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Research Article

Existence of Positive Solutions of Semilinear Biharmonic Equations

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This paper is concerned with the existence of positive solutions of semilinear biharmonic problem whose associated functionals do not satisfy the Palais-Smale condition.

1. Introduction

We consider the semilinear biharmonic problem

$$\begin{aligned} \Delta^2 u &= f(x,u) \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 5$.

Problems of this type have been studied in [1–7]. In [6] Liu and Wang studied (1) when $f(x,t)$ is asymptotically linear with respect to t at infinity. In order to find critical points of the functional Φ associated with (1), one usually applies the Mountain Pass Theorem proposed by Ambrosetti and Rabinowitz [8]. For applying the theorem, one often requires the following condition, that is, for some $\theta > 2$ and $M > 0$:

$$0 < \theta F(x,t) \leq t f(x,t) \quad \text{for } |t| \geq M, \tag{2}$$

where $F(x,t) = \int_0^t f(x,s)ds$. Condition (2) is important for ensuring that each Palais-Smale sequence is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. We say Φ satisfies the Palais-Smale condition (henceforth denoted by (PS)) if any sequence $\{u_n\}$ for which $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. Note that the nonlinear term $f(x,t)$ is asymptotically linear, not superlinear, with respect to t at infinity, which means that condition (2) cannot be assumed in their case. Lin et al. use some tricks and techniques to prove that the (PS) sequence is bounded. Then they use Mountain Pass Theorem to get a positive solution to (1).

In [2] Ramos and Rodrigues considered (1) with the nonlinearity $f(x,t) = \mu t + a(x)g(t)$, where μ is a real parameter, $a \in C^1(\overline{\Omega})$ changes sign in Ω, and $g \in C^1(\mathbb{R})$ is subcritical and has a superlinear behavior both at zero and at infinity. They extended for the biharmonic operator results that were obtained for the corresponding second order problem in [9]. Their assumptions on f do not seem to imply suitable compactness properties (namely, the so-called Palais-Smale condition) for the corresponding functional, if one uses a variational argument. Moreover, due to the absence of sign in the nonlinear term, it is not clear whether the geometric structure of the functional associated with (1) falls into one of the usual schemes used in critical point theory.

In this paper, we suppose that f satisfies the following:

(H1) $f \in C^1(\overline{\Omega} \times \mathbb{R})$, $\partial f_t'/\partial x_i \in C(\overline{\Omega} \times \mathbb{R})(1 \leq i \leq n)$, and $f(x,t) \geq 0$ if $t \geq 0$;

(H2) $f(x,0) = f_t'(x,0) = 0$ for all $x \in \overline{\Omega}$;

(H3) there exist $T > 0$ and $1 < p < (N+4)/(N-4)$ such that

$$\left|f_t'(x,t)\right| \leq C|t|^{p-1}, \qquad \left|\nabla_x f(x,t)\right| \leq C|t|^p, \qquad \left|\nabla_x f_t'(x,t)\right| \leq C|t|^{p-1} \tag{3}$$

for $|t| \geq T$ and $x \in \overline{\Omega}$;

(H4) there exist $\mu > 0$ and $T > 0$ such that $f(x,t) \geq \mu t^p$ for all $x \in \overline{\Omega}$ and $t \geq T$.

This type of hypotheses assumed here does not imply the (PS) condition and does not fit in the condition that implies a priori bounds. Recently, de Figueiredo and Yang [10], Liu et al. [11], and Ramos et al. [9] have considered semilinear second order elliptic problems without the (PS) condition. Our assumptions (H1)–(H4) exactly come from [10]. In [9, 10] the link between the Morse index and the L^∞ bounds of solutions is shown. In [12] Bahri and Lions mentioned that bounds on Morse indices are useful in some problems to prove the Palais-Smale compactness condition.

For the reader's convenience, we give an example: $f(x,t) = t^p[2 + \sin(\ln t)\cos|x|^2]$ for $x \in \overline{\Omega}$, $t \geq 0$, where $p > 1$. Due to advances of our method and our interest in positive solutions, without loss of generality, we may assume that $f(x,t) = 0$ for $t \leq 0$. It is easy to see that $f(x,t)$ satisfies the conditions (H1)–(H4). Moreover, it is obvious that $f(x,t)$ does not satisfy the hypothetical conditions on nonlinearity in [2].

Our main result is the following.

Theorem 1. *Suppose f satisfies (H1)–(H4). Problem* (1) *has at least a positive solution.*

The organization of the paper is as follows. In Section 2 we prove some new nonlinear Liouville type theorems which may be useful in other situations. In Section 3 we prove Theorem 1. Firstly, we apply the Mountain Pass Theorem to a suitable sequence of truncated problems. In particular, it follows that the Morse index of the solutions of the truncated problems is finite. We use this fact and the blow-up argument to show that the sequence of the truncated problems is bounded. A version of the well-known Pohozaev identity is in turn essential. Throughout this paper, the constant C will denote various generic constants.

2. Liouville Type Theorems

For $R > 2r > 0$, let $\psi_{r,R} \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfying

$$\begin{gathered}\psi_{r,R}(x) = 1, \quad x \in B_R \setminus B_{2r},\\ \psi_{r,R}(x) = 0, \quad x \in B_r \cup B_{2R}^c,\\ \left|\nabla\psi_{r,R}\right| \leq \frac{C}{R}, \quad x \in B_R^c,\\ \left|D^2\psi_{r,R}(x)\right| \leq \frac{C}{R^2}, \quad x \in B_R^c.\end{gathered} \tag{4}$$

Define

$$J'(u)v = \int_{\mathbb{R}^N} \Delta u \Delta v\,dx - \int_{\mathbb{R}^N} Q(x)u^p v\,dx, \quad \forall v \in C_0^\infty\left(\mathbb{R}^N\right), \tag{5}$$

and then

$$J''(u)\varphi^2 = \int_{\mathbb{R}^N} |\Delta\varphi|^2 dx - p\int_{\mathbb{R}^N} Q(x)u^{p-1}\varphi^2 dx, \quad \forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{6}$$

Lemma 2. *Suppose that Q is a function satisfying $0 < \mu \leq Q \leq C$, where μ and C are constants. Let u be a nonnegative solution of the following problem:*

$$\Delta^2 u = Q(x)u^p \quad \text{in } \mathbb{R}^N, \tag{7}$$

with finite Morse index, where $1 < p < (N+4)/(N-4)$. Then there exists $r_0 > 0$ such that $J''(u)(\psi_{r_0,R}^2 u)^2 \geq 0$, $\forall R > 2r_0$.

The Morse index of solutions of (7) is defined as the dimension of the negative space corresponding to the spectral decomposition of the operator $\Delta^2 - pQu^{p-1}$.

Proof of Lemma 2. Suppose the assertion is false. Then for $r_1 > 0$, there exists $R_1 > 2r_1$ such that $J''(u)(\psi_{r_1,R_1}^2 u)^2 < 0$ and for $r_2 > 2R_1$, we may find $R_2 > 2r_2$ such that $J''(u)(\psi_{r_2,R_2}^2 u)^2 < 0$. Then the supports of $\psi_{r_1,R_1}^2 u$ and $\psi_{r_2,R_2}^2 u$ are disjoint, so the Morse index of u is larger than or equal to 2. Iterating the argument, we may get a contradiction since the Morse index of u is supposed to be finite. □

Proposition 3. *Let u be a bounded nonnegative solution with finite Morse index of* (7). *Then both $\|\Delta u\|_{L^2(\mathbb{R}^N)}$ and $\|u\|_{L^{p+1}(\mathbb{R}^N)}$ are finite.*

Proof. By Lemma 2, there exists a $r_0 > 0$ such that

$$J''(u)\left(\psi_{r_0,R}^2 u\right)^2 \geq 0, \quad \forall R > 2r_0. \tag{8}$$

