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Research Article

Viscosity Approximation Methods for a Family of Nonexpansive Mappings in CAT(0) Spaces

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The purpose of this paper is using the viscosity approximation method to study the strong convergence problem for a family of nonexpansive mappings in CAT(0) spaces. Under suitable conditions, some strong convergence theorems for the proposed implicit and explicit iterative schemes to converge to a common fixed point of the family of nonexpansive mappings are proved which is also a unique solution of some kind of variational inequalities. The results presented in this paper extend and improve the corresponding results of some others.

1. Introduction

Throughout this paper, we assume that X is a CAT(0) space, $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, and C is a nonempty closed and convex subset of a complete CAT(0) space X.

A mapping $T : C \to C$ is called a *nonexpansive mapping*, if

$$d(T(x), T(y)) \le d(x, y), \quad \forall x, y \in C. \tag{1}$$

It is well-known that one classical way to study nonexpansive mappings is to use the contractions to approximate nonexpansive mappings. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t = tu + (1 - t) Tx, \quad \forall x \in C, \tag{2}$$

where $u \in C$ is an arbitrary fixed element. In the case of T having a fixed point, Browder [1] proved that x_t converged strongly to a fixed point of T that is nearest to u in the framework of Hilbert spaces. Reich [2] extended Browder's result to the setting of a uniformly smooth Banach space and proved that x_t converged strongly to a fixed point of T.

Halpern [3] introduced the following explicit iterative scheme (3) for a nonexpansive mapping T on a subset C of a Hilbert space:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Tx_n. \tag{3}$$

He proved that the sequence $\{x_n\}$ converged to a fixed point of T.

Fixed-point theory in CAT(0) spaces was first studied by Kirk (see [4, 5]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed. In 2012, using Moudafi's viscosity approximation methods, Shi and Chen [6] studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T:

$$x_t = tf(x_t) \oplus (1 - t) Tx_t, \tag{4}$$

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) Tx_n. \tag{5}$$

They proved that $\{x_t\}$ defined by (4) and $\{x_n\}$ defined by (5) converged strongly to a fixed point of T in the framework of CAT(0) space which satisfies the property $\mathscr{P}$.

Motivated and inspired by the researches going on in this direction, especially inspired by Shi and Chen [6], the purpose of this paper is to study the strong convergence theorems of Moudafi's viscosity approximation methods for a family of nonexpansive mappings in CAT(0) spaces. We prove that the implicit and explicit iteration algorithms both converge strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{\mathscr{F}} f(\tilde{x})$, which is the unique solution to the variational inequality (35), where $\mathscr{F}$ is the set of common fixed points of the family of nonexpansive mappings.

2. Preliminaries and Lemmas

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y). \tag{6}$$

Lemma 1 (see [7]). *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$\begin{aligned} &d^2((1-t)x \oplus ty, z) \\ &\quad \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \end{aligned} \tag{7}$$

is satisfied for all $x, y, z \in X$ and $t \in [0,1]$. In particular, if x, y, z are points in a CAT(0) space and $t \in [0,1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z). \tag{8}$$

Lemma 2 (see [8]). *Let X be a CAT(0) space, $p, q, r, s \in X$, and $\lambda \in [0,1]$. Then*

$$\begin{aligned} &d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \\ &\quad \le \lambda d(p,r) + (1-\lambda)d(q,s). \end{aligned} \tag{9}$$

By induction, one writes

$$\begin{aligned} \bigoplus_{m=1}^{n} \lambda_m x_m := (1-\lambda_n)\Big(&\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \\ &\oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}\Big) \oplus \lambda_n x_n. \end{aligned} \tag{10}$$

Lemma 3. *Let X be a CAT(0) space; then, for any sequence $\{\lambda_m\}_{m=1}^{n}$ in $[0,1]$ satisfying $\sum_{m=1}^{n}\lambda_m = 1$ and for any $\{x_m\}_{m=1}^{n} \subset X$, the following conclusions hold:*

$$d\Big(\bigoplus_{m=1}^{n}\lambda_m x_m, x\Big) \le \sum_{m=1}^{n}\lambda_m d(x_m, x), \quad x \in X; \tag{11}$$

$$\begin{aligned} &d^2\Big(\bigoplus_{m=1}^{n}\lambda_m x_m, x\Big) \\ &\quad \le \sum_{m=1}^{n}\lambda_m d^2(x_m, x) - \lambda_1\lambda_2 d^2(x_1, x_2), \quad x \in X. \end{aligned} \tag{12}$$

