begin{aligned}
&L_q(f;z) \\
&= \sum_{m=2}^{\infty} a_m\left(\frac{q\left([m]_q - [m]_{q^{-1}}\right)}{q-1}\right)z^{m-1}(1-z) \\
&= \sum_{m=2}^{\infty} a_m\left(\frac{q\left([m]_q - m\right)}{q-1} + \frac{[m]_{q^{-1}} - m}{q^{-1}-1}\right)z^{m-1}(1-z) \\
&= \sum_{m=2}^{\infty} a_m\left(q\sum_{i=1}^{m-1}[i]_q + \sum_{i=1}^{m-1}[i]_{q^{-1}}\right)z^{m-1}(1-z),
\end{aligned} \tag{43}
$$

where $f(z) = \sum_{m=0}^{\infty} a_m z^m$. □

4. Proofs of the Main Results

Firstly we prove that $U_{n,q}(f;z) = \sum_{m=0}^{\infty} a_m U_{n,q}(e_m, z)$. Indeed denoting $f_k(z) = \sum_{j=0}^{k} a_j z^j$, $|z| \le r$ with $m \in \mathbb{N}$, by the linearity of $U_{n,q}$, we have

$$
U_{n,q}(f_k, z) = \sum_{m=0}^{k} a_m U_{n,q}(e_m, z), \tag{44}
$$

and it is sufficient to show that, for any fixed $n \in \mathbb{N}$ and $|z| \le r$ with $r \ge 1$, we have $\lim_{k\to\infty} U_{n,q}(f_k, z) = U_{n,q}(f;z)$. But this is immediate from $\lim_{k\to\infty}\|f_k - f\|_r = 0$, the norm being defined as $\|f\|_r = \max\{|f(z)| : |z| \le r\}$, and from the inequality

$$
\begin{aligned}
&\left|U_{n,q}(f_k, z) - U_{n,q}(f, z)\right| \\
&\le \left|f_k(0) - f(0)\right| \cdot \left|(1-z)^n\right| + \left|f_k(1) - f(1)\right| \cdot \left|z^n\right| \\
&\quad + [n+1]_{q^{-1}}\sum_{j=1}^{n-1}\left|p_{n,j}(q;z)\right| q^{j-1} \\
&\quad \times \int_0^1 p_{n-2,j-1}\left(q^{-1}, q^{-1}t\right)\left|f_k(t) - f(t)\right| d_{q^{-1}}t \\
&\le C_{r,n}\|f_k - f\|_r,
\end{aligned} \tag{45}
$$

valid for all $|z| \le r$, where

$$
\begin{aligned}
C_{r,n} &= (1+r)^n + r^n + [n+1]_{q^{-1}} \\
&\quad \times \sum_{j=1}^{n-1}\begin{bmatrix} n \\ j \end{bmatrix}_q (1+r)^{n-j} r^j q^{j-1} \\
&\quad \times \int_0^1 p_{n-2,j-1}\left(q^{-1}; q^{-1}t\right) d_{q^{-1}}t \\
&= (1+r)^n + r^n \\
&\quad + \sum_{j=1}^{n-1}\begin{bmatrix} n \\ j \end{bmatrix}_q \left(1 + q^{n-j}r\right)^{n-j} r^j q^{j-1}.
\end{aligned} \tag{46}
$$

Therefore we get

$$\begin{aligned}&\left|U_{n,q}(f;z)-f(z)\right|\\&\quad\leq\sum_{m=0}^{\infty}|a_m|\left|U_{n,q}(e_m,z)-e_m(z)\right|=\sum_{m=2}^{\infty}|a_m|\\&\quad\quad\times\left|U_{n,q}(e_m,z)-e_m(z)\right|,\end{aligned}\tag{47}$$

as $U_{n,q}(e_0,z)=e_0(z)$ and $U_{n,q}(e_1,z)=e_1(z)$.