That is,

$$\begin{aligned}\int_{\mathbb{R}^N} \Big[&4\left(\psi_{r_0,R}\Delta\psi_{r_0,R} + \left|\nabla\psi_{r_0,R}\right|^2\right)^2 u^2\\ &+ 16\psi_{r_0,R}^2\left(\nabla\psi_{r_0,R}\nabla u\right)^2 + \psi_{r_0,R}^4|\Delta u|^2\\ &+ 16\left(\psi_{r_0,R}\Delta\psi_{r_0,R} + \left|\nabla\psi_{r_0,R}\right|^2\right)\\ &\times u\psi_{r_0,R}\nabla\psi_{r_0,R}\nabla u + 8\psi_{r_0,R}^3\Delta u\nabla\psi_{r_0,R}\nabla u\\ &+4\left(\psi_{r_0,R}\Delta\psi_{r_0,R} + \left|\nabla\psi_{r_0,R}\right|^2\right)u\psi_{r_0,R}^2\Delta u\Big]\,dx\\ \geq p&\int_{\mathbb{R}^N} Q(x)u^{p+1}\psi_{r_0,R}^4 dx.\end{aligned} \tag{9}$$

Multiplying (7) by $u\psi_{r_0,R}^4$, we find

$$\begin{aligned}\int_{\mathbb{R}^N} \Big[&\psi_{r_0,R}^4 + 8\psi_{r_0,R}^3\Delta u\nabla u\nabla\psi_{r_0,R}\\ &+4\left(3\psi_{r_0,R}^2\left|\nabla\psi_{r_0,R}\right|^2 + \psi_{r_0,R}^3\Delta\psi_{r_0,R}\right)u\Delta u\Big]\,dx\\ =&\int_{\mathbb{R}^N} Q(x)u^{p+1}\psi_{r_0,R}^4 dx.\end{aligned} \tag{10}$$

From (9) and (10), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Big[4\Big(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\Big)^2u^2\\
&\quad+16\psi_{r_0,R}^2\big(\nabla\psi_{r_0,R}\nabla u\big)^2\\
&\quad+16\left(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\right)u\psi_{r_0,R}\nabla\psi_{r_0,R}\nabla u\Big]\,dx\\
&\geq(p-1)\int_{\mathbb{R}^N}\left(\psi_{r_0,R}^4|\Delta u|^2+8\psi_{r_0,R}^3\Delta u\nabla u\nabla\psi_{r_0,R}\right.\\
&\qquad\left.+4u\psi_{r_0,R}^3\Delta\psi_{r_0,R}\Delta u\right)dx\\
&\quad+(12p-4)\int_{\mathbb{R}_N}\psi_{r_0,R}^2\left|\nabla\psi_{r_0,R}\right|^2u\Delta u\,dx\\
&\geq(p-1)\int_{\mathbb{R}^N}\Big[\psi_{r_0,R}^4|\Delta u|^2\\
&\qquad-4\left(\varepsilon\psi_{r_0,R}^4|\Delta u|^2+\frac{1}{\varepsilon}\psi_{r_0,R}^2|\nabla u|^2\left|\nabla\psi_{r_0,R}\right|^2\right)\\
&\qquad-2\left(\varepsilon\psi_{r_0,R}^4|\Delta u|^2+\frac{1}{\varepsilon}\psi_{r_0,R}^2u^2\left|\Delta\psi_{r_0,R}\right|^2\right)\Big]\,dx\\
&\quad-(6p-2)\int_{\mathbb{R}^N}\left(\varepsilon\psi_{r_0,R}^4|\Delta u|^2+\frac{1}{\varepsilon}\left|\nabla\psi_{r_0,R}\right|^4u^2\right)dx,
\end{aligned}\tag{11}$$

where $\varepsilon>0$ is small enough. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\psi_{r_0,R}^4|\Delta u|^2dx\\
&\quad\leq C\int_{\mathbb{R}^N}\Big[\Big(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\Big)^2u^2\\
&\qquad+\psi_{r_0,R}^2\left|\nabla\psi_{r_0,R}\right|^2|\nabla u|^2\\
&\qquad+\left(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\right)u\psi_{r_0,R}\nabla\psi_{r_0,R}\nabla u\\
&\qquad+\psi_{r_0,R}^2u^2\left|\Delta\psi_{r_0,R}\right|^2+\left|\nabla\psi_{r_0,R}\right|^4u^2\Big]\,dx.
\end{aligned}\tag{12}$$

Using the value of $\psi_{r_0,R}$, we get

$$\begin{aligned}
&\int_{B_{2R}}\psi_{r_0,R}^4|\Delta u|^2dx\\
&\quad\leq C\int_{B_{2R}}\Big[\psi_{r_0,R}^2\left|\Delta\psi_{r_0,R}\right|^2u^2+\left|\nabla\psi_{r_0,R}\right|^4u^2\\
&\qquad+\psi_{r_0,R}^2|\nabla u|^2\left|\nabla\psi_{r_0,R}\right|^2\Big]\,dx\\
&\quad=C\Big[\int_{B_{2r_0}\setminus B_{r_0}}\left(\psi_{r_0,R}^2\left|\Delta\psi_{r_0,R}\right|^2+C\left|\nabla\psi_{r_0,R}\right|^4\right)u^2dx\\
&\qquad+\int_{B_{2R}\setminus B_R}\left(\psi_{r_0,R}^2\left|\Delta\psi_{r_0,R}\right|^2+\left|\nabla\psi_{r_0,R}\right|^4\right)u^2dx\\
&\qquad+\int_{B_{2R}}\psi_{r_0,R}^2|\nabla u|^2\left|\nabla\psi_{r_0,R}\right|^2dx\Big]\\
&\quad\leq C\left(1+\frac{1}{R^4}\int_{B_{2R}}u^2dx+\frac{1}{R^2}\int_{B_{2R}}\psi_{r_0,R}^2|\nabla u|^2dx\right).
\end{aligned}\tag{13}$$

Using the interpolation inequality (see [13]), we obtain

$$\int_{B_{2R}}\psi_{r_0,R}^4|\Delta u|^2dx\leq C\left(1+\frac{1}{R^4}\int_{B_{2R}}u^2dx\right).\tag{14}$$

From (9) and (10), it follows that

$$\begin{aligned}
&(p-1)\int_{\mathbb{R}^N}Q(x)\,u^{p+1}\psi_{r_0,R}^4dx\\
&\leq\int_{\mathbb{R}^N}\Big[4\Big(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\Big)^2u^2\\
&\qquad+16\psi_{r_0,R}^2\big(\nabla\psi_{r_0,R}\nabla u\big)^2\\
&\qquad+16\left(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\right)u\psi_{r_0,R}\nabla\psi_{r_0,R}\nabla u\\
&\qquad-8\psi_{r_0,R}^2\left|\nabla\psi_{r_0,R}\right|^2u\Delta u\Big]\,dx.
\end{aligned}\tag{15}$$

Using the value of $\psi_{r_0,R}$ again, we have

$$\begin{aligned}
&\int_{B_R}Q(x)\,u^{p+1}\psi_{r_0,R}^4dx\\
&\quad\leq C\int_{B_{2R}}\Big[\Big(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\Big)^2u^2\\
&\qquad+\psi_{r_0,R}^2\big(\nabla\psi_{r_0,R}\nabla u\big)^2\\
&\qquad+\left(\psi_{r_0,R}\Delta\psi_{r_0,R}+\left|\nabla\psi_{r_0,R}\right|^2\right)u\psi_{r_0,R}\nabla\psi_{r_0,R}\nabla u\\
&\qquad-\psi_{r_0,R}^2\left|\nabla\psi_{r_0,R}\right|^2u\Delta u\Big]\,dx.
\end{aligned}\tag{16}$$

Estimating the right side of (16) by the argument exactly as above and using (14), we have

$$\int_{B_R}Q(x)\,u^{p+1}\psi_{r_0,R}^4dx\leq C\left(1+\frac{1}{R^4}\int_{B_{2R}}u^2dx\right),\tag{17}$$

where C does not depend on R.