Proof. It is obvious that (11) holds for $n = 2$. Suppose that (11) holds for some $k \ge 2$. Next we prove that (11) is also true for $k+1$. From (8) and (10) we have

$$\begin{aligned} &d\Big(\bigoplus_{m=1}^{k+1}\lambda_m x_m, x\Big) \\ &\quad = d\Big((1-\lambda_{k+1})\Big(\frac{\lambda_1}{1-\lambda_{k+1}}x_1 \oplus \frac{\lambda_2}{1-\lambda_{k+1}}x_2 \\ &\qquad\qquad \oplus \cdots \oplus \frac{\lambda_k}{1-\lambda_{k+1}}x_k\Big) \\ &\qquad \oplus \lambda_{k+1}x_{k+1}, x\Big) \\ &\quad \le (1-\lambda_{k+1})d\Big(\frac{\lambda_1}{1-\lambda_{k+1}}x_1 \oplus \frac{\lambda_2}{1-\lambda_{k+1}}x_2 \\ &\qquad\qquad \oplus \cdots \oplus \frac{\lambda_k}{1-\lambda_{k+1}}x_k, x\Big) \\ &\qquad + \lambda_{k+1}d(x_{k+1}, x) \\ &\quad \le \lambda_1 d(x_1, x) + \lambda_2 d(x_2, x) \\ &\qquad + \cdots + \lambda_k d(x_k, x) + \lambda_{k+1}d(x_{k+1}, x) \\ &\quad = \sum_{m=1}^{k+1}\lambda_m d(x_m, x). \end{aligned} \tag{13}$$

This implies that (11) holds.

Next, we prove that (12) holds.

Indeed, it is obvious that (12) holds for $n = 2$. Suppose that (12) holds for some $k \ge 2$. Next we prove that (12) is also true for $k+1$.

In fact, we have

$$d^2\Big(\bigoplus_{m=1}^{k+1}\lambda_m x_m, x\Big) = d^2\Big(\bigoplus_{m=1}^{k}\lambda_m x_m \oplus \lambda_{k+1}x_{k+1}, x\Big). \tag{14}$$

From (7) and (10) and the assumption of induction, we have

$$\begin{aligned} &d^2\Big(\bigoplus_{m=1}^{k+1}\lambda_m x_m, x\Big) \\ &\quad = d^2\Big(\bigoplus_{m=1}^{k}\lambda_m x_m \oplus \lambda_{k+1}x_{k+1}, x\Big) \\ &\quad = d^2\Big((1-\lambda_{k+1})\bigoplus_{m=1}^{k}\frac{\lambda_m}{1-\lambda_{k+1}}x_m \oplus \lambda_{k+1}x_{k+1}, x\Big) \\ &\quad \le (1-\lambda_{k+1})d^2\Big(\bigoplus_{m=1}^{k}\frac{\lambda_m}{1-\lambda_{k+1}}x_m, x\Big) \\ &\qquad + \lambda_{k+1}d^2(x_{k+1}, x) \end{aligned}$$

$$\leq (1-\lambda_{k+1}) \sum_{m=1}^{k} \frac{\lambda_m}{1-\lambda_{k+1}} d^2(x_m, x) - \lambda_1\lambda_2 d^2(x_1, x_2) + \lambda_{k+1} d^2(x_{k+1}, x) = \sum_{m=1}^{k+1} \lambda_m d^2(x_m, x) - \lambda_1\lambda_2 d^2(x_1, x_2). \tag{15}$$

This completes the proof of (12). □

Clearly, every CAT(0) space X is strictly convex: if, in X, $d(u, y_0) = d(v, y_0)$ and $x = \alpha u \oplus \beta v \in [u, v]$, then $u = x = v$ whenever $d(x, y_0) = d(v, y_0)$. Dhompongsa et al. [9] showed the following conclusion which is called *Condition (A)*:

(A) if y_0 and v_n belong to X and $d(v_n, y_0) = d(x, y_0)$ for all n, where $x = \bigoplus_{n=1}^{\infty} \lambda_n v_n$, then $v_n = x$ for all n.

The concept of Δ-convergence introduced by Lim [10] in 1976 was shown by Kirk and Panyanak [11] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting. Now, we give the concept of Δ-convergence.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n). \tag{16}$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf_{x\in X} \{r(x, \{x_n\})\}, \tag{17}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{18}$$

It is known from Proposition 7 of [12] that, in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

The uniqueness of an asymptotic center implies that a CAT(0) space X satisfies *Opial's property*; that is, for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y). \tag{19}$$

Lemma 4 (see [11]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Berg and Nikolaev [13] introduced the concept of quasilinearization as follows. Let one denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle\cdot,\cdot\rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right) \quad (a, b, c, d \in X). \tag{20}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle \overrightarrow{cd}, \overrightarrow{ab}\rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd}\rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd}\rangle + \langle \overrightarrow{xb}, \overrightarrow{cd}\rangle = \langle \overrightarrow{ab}, \overrightarrow{cd}\rangle$ for all $a, b, c, d \in X$. One says that X satisfies the Cauchy-Schwarz inequality if

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \leq d(a, b)\, d(c, d), \tag{21}$$

for all $a, b, c, d \in X$.