Proof of Theorem 2. From the recurrence formula (34) and the inequality (29) for $m\geq 2$ we get

$$\begin{aligned}&\left|U_{n,q}(e_m;z)-z^m\right|\\&\quad\leq\frac{q^{m-1}z(1-z)}{q^{m-2}[n+1]_q+[m-2]_q}\left|D_qU_{n,q}(e_{m-1};z)\right|\\&\qquad+\frac{q^{m-1}[n]z+[m-1]_q}{q^{m-1}[n]_q+[m-1]_q}\\&\qquad\times\left|U_{n,q}(e_{m-1};z)-z^{m-1}\right|\\&\qquad+\frac{[m-1]_q}{q^{m-2}[n+1]_q+[m-2]_q}|1-z||z|^{m-1}.\end{aligned}\tag{48}$$

It is known that, by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$\left|P_k'(z)\right|\leq\frac{k}{qr_1}\|P_k\|_{qr},\quad\forall|z|\leq qr,\ r\geq1,\tag{49}$$

which, combined with the mean value theorem in complex analysis, implies

$$\left|D_q(P_k;z)\right|\leq\left\|P_k'\right\|_{qr}\leq\frac{k}{qr}\|P_k\|_{qr},\tag{50}$$

for all $|z|\leq qr$, where $P_k(z)$ is a complex polynomial of degree $\leq k$. It follows that

$$\begin{aligned}&\left|U_{n,q}(e_m;z)-z^m\right|\\&\quad\leq\frac{q^{m-1}r(1+r)}{q^{m-2}[n+1]_q+[m-2]_q}\frac{m-1}{qr}\left\|U_{n,q}(e_{m-1})\right\|_{qr}\\&\qquad+r\left|U_{n,q}(e_{m-1};z)-z^{m-1}\right|+\frac{[m-1]_{1/q}}{[n+1]_q}(1+r)r^{m-1}\\&\quad\leq\frac{(m-1)}{[n+1]_q}(1+r)q^{m-1}r^{m-1}\\&\qquad+r\left|U_{n,q}(e_{m-1};z)-z^{m-1}\right|+\frac{[m-1]_{1/q}}{[n+1]_q}(1+r)r^{m-1}\\&\quad\leq2q(m-1)\frac{r(1+r)}{[n+1]_q}(qr)^{m-2}\\&\qquad+r\left|U_{n,q}(e_{m-1};z)-z^{m-1}\right|.\end{aligned}\tag{51}$$

By writing the last inequality for $m=2,3,\ldots$, we easily obtain, step by step, the following:

$$\begin{aligned}&\left|U_{n,q}(e_m;z)-z^m\right|\\&\leq r\left(r\left|U_{n,q}(e_{m-2};z)-z^{m-2}\right|+2\frac{(m-2)}{[n+1]_q}r(1+r)(qr)^{m-3}\right)\\&\quad+2\frac{(m-1)}{[n+1]_q}r(1+r)(qr)^{m-2}\\&=r^2\left|U_{n,q}(e_{m-2};z)-z^{m-2}\right|\\&\quad+2\frac{r(1+r)}{[n+1]_q}r^{m-2}(m-1+m-2)\\&\leq\cdots\leq\frac{r(1+r)}{[n+1]_q}m(m-1)q^{m-2}r^{m-2}.\end{aligned}\tag{52}$$

It follows that

$$\begin{aligned}\left|U_{n,q}(f;z)-f(z)\right|&\leq\sum_{m=2}^{\infty}|a_m|\left|U_{n,q}(e_m;z)-z^m\right|\\&\leq\frac{r(1+r)}{[n+1]_q}\sum_{m=2}^{\infty}|a_m|m(m-1)q^{m-2}r^{m-2}.\end{aligned}\tag{53}$$

□

The second main result of the paper is the Voronovskaja-type theorem with a quantitative estimate for the complex version of genuine q-Bernstein-Durrmeyer polynomials.