Since $\psi_{r_0,R}=1$ over $B_R\setminus B_{2r_0}$,

$$\begin{aligned}
&\int_{B_R}Q(x)\,u^{p+1}\psi_{r_0,R}^4dx\\
&\quad=\int_{B_{2r_0}}Q(x)\,u^{p+1}\psi_{r_0,R}^4dx+\int_{B_R\setminus B_{2r_0}}Q(x)\,u^{p+1}\psi_{r_0,R}^4dx\\
&\quad=\int_{B_R}Q(x)\,u^{p+1}dx-\int_{B_{2r_0}}Q(x)\,u^{p+1}\left(1-\psi_{r_0,R}^4\right)dx.
\end{aligned}\tag{18}$$

By (17), we get

$$\int_{B_R} Q(x)\,u^{p+1}dx \le \int_{B_{2r_0}} Q(x)\,u^{p+1}\left(1-\psi_{r_0,R}^4\right)dx + C\left(1+\frac{1}{R^4}\int_{B_{2R}} u^2dx\right). \quad (19)$$

Hence

$$\int_{B_R} u^{p+1}dx \le C\left(1+\frac{1}{R^4}\int_{B_{2R}} u^2dx\right), \quad (20)$$

since $\mu \le Q(x) \le C$ and u is bounded.

We will prove $\int_{\mathbb{R}^N}|u|^{p+1}dx < +\infty$. Assume to the contrary that $\int_{\mathbb{R}^N}|u|^{p+1}dx$ is not finite; by (20), we obtain

$$\int_{B_R}|u|^{p+1}dx \le \frac{C}{R^4}\int_{B_{2R}} u^2dx, \quad (21)$$

for large R.

Using Hölder's inequality, we get

$$\int_{B_{2R}} u^2dx \le C\left(\int_{B_{2R}}|u|^{p+1}dx\right)^{2/(p+1)} R^{N(p-1)/(p+1)}, \quad (22)$$

Substituting (22) for the right-hand side in (21) gives.

$$\int_{B_R}|u|^{p+1}dx \le CR^{-4+(N(p-1)/(p+1))}\left(\int_{B_{2R}}|u|^{p+1}dx\right)^{2/(p+1)}. \quad (23)$$

Let $\alpha = -4+(N(p-1)/(p+1))$, $\beta = 2/(p+1)$, and $I(R) = \int_{B_R}|u|^{p+1}dx$. Then iterating (23), we get

$$I(R) \le C^{\gamma}2^{\alpha\delta}R^{\alpha\gamma}I\left(2^{k+1}R\right)^{\beta^{k+1}}, \quad (24)$$

where $\gamma = 1+\beta+\cdots+\beta^k$, $\delta = \beta+2\beta^2+\cdots+k\beta^k$.

Since u is bounded, the left side of (24) is of the order R^N, while the right side is of the order R^M, where

$$M = \alpha\frac{1-\beta^{k+1}}{1-\beta} + N\beta^{k+1} \longrightarrow \frac{\alpha}{1-\beta}, \quad \text{as } k \longrightarrow +\infty, \quad (25)$$

which yields a contradiction since $\alpha < 0$.

As above using (14), we obtain

$$\int_{B_R}|\Delta u|^2dx \le C\left(1+\frac{1}{R^4}\int_{B_{2R}} u^2dx\right). \quad (26)$$

Combining (26) and (22), we have

$$\int_{B_R}|\Delta u|^2dx \le C\left[1+R^{-4+(N(p-1)/(p+1))}\left(\int_{B_{2R}}|u|^{p+1}dx\right)^{2/(p+1)}\right]. \quad (27)$$

Using the already proved fact that $\int_{\mathbb{R}^N}|u|^{p+1}dx$ is finite, we obtain $\int_{\mathbb{R}^N}|\Delta u|^2dx < +\infty$. □

Using an approach similar to the method used in the proof of Proposition 3, we prove the following proposition.

Proposition 4. *Suppose that Q is a function satisfying $0 < \mu \le Q \le C$, where μ and C are constants. Let u be a bounded nonnegative solution with finite Morse index of*

$$\Delta^2 u = Q(x)\,u^p \quad \text{in } \mathbb{R}_+^N, \qquad u = \Delta u = 0 \quad \text{on } \partial\mathbb{R}_+^N, \quad (28)$$

where $1 < p < (N+4)/(N-4)$ and $\mathbb{R}_+^N = \{x = (x_1,\ldots,x_N) \in \mathbb{R}_+^N : x_N > 0\}$. Then both $\|\Delta u\|_{L^2(\mathbb{R}_+^N)}$ and $\|u\|_{L^{p+1}(\mathbb{R}_+^N)}$ are finite.

Proposition 5. *Let u be as in Proposition 3. Suppose that*

$$\int_{\mathbb{R}^N}|\Delta u|^2dx \ge \nu\int_{\mathbb{R}^N} Q(x)\,u^{p+1}dx, \quad (29)$$

where $\nu > 1$. Then $u \equiv 0$.

Proof. By Proposition 3 and (7), we obtain

$$\int_{\mathbb{R}^N}|\Delta u|^2dx = \int_{\mathbb{R}^N} Q(x)\,u^{p+1}dx. \quad (30)$$

By (29), it yields

$$\int_{\mathbb{R}^N} Q(x)\,u^{p+1}dx \ge \nu\int_{\mathbb{R}^N} Q(x)\,u^{p+1}dx. \quad (31)$$

Then $u \equiv 0$. □

Similar arguments are used to prove the following proposition.

Proposition 6. *Let u be as in Proposition 4. Suppose that*

$$\int_{\mathbb{R}_+^N}|\Delta u|^2dx \ge \nu\int_{\mathbb{R}_+^N} Q(x)\,u^{p+1}dx, \quad (32)$$

where $\nu > 1$. Then $u \equiv 0$.

For future reference, we have the following result at the end of this section.

Lemma 7. *Let u be a nonnegative solution of problem (7) with finite Morse index. Then there exists $r_0 > 0$ such that for $R > 2r_0$ one has*

$$R\int_{\partial B_R}\left(|\nabla u|^2+|\Delta u|^2+|\nabla(\Delta u)|^2+Q(x)\,u^{p+1}\right)dS \le CR^{(N(p-1)/(p+1))-4}\left(\int_{\mathbb{R}^N} u^{p+1}\right)^{2/(p+1)}. \quad (33)$$

Proof. By Proposition 3, $\int_{\mathbb{R}^N} u^{p+1}dx < \infty$. We proceed as in [10, 12]. Denote by $i(Q,u)$ the Morse index of u with respect to the operator $\Delta^2 - pQu^{p-1}$. Suppose that $0 \le i(Q,u) < m$. Let ψ_i, $i = 1,2,\ldots$, be smooth functions such that

$$\begin{gathered}
\psi_1 = 1 \quad \text{on } B_1, \\
\operatorname{supp}\psi_1 \subset B_{3/2}, \\
\psi_1 = 1 \quad \text{on } D_i := \left\{x \in \mathbb{R}^N : 2(i-1) \le |x| \le 2i-1\right\}, \\
\text{for } i \ge 2, \\
\operatorname{supp}\psi_i \subset A_i := \left\{x \in \mathbb{R}^N : 2i - \frac{5}{2} < |x| < 2i - \frac{1}{2}\right\}, \\
\text{for } i \ge 2, \\
|\nabla\psi_i| \le C, \quad |\Delta\psi_i| \le C, \quad \text{for } i \ge 1.
\end{gathered} \tag{34}$$

Denote $\psi_{i,R} = \psi_i(x/R)$ for $R > 0$. For each $R > 0$, there exists $i = i(R)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big[4\left(\psi_{i,R}\Delta\psi_{i,R} + |\nabla\psi_{i,R}|^2\right)^2 u^2 \\
&\quad + 16\psi_{i,R}^2(\nabla\psi_{i,R}\nabla u)^2 + \psi_{i,R}^4|\Delta u|^2 \\
&\quad + 16\left(\psi_{i,R}\Delta\psi_{i,R} + |\nabla\psi_{i,R}|^2\right)u\psi_{i,R}\nabla\psi_{i,R}\nabla u \\
&\quad + 8\psi_{i,R}^3\Delta u\nabla\psi_{i,R}\nabla u \\
&\quad + 4\left(\psi_{i,R}\Delta\psi_{i,R} + |\nabla\psi_{i,R}|^2\right)u\psi_{i,R}^2\Delta u\Big]\,dx \\
&\ge p\int_{\mathbb{R}^N} Q(x)\,u^{p+1}\psi_{i,R}^4 dx.
\end{aligned} \tag{35}$$

So we deduce as (12) and (15) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \psi_{i,R}^4|\Delta u|^2 dx \\
&\le C\int_{\mathbb{R}^N} \Big[\left(\psi_{i,R}\Delta\psi_{i,R} + |\nabla\psi_{i,R}|^2\right)^2 u^2 \\
&\qquad + \psi_{i,R}^2(\nabla\psi_{i,R}\nabla u)^2 + \left(\psi_{i,R}\Delta\psi_{i,R} + |\nabla\psi_{i,R}|^2\right) \\
&\qquad \times u\psi_{i,R}|\nabla\psi_{i,R}|\cdot|\nabla u| + \psi_{i,R}^2|\nabla u|^2|\nabla\psi_{i,R}|^2 \\
&\qquad + \psi_{i,R}^2u^2|\Delta\psi_{i,R}|^2 + |\nabla\psi_{i,R}|^4u^2\Big]\,dx, \\
&\int_{\mathbb{R}^N} Q(x)\,u^{p+1}\psi_{i,R}^4 dx \\
&\le C\int_{\mathbb{R}^N} \Big[\left(\psi_{i,R}\Delta\psi_{i,R} + |\nabla\psi_{i,R}|^2\right)^2 u^2 + \psi_{i,R}^2(\nabla\psi_{i,R}\nabla u)^2 \\
&\qquad + \left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)u\psi_{i,R}|\nabla\psi_{i,R}|\cdot|\nabla u| \\
&\qquad + 8\psi_{i,R}^2|\nabla\psi_{i,R}|^2u|\Delta u|\Big]\,dx.
\end{aligned} \tag{36}$$

By (36), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\psi_{i,R}^4|\Delta u|^2 + Q(x)\,u^{p+1}\psi_{i,R}^4\right)dx \\
&\le C\int_{\mathbb{R}^N} \Big[\left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)^2 u^2 + \psi_{i,R}^2|\nabla\psi_{i,R}|^2|\nabla u|^2 \\
&\qquad + \left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)u\psi_{i,r}|\nabla\psi_{i,R}|\cdot|\nabla u| \\
&\qquad + \psi_{i,R}^2|\nabla\psi_{i,R}|^2u|\Delta u| + \psi_{i,R}^2|\Delta\psi_{i,R}|^2u^2 \\
&\qquad + |\nabla\psi_{i,R}|^4u^2\Big]\,dx \\
&\le C\int_{\mathbb{R}^N} \Big[\left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)^2 u^2 \\
&\qquad + \psi_{i,R}^2|\nabla\psi_{i,R}|^2|\nabla u|^2 \\
&\qquad + \left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)u\psi_{i,R}|\nabla\psi_{i,R}| \\
&\qquad \cdot|\nabla u| + \psi_{i,R}^2|\Delta\psi_{i,R}|^2u^2 + |\nabla\psi_{i,R}|^4u^2 \\
&\qquad + \frac{1}{2}\varepsilon\psi_{i,R}^4|\Delta u|^2 + \frac{1}{2\varepsilon}|\nabla\psi_{i,R}|^4u^2\Big]\,dx,
\end{aligned} \tag{37}$$

where $\varepsilon > 0$ is small enough.

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\psi_{i,R}^4|\Delta u|^2 + Q(x)\,u^{p+1}\psi_{i,R}^4\right)dx \\
&\le C\int_{\mathbb{R}^N} \Big[\left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)^2 u^2 + \psi_{i,R}^2|\nabla\psi_{i,R}|^2|\nabla u|^2 \\
&\qquad + \left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)u\psi_{i,R}|\nabla\psi_{i,R}| \\
&\qquad \cdot|\nabla u| + \psi_{i,R}^2|\Delta\psi_{i,R}|^2u^2 + |\nabla\psi_{i,R}|^4u^2\Big]\,dx \\
&\le C\int_{\mathbb{R}^N} \Big[\left(\psi_{i,R}|\Delta\psi_{i,R}| + |\nabla\psi_{i,R}|^2\right)^2 u^2 + \psi_{i,R}^2|\nabla\psi_{i,R}|^2|\nabla u|^2 \\
&\qquad + \psi_{i,R}^2|\Delta\psi_{i,R}|^2u^2 + |\nabla\psi_{i,R}|^4u^2\Big]\,dx.
\end{aligned} \tag{38}$$

Integrating by parts, we obtain

$$\int_{\mathbb{R}^N} u\psi_{i,R}^4\Delta u\,dx = -\int_{\mathbb{R}^N} \left(|\nabla u|^2\psi_{i,R}^4 + 4\psi_{i,R}^3u\nabla u\nabla\psi_{i,R}\right)dx. \tag{39}$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2\psi_{i,R}^4 dx \\
&\le C\left(\int_{\mathbb{R}^N} |\Delta u|\,u\psi_{i,R}^4 dx + \int_{\mathbb{R}^N} \psi_{i,R}^3u|\nabla u|\cdot|\nabla\psi_{i,R}|\,dx\right) \\
&\le C\left(\int_{\mathbb{R}^N} \left(\psi_{i,R}^4|\Delta u|^2 + u^2\psi_{i,R}^4\right)dx + \varepsilon\int_{\mathbb{R}^N} |\nabla u|^2\psi_{i,R}^4 dx\right. \\
&\qquad \left. + \varepsilon^{-1}\int_{\mathbb{R}^N} \psi_{i,R}^2u^2|\nabla\psi_{i,R}|^2 dx\right).
\end{aligned} \tag{40}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2 \psi_{i,R}^4 \\
&\le C\left(\int_{\mathbb{R}^N} \psi_{i,R}^4 |\Delta u|^2 dx + \int_{\mathbb{R}^N} u^2 \psi_{i,R}^4 dx \right.\\
&\qquad \left. + \int_{\mathbb{R}^N} u^2 |\nabla \psi_{i,R}|^4 dx\right) \\
&\le C\left(\int_{\mathbb{R}^N} \left(\psi_{i,R}^4 |\Delta u|^2 + u^2 |\nabla \psi_{i,R}|^4\right) dx \right.\\
&\qquad \left. + \int_{\mathbb{R}^N} \left|\Delta\left(u\psi_{i,R}^2\right)\right|^2 dx\right) \\
&\le C\left(\int_{\mathbb{R}^N} \left(\psi_{i,R}^4 |\Delta u|^2 + u^2 |\nabla \psi_{i,R}|^4\right) dx \right.\\
&\qquad + \int_{\mathbb{R}^N} \left(\psi_{i,R}^2 \Delta u + 2u\psi_{i,R}\Delta\psi_{i,R} + 2u\psi_{i,R}\Delta\psi_{i,R}\right.\\
&\qquad\qquad \left.\left. + 2u|\nabla\psi_{i,R}|^2 + 4\psi_{i,R}\nabla u \nabla\psi_{i,R}\right)^2 dx\right) \\
&\le C\int_{\mathbb{R}^N} \left(\psi_{i,R}^4 |\Delta u|^2 + u^2 |\nabla\psi_{i,R}|^4 + u^2\psi_{i,R}^2 |\Delta\psi_{i,R}|^2 \right.\\
&\qquad \left. + \psi_{i,R}^2 |\nabla u|^2 |\nabla\psi_{i,R}|^2\right) dx.
\end{aligned} \tag{41}$$