Proof of Theorem 3. By Lemma 11 we have

$$\begin{aligned}&\Theta_{n,m}(q;z)\\&\quad=\frac{q^{m-1}z(1-z)}{[n+m-1]_q}D_q\left(U_{n,q}(e_{m-1};z)-z^{m-1}\right)\\&\qquad+\frac{q^{m-1}[n]z+[m-1]_q}{[n+m-1]_q}\Theta_{n,m-1}(q;z)+R_{n,m}(q;z),\end{aligned}\tag{54}$$

where

$$\begin{aligned}&R_{n,m}(q;z)\\&\quad=\frac{[m-1]_q}{[n+m-1]_q[n+1]_q}\\&\qquad\times\left[\left(1+q^{m-1}\right)+\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right)(z+1)\right]\\&\qquad\times z^{m-2}(1-z).\end{aligned}\tag{55}$$

It follows that

$$\begin{aligned}
&\left|R_{n,m}(q;z)\right| \\
&\leq \frac{[m-1]_q}{[n+1]_q^2} \\
&\quad\times\left(\left(1+q^{m-1}\right)r+\left(q\sum_{i=1}^{m-2}[i]_q+\sum_{i=1}^{m-2}[i]_{q^{-1}}\right)(1+r)\right) \\
&\quad\times(1+r)\,r^{m-2} \\
&\leq \frac{[m-1]_q}{[n+1]_q^2} \\
&\quad\times\left(\left(1+q^{m-1}\right)+\left(q(m-2)[m-2]_q+(m-2)^2\right)\right) \\
&\quad\times(1+r)^2 r^{m-2} \\
&= \frac{q^{m-2}[m-1]_{q^{-1}}}{[n+1]_q^2}q^{m-2} \\
&\quad\times\left(\left(\frac{1}{q^{m-2}}+q\right)+(m-2)[m-2]_{q^{-1}}+\frac{1}{q^{m-2}}(m-2)^2\right) \\
&\quad\times(1+r)^2 r^{m-2} \\
&\leq \frac{3}{[n+1]_q^2}(m-1)(m-2)^2(1+r)^2\left(q^2r\right)^{m-2}
\end{aligned} \tag{56}$$

for all $m \geq 2$, $n \in \mathbb{N}$, and $z \in \mathbb{C}$. Equation (54) implies that for $|z| \leq r$

$$\begin{aligned}
&\left|\Theta_{n,m}(q;z)\right| \\
&\quad\leq r\left|\Theta_{n,m-1}(q;z)\right|+\frac{q^{m-1}r(1+r)}{q^{m-2}[n+1]_q}\frac{m-1}{qr} \\
&\qquad\times\left\|U_{n,q}(e_{m-1})-e_{m-1}\right\|_{qr} \\
&\qquad+\frac{3}{[n+1]_q^2}(m-1)(m-2)^2(1+r)^2\left(q^2r\right)^{m-2} \\
&\quad\leq r\left|\Theta_{n,m-1}(q;z)\right|+\frac{r^2(1+r)^2}{[n+1]_q^2} \\
&\qquad\times(m-1)^2(m-2)\left(q^2r\right)^{m-3} \\
&\qquad+\frac{3}{[n+1]_q^2}(m-1)(m-2)^2(1+r)^2\left(q^2r\right)^{m-2} \\
&\quad\leq r\left|\Theta_{n,m-1}(q;z)\right|+\frac{4r^2(1+r)^2}{[n+1]_q^2} \\
&\qquad\times(m-1)^2(m-2)\left(q^2r\right)^{m-2}.
\end{aligned} \tag{57}$$

By writing the last inequality for $m = 3, 4, \ldots$, we easily obtain, step by step, the following:

$$\begin{aligned}
&\left|U_{n,q}(f;z)-f(z)-\frac{1}{[n+1]_q}L_q(f;z)\right| \\
&\quad\leq \frac{4r^2(1+r)^2}{[n+1]_q^2}\sum_{m=2}^{\infty}\left|a_m\right|\left(q^2r\right)^{m-2} \\
&\qquad\times\sum_{j=2}^{m}(j-1)^2(j-2)\leq\frac{4r^2(1+r)^2}{[n+1]_q^2} \\
&\qquad\times\sum_{m=2}^{\infty}\left|a_m\right|(m-1)^2(m-2)^2\left(q^2r\right)^{m-2}.
\end{aligned} \tag{58}$$