Multiplying (7) by $\psi_{i,R}^4 \Delta u$ and integrating by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \psi_{i,R}^4 |\nabla(\Delta u)|^2 dx &= -4\int_{\mathbb{R}^N} \psi_{i,R}^3 \Delta u \nabla\psi_{i,R} \cdot \nabla(\Delta u)\, dx \\
&\quad - \int_{\mathbb{R}^N} Q u^p \psi_{i,R}^4 \Delta u dx.
\end{aligned} \tag{42}$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \psi_{i,R}^4 |\nabla(\Delta u)|^2 dx \\
&\le C\int_{\mathbb{R}^N} \left[\psi_{i,R}^3 |\Delta u| \cdot |\nabla\psi_{i,R}| \cdot |\nabla(\Delta u)| + Q u^p \psi_{i,R}^4 |\Delta u|\right] dx \\
&\le C\int_{\mathbb{R}^N} \left[\varepsilon\psi_{i,R}^4 |\nabla(\Delta u)|^2 + \varepsilon^{-1}\psi_{i,R}^2 |\Delta u|^2 |\nabla\psi_{i,R}|^2 \right.\\
&\qquad \left. + Q^2 u^{2p}\psi_{i,R}^4 + \psi_{i,R}^4 |\Delta u|^2\right] dx.
\end{aligned} \tag{43}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^N} \psi_{i,R}^4 |\nabla(\Delta u)|^2 dx \\
&\quad \le C\left[\frac{1}{R^2}\int_{\mathbb{R}^N} \psi_{i,R}^2 |\Delta u|^2 dx \right.\\
&\qquad \left. + \int_{\mathbb{R}^N} \left(Q u^{p+1}\psi_{i,R}^4 + \psi_{i,R}^4 |\Delta u|^2\right) dx\right].
\end{aligned} \tag{44}$$

By (38)–(44), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\psi_{i,R}^4 + \psi_{i,R}^4 + \psi_{i,R}^4 |\nabla(\Delta u)|^2 + Q(x) u^{p+1}\psi_{i,R}^4\right) dx \\
&\quad \le \left[\frac{1}{R^2}\int_{\mathbb{R}^N} \psi_{i,R}^2 |\Delta u|^2 dx \right.\\
&\qquad + \int_{\mathbb{R}^N} \left(\psi_{i,R}^4 |\Delta\psi_{i,R}|^2 + |\nabla\psi_{i,R}|^4\right) u^2 dx \\
&\qquad \left. + \int_{\mathbb{R}^N} \psi_{i,R}^2 |\nabla\psi_{i,R}|^2 |\nabla u|^2 dx\right].
\end{aligned} \tag{45}$$

By the definition of $\psi_{i,R}$, it follows that

$$\begin{aligned}
&\int_{RD_i} \left(|\nabla u|^2 + |\Delta u|^2 + |\nabla(\Delta u)|^2 + Q(x) u^{p+1}\right) dx \\
&\quad \le \frac{C}{R^4}\int_{RA_i \setminus RD_i} u^2 dx.
\end{aligned} \tag{46}$$

This implies by Hölder's inequality that

$$\begin{aligned}
&\int_{RD_i} \left(|\nabla u|^2 + |\Delta u|^2 + |\nabla(\Delta u)|^2 + Q(x) u^{p+1}\right) dx \\
&\quad \le \frac{1}{R^4}\left(\int_{RA_i \setminus RD_i} u^{p+1} dx\right)^{2/(p+1)} \\
&\qquad \times \left[\operatorname{mes}\left(RA_i \setminus RD_i\right)\right]^{(p-1)/(p+1)} \\
&\quad \le CR^{(N(p-1)/(p+1))-4}\left(\int_{\mathbb{R}^N} u^{p+1} dx\right)^{2/(p+1)}.
\end{aligned} \tag{47}$$

Hence, there exists $\overline{R} \in (R, (2m-1)R)$ such that

$$\begin{aligned}
&R\int_{\partial B_{\overline{R}}} \left(|\nabla u|^2 + |\Delta u|^2 + |\nabla(\Delta u)|^2 + Q(x) u^{p+1}\right) dS \\
&\quad \le CR^{(N(p-1)/(p+1))-4}\left(\int_{\mathbb{R}^N} u^{p+1} dx\right)^{2/(p+1)}.
\end{aligned} \tag{48}$$

The assertion follows. □

3. Proof of Theorem 1

Let us first note that hypotheses (H1)–(H4) imply that there exist a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to \infty$ and a continuous function c such that

$$f_t'(x, t_n) > 0, \qquad \lim_{n\to\infty} \frac{f_t'(x, t_n)}{t_n^{p-1}} = c(x) \quad \text{uniformly in } x \in \Omega. \tag{49}$$

Using L'Hospital's rule, we obtain

$$\lim_{n\to\infty} \frac{f(x, t_n)}{t_n^p} = \lim_{n\to\infty} \frac{f_t'(x, t_n)}{p t_n^{p-1}} = \frac{1}{p} c(x). \tag{50}$$

Without loss of generality, we may assume that $f(x,t)=0$, if $t\leq 0$. Define a truncation of f by

$$f_n(x,t)=\begin{cases} f(x,t_n)-\dfrac{1}{p}t_nf_t'(x,t_n) \\ \quad +\dfrac{1}{p}\dfrac{f_t'(x,t_n)}{t_n^{p-1}}t^p & \text{if } t>t_n, \\ f(x,t) & \text{if } 0\leq t\leq t_n, \\ 0 & \text{if } t<0. \end{cases} \tag{51}$$

Note that $f_n\in C^1$ in the variable t.

Let us consider the truncated problem

$$\begin{gathered} \Delta^2u=f_n(x,u) \quad \text{in } \Omega, \\ u=\Delta u=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{52}$$

Lemma 8. *Problem* (52) *possesses at least one positive solution with finite Morse index.*

Proof. Consider the functional associated with problem (52)

$$J_n(u)=\frac{1}{2}\int_\Omega|\Delta u|^2dx-\int_\Omega F_n(x,u)\,dx, \tag{53}$$

where $F_n(x,t)=\int_0^t f_n(x,s)ds$. We will use the Mountain Pass Theorem by Ambrosetti and Rabinowitz [8] to obtain existence result for problem (52). One can easily check that there exist $\theta>2$ and $M>0$ such that for $|t|\geq$ M,

$$0<\theta F_n(x,t)\leq tf_n(x,t). \tag{54}$$

We note that condition (54) is important for ensuring that J_n has a Mountain Pass geometry and satisfies the Palais-Smale condition. So, using the Mountain Pass Theorem, we obtain a nontrivial weak solution u_n of (52). By Lemma B3 in [14], u_n is a classical solution of (52). By the maximum principle for Δ^2 with Navier boundary conditions we get that u_n is positive. The geometry of the Mountain Pass, described in [15, 16], implies that the Morse indices of u_n are less than or equal to 1. Thus Lemma 8 is proven. □

Let g be a function satisfying (H1)–(H3), and consider the problem

$$\begin{gathered} \Delta^2u=g(x_0+ax,bu) \quad \text{in } \Omega, \\ u=\Delta u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{55}$$

where a,b are positive constants and $x_0\in\mathbb{R}^N$.

Lemma 9. *Let u be a solution of* (55). *Then for any ball $B_R(0)\subset\Omega$ one has*

$$\begin{aligned} &b^{-1}N\int_{B_R}G(x_0+ax,bu)\,dx \\ &\quad +R\int_{\partial B_R}\left[2\frac{\partial u}{\partial\mathbf{n}}\frac{\partial(\Delta u)}{\partial\mathbf{n}}-\langle\nabla(\Delta u),\nabla u\rangle\right]dS \\ &\quad +(N-2)\int_{\partial B_R}\Delta u\frac{\partial u}{\partial\mathbf{n}}dS+\frac{1}{2}R\int_{\partial B_R}|\Delta u|^2dS \\ &\quad +ab^{-1}\int_{B_R}\langle x,\nabla_xG(x_0+ax,bu)\rangle\,dx \\ &\quad =\frac{N-4}{2}\int_{B_R}|\Delta u|^2dx+b^{-1}R\int_{\partial B_R}G(x_0+ax,bu)\,dS, \end{aligned} \tag{56}$$

where $G(x,u)=\int_0^u g(x,t)dt$ and $\mathbf{n}$ denotes the unit outward normal to ∂B_R.

Proof. By standard procedures, one can prove the Pohozaev type identity. We give the proof for completeness and for the reader's convenience.

By Proposition 2.2 in [17], we have

$$\begin{aligned} &\int_{B_R}\Delta^2u\langle x,\nabla u\rangle\,dx \\ &\quad =-\frac{1}{2}\int_{\partial B_R}|\Delta u|^2\langle x,\mathbf{n}\rangle\,dS \\ &\qquad +\int_{\partial B_R}\left\{\frac{\partial(\Delta u)}{\partial\mathbf{n}}\langle x,\nabla u\rangle+\frac{\partial u}{\partial\mathbf{n}}\langle x,\nabla(\Delta u)\rangle\right. \\ &\qquad\qquad \left.-\langle\nabla(\Delta u),\nabla u\rangle\langle x,\mathbf{n}\rangle\right\}dS+\frac{N}{2}\int_{B_R}|\Delta u|^2dx \\ &\qquad +(N-2)\int_{B_R}\langle\nabla(\Delta u),\nabla u\rangle\,dx. \end{aligned} \tag{57}$$

It is clear that

$$\begin{aligned} &g(x_0+ax,bu)\langle x,\nabla u\rangle \\ &\quad =b^{-1}\ \operatorname{div}(xG(x_0+ax,bu))-b^{-1}NG(x_0+ax,bu) \\ &\qquad -ab^{-1}\langle x,\nabla_xG(x_0+ax,bu)\rangle. \end{aligned} \tag{58}$$

By (55), we have

$$\int_{B_R}\Delta^2u\langle x,\nabla u\rangle dx=\int_{B_R}g(x_0+ax,bu)\langle x,\nabla u\rangle dx. \tag{59}$$

Substituting (57) and (58) into (59) and using the divergence theorem, we find

$$\begin{aligned} &-\frac{1}{2}\int_{\partial B_R}|\Delta u|^2\langle x,\mathbf{n}\rangle\,dS \\ &\quad +\int_{\partial B_R}\left\{\frac{\partial(\Delta u)}{\partial\mathbf{n}}\langle x,\nabla u\rangle+\frac{\partial u}{\partial\mathbf{n}}\langle x,\nabla(\Delta u)\rangle\right. \\ &\qquad\qquad \left.-\langle\nabla(\Delta u),\nabla u\rangle\langle x,\mathbf{n}\rangle\right\}dS+\frac{N}{2}\int_{B_R}|\Delta u|^2dx \\ &\quad +(N-2)\int_{B_R}\langle\nabla(\Delta u),\nabla u\rangle\,dx \\ &\quad =b^{-1}\int_{\partial B_R}G(x_0+ax,bu)\langle x,\mathbf{n}\rangle\,dS \end{aligned}$$

$$
\begin{aligned}
&- b^{-1} N \int_{B_R} G\left(x_0 + ax, bu\right) dx \\
&- ab^{-1} \int_{B_R} \left\langle x, \nabla_x G\left(x_0 + ax, bu\right)\right\rangle dx.
\end{aligned} \tag{60}
$$

Using the Green's formula, we obtain

$$
\int_{B_R} \langle \nabla (\Delta u), \nabla u\rangle \, dx = -\int_{B_R} \Delta u \cdot \Delta u \, dx + \int_{\partial B_R} \Delta u \frac{\partial u}{\partial \mathbf{n}} dS. \tag{61}
$$

Substituting (61) into (60) and using $\mathbf{n} = x/R$ on ∂B_R, we obtain (56). ☐

If it happens that $\|u_n\| \leq t_n$, for some n, then u_n is also a solution of (1), and the proof of Theorem 1 will be completed. Thus it suffices to prove the following proposition. We prove the proposition by the blow-up technique of Gidas and Spruck [18].

Proposition 10. *Suppose that u_n is a solution of* (52) *with finite Morse index. Then there exists a $t_n > 0$ such that $\|u_n\|_\infty \leq t_n$.*

Proof. Assume by contradiction that there does not exist such a t_n. So we should have $\|u_n\| > t_n$, for all n. Then $\|u_n\| \to \infty$ as $n \to \infty$.

Let $M_n = \max_{\overline{\Omega}} u_n(x) = u_n(x_n)$, $x_n \in \Omega$. Define

$$
\begin{gathered}
v_n(y) = M_n^{-1} u_n\left(x_n + M_n^{-((p-1)/4)} y\right), \\
y \in \Omega_n := M_n^{(p-1)/4}\left(\Omega - x_n\right),
\end{gathered} \tag{62}
$$

which satisfies

$$
\begin{gathered}
\Delta^2 v_n = M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right) \quad \text{in } \Omega_n, \\
v_n = \Delta v_n = 0 \quad \text{on } \partial\Omega_n,
\end{gathered} \tag{63}
$$

and $v_n \leq 1$ in $\overline{\Omega}_n$, $v_n(0) = 1$.

Due to compactness of $\overline{\Omega}$ we may also assume that $x_n \to x_0 \in \overline{\Omega}$. So there are two cases to be considered, $x_0 \in \Omega$ and $x_0 \in \partial\Omega$.

Case 1. $x_0 \in \Omega$.

Given $R > 0$ there is an $n_0 \in \mathbb{N}$ such that $\overline{B}_{2R}(0) \subset \Omega_n$ for all $n \geq n_0$. By the L^p-estimates due to Agmon et al. [19], we have that for all $\gamma > 1$

$$
\begin{aligned}
\|v_n\|_{W^{4,\gamma}(B_R)} \leq C \Big\{ &\left\| M_n^{-p} f_n(x_n + M_n^{-((p-1)/4)} y, M_n v_n)\right\|_{L^\gamma(B_{2R})} \\
&+ \|v_n\|_{L^\gamma(B_{2R})} \Big\}.
\end{aligned} \tag{64}
$$

By assumptions (H1)–(H3) and the definition of f_n, it follows that

$$
\left|f_n(x,t)\right| \leq C\left(1 + |t|^p\right), \tag{65}
$$

and then for large n,

$$
\begin{aligned}
\left|M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right)\right| &\leq M_n^{-p} C\left(1 + \left|M_n v_n\right|^p\right) \\
&\leq C\left(M_n^{-p} + \left|v_n\right|^p\right) \leq 2C.
\end{aligned} \tag{66}
$$

So we obtain that

$$
\|v_n\|_{W^{4,\gamma}(B_R)} \leq C \quad \text{uniformly in } n. \tag{67}
$$

Choosing $\gamma > n$, it follows from standard embedding theorems that $\{v_n\}$ is uniformly bounded in $C^{3,\alpha}(\overline{B}_R)$, $0 < \alpha < 1$. By the Schauder estimate of Agmon et al. [19] one has

$$
\begin{aligned}
\|v_n\|_{4,\alpha,B_{(1/2)R}} \leq C \Big\{ &\left\| M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right)\right\|_{\alpha,B_R} \\
&+ \|v_n\|_{0,B_R} \Big\}.
\end{aligned} \tag{68}
$$

Next we claim that

$$
\left\| M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right)\right\|_{\alpha,B_R} \leq C. \tag{69}
$$

In order to do that we write

$$
\begin{aligned}
& f_n\left(x_n + m_n^{-((p-1)/4)} y, M_n v_n(y)\right) \\
&\quad - f_n\left(x_n + m_n^{-((p-1)/4)} z, M_n v_n(z)\right) \\
&= \Big[f_n\left(x_n + m_n^{-((p-1)/4)} y, M_n v_n(y)\right) \\
&\qquad - f_n\left(x_n + m_n^{-((p-1)/4)} y, M_n v_n(z)\right)\Big] \\
&\quad + \Big[f_n\left(x_n + m_n^{-((p-1)/4)} y, M_n v_n(z)\right) \\
&\qquad - f_n\left(x_n + m_n^{-((p-1)/4)} z, M_n v_n(z)\right)\Big] \\
&:= I_1 + I_2.
\end{aligned} \tag{70}
$$

Then we have

$$
\begin{aligned}
\left|I_1\right| &\leq \Big| \frac{\partial f_n}{\partial t}\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right) \\
&\qquad \times M_n\left(v_n(y) - v_n(z)\right)\Big| \\
&\leq C\left(1 + \left|M_n w_n\right|^{p-1}\right) M_n \left|v_n(y) - v_n(z)\right| \\
&\leq C\left(1 + M_n^p\right) \left|y - z\right|^\alpha.
\end{aligned} \tag{71}
$$

According to the definition of f_n, we divide the estimate of I_1 into three cases.