□

Proof of Theorem 4. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we get

$$\begin{aligned}
&U_{n,q}(f;z)-f(z) \\
&\quad=\frac{1}{[n+1]_q}\left\{L_q(f;z)+[n+1]_q\right. \\
&\qquad\left.\times\left(U_{n,q}(f;z)-f(z)-\frac{1}{[n+1]_q}L_q(f;z)\right)\right\}.
\end{aligned} \tag{59}$$

It follows that

$$\begin{aligned}
&\left\|U_{n,q}(f)-f\right\|_r \\
&\quad\geq\frac{1}{[n+1]_q}\left\{\left\|L_q(f;z)\right\|_r-[n+1]_q\right. \\
&\qquad\left.\times\left\|U_{n,q}(f)-f-\frac{1}{[n+1]_q}L_q(f;z)\right\|_r\right\}.
\end{aligned} \tag{60}$$

Because by hypothesis f is not a polynomial of degree ≤1 in $\mathbb{D}_R$, it follows $\|L_q(f;z)\|_r > 0$. Indeed, assuming the contrary it follows that $L_q(f;z) = 0$ for all $z \in \overline{\mathbb{D}_r}$; that is, $D_q f(z) = D_{q^{-1}} f(z)$ for all $z \in \overline{\mathbb{D}_r}$. Thus $a_m = 0$, $m = 2, 3, \ldots$ and f is linear, which is a contradiction with the hypothesis.

Now, by Theorem 3, we have

$$\begin{aligned}
&[n+1]_q\left|U_{n,q}(f;z)-f(z)-\frac{1}{[n+1]_q}L_q(f;z)\right| \\
&\quad\leq\frac{4r^2(1+r)^2}{[n+1]_q}\sum_{m=3}^{\infty}\left|a_m\right|(m-1)^2(m-2)^2\left(q^2r\right)^{m-2} \\
&\quad\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{61}$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\begin{aligned}
&\left\|L_q(f;z)\right\|_r-[n+1]_q\left\|U_{n,q}(f)-f-\frac{1}{[n+1]_q}L_q(f;z)\right\|_r \\
&\quad\geq\frac{1}{2}\left\|L_q(f;z)\right\|_r,
\end{aligned} \tag{62}$$

which implies that

$$\left\|U_{n,q}(f)-f\right\|_r \geq \frac{1}{2[n+1]_q}\left\|L_q(f;z)\right\|_r, \quad \forall n \geq n_1. \quad (63)$$

For $1 \leq n \leq n_1 - 1$ we have

$$\begin{aligned}\left\|U_{n,q}(f)-f\right\|_r &\geq \frac{1}{[n+1]_q}\left([n+1]_q\left\|U_{n,q}(f)-f\right\|_r\right)\\ &= \frac{1}{[n+1]_q}M_{r,n,q}(f) > 0,\end{aligned} \quad (64)$$

which finally implies that

$$\left\|U_{n,q}(f)-f\right\|_r \geq \frac{1}{[n+1]_q}C_{r,q}(f), \quad (65)$$

for all n, with $C_{r,q}(f) = \min\{M_{r,1,q}(f),\ldots,M_{r,n_1-1,q}(f), (1/2)\|L_q(f;z)\|_r\}$, which ends the proof. □

Proof of Theorem 6. Let $1 \leq r < R$, $1 < q_0 < R/r$ be fixed. Then, by Lemma 12 for any $1 \leq q \leq q_0$ and $|z| \leq r$, we have

$$\begin{aligned}L_q(f;z) &= \sum_{m=2}^{\infty} a_m\left(q\sum_{i=1}^{m-1}[i]_q + \sum_{i=1}^{m-1}[i]_{q^{-1}}\right)z^{m-1}(1-z),\\ L_1(f;z) &= \sum_{m=2}^{\infty} a_m m(m-1)z^{m-1}(1-z).\end{aligned} \quad (66)$$