(i) If $M_n v_n(z) \leq T$, then since f_n is C^1 we get

$$
\left|I_2\right| \leq C M_n^{-((p-1)/4)} \left|y - z\right|. \tag{72}
$$

(ii) If $T \leq M_n v_n(z) \leq t_n$, we use condition (H3) to get

$$
\begin{aligned}
\left|I_2\right| &\leq \left|\nabla_x f\left(x_n + M_n^{-((p-1)/4)} w_n, M_n v_n(z)\right)\right| \\
&\qquad \cdot M_n^{-((p-1)/4)} \left|y - z\right| \\
&\leq C M_n^{-((p-1)/4)} \left|M_n v_n(z)\right|^p \cdot \left|y - z\right|.
\end{aligned} \tag{73}
$$

(iii) If $t_n \le M_n v_n(z)$, by the definition of f_n, we have

$$
\begin{aligned}
|I_2| \le & \left| f\left(x_n + M_n^{-((p-1)/4)} y, t_n\right) \right. \\
& \left. - f\left(x_n + M_n^{-((p-1)/4)} z, t_n\right) \right| \\
& + \frac{1}{p} t_n \left| f_t'\left(x_n + M^{-((p-1)/4)} y, t_n\right) \right. \\
& \left. - f_t'\left(x_n + M^{-((p-1)/4)} z, t_n\right) \right| \\
& + \frac{1}{p} (M_n v_n(z))^p \left| \frac{f_t'\left(x_n + M_n^{-((p-1)/4)} y, t_n\right)}{t_n^{p-1}} \right. \\
& \left. - \frac{1}{p} \frac{f_t'\left(x_n + M_n^{-((p-1)/4)} z, t_n\right)}{t_n^{p-1}} \right| \\
\le & \left| \nabla_x f\left(x_n + M_n^{-((p-1)/4)} w_n, t_n\right) \right| \cdot M_n^{-((p-1)/4)} |y - z| \\
& + \frac{1}{p} t_n \left| \nabla_x f_t'\left(x_n + M_n^{-((p-1)/4)} \widetilde{w}_n, t_n\right) \right| \\
& \cdot M_n^{-((p-1)/4)} |y - z| \\
& + \frac{1}{p} \cdot \frac{1}{t_n^{p-1}} M_n^p \left| \nabla_x f_t'\left(x_n + M_n^{-((p-1)/4)} \widetilde{w}_n, t_n\right) \right| \\
& \cdot M_n^{-((p-1)/4)} |y - z| \\
\le & C M_n^{(3p+1)/4} |y - z| .
\end{aligned}
\tag{74}
$$

By (71)–(74), we obtain

$$
|I_1 + I_2| \le M_n^p |y - z| , \tag{75}
$$

which proves (69), and therefore

$$
\|v_n\|_{4,\alpha,B_{(1/2)R}} \le C \quad \text{uniformly in } n. \tag{76}
$$

Using Arzela-Ascoli Theorem and (69) and (76), we obtain a subsequence of v_n still denoted by v_n, such that

$$
v_n \longrightarrow v \quad \text{in } C^{4,\alpha'}\left(B_{(1/2)R}\right), \tag{77}
$$

$$
\begin{aligned}
M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n(y)\right) & \longrightarrow A(y) \\
& \text{in } C^{0,\alpha'}\left(B_{(1/2)R}\right),
\end{aligned}
\tag{78}
$$

where $0 < \alpha' < \alpha < 1$, as $n \to \infty$.

Assume that $\beta = \lim_{n\to\infty} t_n / M_n$. By (65), we have

$$
\begin{aligned}
& M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n(y)\right) \\
& \quad \le C M_n^{-p} (M_n v_n)^p = C v_n^p .
\end{aligned}
\tag{79}
$$

By (77), we have

$$
v_n(y) < \beta, \quad y \in \{y : v(y) < \beta\} , \tag{80}
$$

for large n. Consequently,

$$
v_n \le \frac{t_n}{M_n} . \tag{81}
$$

That is,

$$
M_n v_n \le t_n . \tag{82}
$$

Then, by the definition of f_n and the assumption (H4), we get

$$
\mu v_n^p \le M_n^{-p} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n(y)\right) , \tag{83}
$$

for large n. Combining (79) and (83) and letting $n \to \infty$, we obtain

$$
\mu v^p(y) \le A(y) \le C v^p(y) , \quad y \in \{y : v(y) < \beta\} . \tag{84}
$$

Define

$$
Q(y) = \begin{cases}
\liminf\limits_{\substack{z \to y \\ z \in \omega}} \dfrac{A(z)}{v^p(z)} & \text{if } y \in \omega := \{y : v(y) = 0\} , \\
A(y) v^{-p}(y) & \text{if } y \in \{y : 0 < v(y) < \beta\} , \\
\dfrac{1}{p} c(x_0) & \text{if } y \in \{y : v(y) = \beta\} .
\end{cases}
\tag{85}
$$

Then there exist positive constants σ and γ such that $\sigma \le Q(y) \le \gamma$, $\forall \in B_{(1/2)R}$. Passing to the limit in (63) and using (77) and (78), we conclude that v satisfies

$$
\Delta^2 v = Q(y) v^p \quad \text{in } B_{(1/2)R} . \tag{86}
$$

By a diagonal process, it follows that

$$
\Delta^2 v = Q(y) v^p \quad \text{in } \mathbb{R}^N . \tag{87}
$$

Next we claim that the Morse index of v is finite. If $y \in B_{(1/2)R} \setminus \Omega$, by the L'Hospital's rule we have

$$
\begin{aligned}
& Q(y) v^p(y) \\
& = \lim_{n\to\infty} \frac{f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right)}{M_n^p} \\
& = \lim_{n\to\infty} \frac{1}{p M_n^{p-1}} \left[v_n \frac{\partial}{\partial t} f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right) \right. \\
& \qquad - \frac{p-1}{4} \nabla_x f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right) \\
& \qquad \left. \times M_n^{-((p+3)/4)} y \right] .
\end{aligned}
\tag{88}
$$

By (H3) for $y \in B_{(1/2)R}$,

$$
\begin{aligned}
& \left| M_n^{-p+1} \nabla_x f_n\left(x_n + M_n^{-((p-1)/4)} y, M_n v_n\right) M_n^{-((p+3)/4)} y \right| \\
& \quad \le C M_n^{-((p-1)/4)} .
\end{aligned}
\tag{89}
$$

Then, by (88), we get

$$M_n^{-p+1}\frac{\partial}{\partial t}f_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)\longrightarrow pQ(y)v^{p-1}(y). \tag{90}$$

If $y\in\omega\cap B_{(1/2)R}$, then $v_n(y)\rightarrow 0$ as $n\rightarrow\infty$. By assumption (H3), we have

$$\begin{aligned}&\left|M_n^{-p+1}\frac{\partial}{\partial t}f_n\left(x_n+M_n^{-((p-1)/4)}\right)y,M_nv_n\right|\\&\qquad\le M_n^{-p+1}C(M_nv_n)^{p-1}=Cv_n^{p-1}\longrightarrow 0\end{aligned} \tag{91}$$

as $n\rightarrow 0$. Therefore, (90) holds for all $x\in B_{(1/2)R}$. By the diagonal process, one knows that (90) holds also in $\mathbb{R}^N$ and it converges uniformly on compact sets of $\mathbb{R}^N$ as $n\rightarrow\infty$.

The uniform convergence of v_n to v on compact sets implies that the Morse index of v is finite. To handle this, we set

$$\begin{aligned}&J_n(u)\varphi^2\\&\quad=\int_{\mathbb{R}^N}|\Delta\varphi|^2dx\\&\qquad-\int_{\mathbb{R}^N}M_n^{-p+1}\frac{\partial}{\partial t}f_n\left(x_n+M_n^{-((p-1)/4)}x,M_nu\right)\varphi^2dx,\\&\qquad\qquad\forall\varphi\in C_0^\infty\left(\mathbb{R}^N\right).\end{aligned} \tag{92}$$

Let $\psi\in C_0^\infty(\mathbb{R}^N)$ be such that $J''(v)\psi^2<0$. The uniform convergence of v_n to v on compact sets and the fact that (90) holds on compact sets imply

$$J_n''(v_n)\psi^2<0 \tag{93}$$

for large n. Since the Morse index of v_n is finite, it follows easily that the Morse index of v is finite. Proposition 3 implies that $\|v\|_{L^{p+1}(\mathbb{R}^N)}$ is finite. We claim that there exists a $\lambda>1$ such that

$$\int_{\mathbb{R}^N}|\Delta v|^2dx\ge\lambda\int_{\mathbb{R}^N}Qv^{p+1}dy. \tag{94}$$

Then Proposition 5 yields $v\equiv 0$, which contradicts $v(0)=1$.

Now, we prove (94). Applying Lemma 9 to (63) in the ball $B_R(0)$ for fixed $R>0$, we obtain

$$\begin{aligned}&M_n^{-p-1}N\int_{B_R}F_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)dy\\&\quad+R\int_{\partial B_R}\left[2\frac{\partial v_n}{\partial\mathbf{n}}\frac{\partial(\Delta v_n)}{\partial\mathbf{n}}-\langle\nabla(\Delta v_n),\nabla v_n\rangle\right]dS\\&\quad+(N-2)\int_{\partial B_R}\Delta v_n\frac{\partial v_n}{\partial\mathbf{n}}+\frac{1}{2}R\int_{\partial B_R}|\Delta v_n|^2dS+M_n^{-((5p+3)/4)}\\&\quad\times\int_{B_R}\left\langle y,\nabla_xF_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)\right\rangle dy\\&=\frac{N-4}{2}\int_{B_R}|\Delta v_n|^2dy+M_n^{-p-1}R\\&\quad\times\int_{\partial B_R}F_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)dS.\end{aligned} \tag{95}$$

By (H3), we estimate

$$\left|\left\langle y,\nabla_xF_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)\right\rangle\right|\le CRM_n^{p+1}. \tag{96}$$

Therefore

$$\begin{aligned}&M_n^{-((5p+3)/4)}\left|\int_{B_R}\left\langle y,\nabla_xF_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)\right\rangle dy\right|\\&\qquad\le CR^{N+1}M_n^{-((p-1)/4)},\end{aligned} \tag{97}$$

which tends to zero as $n\rightarrow\infty$.

Using a similar argument that leads to (69) we can prove

$$\begin{aligned}&\left\|M_n^{-p-1}F_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)\right\|_{\alpha,B_R}\\&\qquad\le C\quad\text{uniformly in } n.\end{aligned} \tag{98}$$

Then its limit exists as $n\rightarrow\infty$. Using L'Hospital's rule as (88) we get

$$\begin{aligned}&M_n^{-p-1}F_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)\longrightarrow\frac{1}{p+1}Q(y)v^{p+1}(y)\\&\qquad\qquad\text{uniformly in } B_R,\end{aligned} \tag{99}$$

as $n\rightarrow\infty$. Therefore, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{n\to\infty}M_n^{-p-1}\int_{B_R}F_n\left(x_n+M_n^{-((p-1)/4)}y,M_nv_n\right)dy\\&\qquad=\frac{1}{p+1}\int_{B_R}Q(y)v^{p+1}dy,\\&\lim_{n\to\infty}M_n^{-p-1}\int_{\partial B_R}F_n\left(x_n+M_n^{-((p-1)/4)}y,Mnv_n\right)dS\\&\qquad=\frac{1}{p+1}\int_{\partial B_R}Q(y)v^{p+1}(y)dS.\end{aligned} \tag{100}$$

Letting $n\rightarrow\infty$ in (95), we get

$$\begin{aligned}&\frac{N}{p+1}\int_{B_R}Q(y)v^{p+1}(y)dy\\&\qquad+R\int_{\partial B_R}\left[2\frac{\partial v}{\partial\mathbf{n}}\frac{\partial(\Delta v)}{\partial\mathbf{n}}-\langle\nabla(\Delta v),\nabla v\rangle\right]dS\\&\qquad+(N-2)\int_{\partial B_R}\Delta v\frac{\partial v}{\partial\mathbf{n}}dS+\frac{1}{2}R\int_{\partial B_R}|\Delta v|^2dS\\&\quad=\frac{N-4}{2}\int_{B_R}|\Delta v|^2dy+\frac{R}{p+1}\int_{\partial B_R}Q(y)v^{p+1}(y)dS.\end{aligned} \tag{101}$$

By Lemma 7, there exists $R\ge 2r_0$ such that

$$\begin{aligned}&R\int_{\partial B_R}\left(|\nabla v|^2+|\Delta v|^2+|\nabla(\Delta v)|^2+Q(y)v^{p+1}\right)dS\\&\qquad\le CR^{(N(p-1)/(p+1))-4}\left(\int_{\mathbb{R}^N}v^{p+1}dy\right)^{2/(p+1)}.\end{aligned} \tag{102}$$

Since $(N(p-1)/(p+1))-4<0$, this implies that

$$\lim_{R\to\infty} R\int_{\partial B_R}\left(|\nabla v|^2+|\Delta v|^2+|\nabla(\Delta v)|^2+Q(y)v^{p+1}\right)dS=0. \tag{103}$$

Taking the limit $R \to \infty$ in (101), one has

$$\frac{2N}{N-4}\cdot\frac{1}{p+1}\int_{\mathbb{R}^N} Q(y)v^{p+1}(y)\,dy=\int_{\mathbb{R}^N}|\Delta v|^2 dy. \tag{104}$$

Assertion (94) follows.

Case 2. $x_0 \in \partial\Omega$.

Two cases may occur: either $d(x_n,\partial\Omega)M_n^{-((p-1)/4)} \to +\infty$ or $d(x_n,\partial\Omega)M_n^{-((p-1)/4)} \to L \geq 0$ as $n \to \infty$.

If $d(x_n,\partial\Omega)M_n^{-((p-1)/4)} \to +\infty$ as $n \to \infty$, then for all $R>0$ the ball B_R is contained in Ω_n for n large enough. We also obtain a contradiction as in Case 1.

If $d(x_n,\partial\Omega)M_n^{-((p-1)/4)} \to L \geq 0$ as $n \to \infty$, by the blow-up argument we get a solution v of

$$\begin{gathered}\Delta^2 v = Q(y)v^p \quad \text{in } \Pi,\\ v=\Delta v=0, \quad \text{on } \partial\Pi,\end{gathered} \tag{105}$$

with $v \leq 1$ in Π, $v(0)=1$, and the Morse index being finite, where $\Pi=\{x\in\mathbb{R}^N : x_N > -L\}$. We may deduce as Case 1 that $v\equiv 0$. This is a contradiction since $v(0)=1$. Thus the proof of Proposition 10 is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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