Using the inequality

$$\begin{aligned}&\left|q\sum_{i=1}^{m-1}[i]_q - \frac{m(m-1)}{2}\right|\\ &= q\sum_{i=2}^{m-1}\left([i]_q - i\right) + (q-1)\frac{m(m-1)}{2}\\ &= q(q-1)\sum_{i=2}^{m-1}\sum_{j=1}^{i}[j]_q + (q-1)\frac{m(m-1)}{2}\\ &\leq q(q-1)[m-1]_q\frac{m(m-1)}{2} + (q-1)\frac{m(m-1)}{2}\\ &= (q-1)\frac{m(m-1)}{2}\left(q[m-1]_q + 1\right)\\ &\leq (q-1)q^{m-1}\frac{m^2(m-1)}{2},\\ &\left|\sum_{i=1}^{m-1}[i]_{q^{-1}} - \frac{m(m-1)}{2}\right| = \sum_{i=2}^{m-1}\left(i - [i]_{q^{-1}}\right)\\ &= \left(1-q^{-1}\right)\sum_{i=2}^{m-1}\sum_{j=1}^{i}[j]_{q^{-1}}\\ &\leq \left(1-q^{-1}\right)\frac{m(m-1)^2}{2},\end{aligned} \quad (67)$$

we get, for $1 \leq q \leq q_0$ and $|z| \leq r$,

$$\begin{aligned}&\left|L_q(f;z) - L_1(f;z)\right|\\ &\leq \sum_{m=2}^{N-1}|a_m|\left|q\sum_{i=1}^{m-1}[i]_q - \frac{m(m-1)}{2}\right|\left|z^{m-1}-z^m\right|\\ &\quad + \sum_{m=N}^{\infty}|a_m|\left|q\sum_{i=1}^{m-1}[i]_q - \frac{m(m-1)}{2}\right|\left|z^{m-1}-z^m\right|\\ &\quad + \sum_{m=2}^{N-1}|a_m|\left|\sum_{i=1}^{m-1}[i]_{q^{-1}} - \frac{m(m-1)}{2}\right|\left|z^{m-1}-z^m\right|\\ &\quad + \sum_{m=N}^{\infty}|a_m|\left|\sum_{i=1}^{m-1}[i]_{q^{-1}} - \frac{m(m-1)}{2}\right|\left|z^{m-1}-z^m\right|\\ &\leq (q-1)\sum_{m=2}^{N-1}|a_m|m^2(m-1)q_0^{m-1}r^m\\ &\quad + 4\sum_{m=N}^{\infty}|a_m|(m-1)^2q_0^m r^m\\ &\quad + \left(1-q^{-1}\right)\sum_{m=2}^{N-1}|a_m|m(m-1)^2r^m\\ &\quad + 2\sum_{m=N}^{\infty}|a_m|m(m-1)r^m.\end{aligned} \quad (68)$$

Since $f \in H(\mathbb{D}_R)$, we can find that $N = N_\varepsilon \in \mathbb{N}$ such that

$$4\sum_{m=N}^{\infty}|a_m|(m-1)^2q_0^m r^m + 2\sum_{m=N}^{\infty}|a_m|m(m-1)r^m < \frac{\varepsilon}{2}. \quad (69)$$

Thus, for q sufficiently close to 1 from the right, we conclude that

$$\lim_{q \to 1^{+1}} L_q(f;z) = L_1(f;z) \quad (70)$$

uniformly on $\mathbb{D}_r$. The proof is finished. □

Proof of Theorem 5. Then, by Theorem 3, we get $L_q(f;z) = \lim_{n\to\infty}[n+1]_q(U_{n,q}(f;z) - f(z)) = 0$ for infinite number of points having an accumulation point on $\mathbb{D}_{R/q^2}$. Since $L_q(f;z) \in H(\mathbb{D}_{R/q^2})$, by the unicity Theorem for analytic functions, we get $L_q(f;z) = 0$ in $\mathbb{D}_{R/q^2}$, and, therefore, by (11), $a_m = 0$, $m = 2, 3, \ldots$. Thus, f is linear. Theorem 5 is proved. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author dedicates this paper to Professor Agamirza E. Bashirov at his 60th anniversary.

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