Let C be a nonempty closed convex subset of CAT(0) space X. The metric projection $P_C : X \to C$ is defined by

$$u = P_C(x) \Longleftrightarrow d(u, x) = \inf\{d(y, x) : y \in C\}, \quad \forall x \in X. \tag{22}$$

Recently, Dehghan and Rooin [14] presented a characterization of metric projection in CAT(0) spaces as follows.

Lemma 5. *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C(x)$ if and only if*

$$\left\langle \overrightarrow{yu}, \overrightarrow{ux} \right\rangle \leq 0, \quad \forall y \in C. \tag{23}$$

Lemma 6 (see [15]). *Let X be a complete CAT(0) space, let $\{x_n\}$ be a sequence in X, and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

Lemma 7 (see [16]). *Let X be a complete CAT(0) space. Then, for all $u, x, y \in X$, the following inequality holds:*

$$d^2(x, u) \leq d^2(y, u) + 2\left\langle \overrightarrow{xy}, \overrightarrow{xu} \right\rangle. \tag{24}$$

Lemma 8 (see [16]). *Let X be a complete CAT(0) space. For any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for any $x, y \in X$, the following inequality holds:*

$$\left\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \right\rangle \leq t\left\langle \overrightarrow{ux}, \overrightarrow{u_t y} \right\rangle + (1-t)\left\langle \overrightarrow{vx}, \overrightarrow{u_t y} \right\rangle. \tag{25}$$

Lemma 9 (see [17]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n$, $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

3. Viscosity Approximation Iteration Algorithms

In this section, we present the strong convergence theorems of Moudafi's viscosity approximation implicit and explicit iteration algorithms for a family of nonexpansive mappings $\{T_n : C \to C\}_{n=1}^{\infty}$ in CAT(0) spaces.

Lemma 10. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $\{\lambda_n\}$ be a given sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$ and $w_1 = T_1$; one defines a sequence $\{w_n : C \to C\}$ as follows:*

$$w_n = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} T_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n} \lambda_i} T_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} T_n, \quad \forall n \geq 2. \tag{26}$$

Then the following holds:

(i) $w_n = (\sum_{i=1}^{n-1} \lambda_i / \sum_{i=1}^{n} \lambda_i) w_{n-1} \oplus (\lambda_n / \sum_{i=1}^{n} \lambda_i) T_n$;

(ii) w_n *is nonexpansive;*

(iii) *for any* $x \in B$, *the sequence* $\{w_n(x)\}$ *converges uniformly to an element* $T(x) \in C$, *writing* $T(x) = \bigoplus_{n=1}^{\infty} \lambda_n T_n(x)$, *where B is a bounded subset of C.*

Proof. (i) For each n we introduce

$$\alpha_i^n = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}, \quad (i = 1, 2, \ldots, n); \tag{27}$$

thus

$$\begin{aligned} w_n &= \alpha_1^n T_1 \oplus \alpha_2^n T_2 \oplus \cdots \oplus \alpha_n^n T_n \\ &= (1 - \alpha_n^n) \left(\frac{\alpha_1^n}{1 - \alpha_n^n} T_1 \oplus \frac{\alpha_2^n}{1 - \alpha_n^n} T_2 \oplus \cdots \oplus \frac{\alpha_{n-1}^n}{1 - \alpha_n^n} T_{n-1} \right) \\ &\quad \oplus \alpha_n^n T_n \\ &= (1 - \alpha_n^n) \\ &\quad \times \left(\frac{\lambda_1}{\sum_{i=1}^{n-1} \lambda_i} T_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n-1} \lambda_i} T_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{\sum_{i=1}^{n-1} \lambda_i} T_{n-1} \right) \\ &\quad \oplus \alpha_n^n T_n \\ &= \frac{\sum_{i=1}^{n-1} \lambda_i}{\sum_{i=1}^{n} \lambda_i} w_{n-1} \oplus \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} T_n. \end{aligned} \tag{28}$$

(ii) We will show by induction that w_n is nonexpansive for all $n \in \mathbb{N}$. Since $w_1 = T_1$, w_1 is nonexpansive. Suppose w_n is nonexpansive. We consider

$$\begin{aligned} &d(w_{n+1}(x), w_{n+1}(y)) \\ &\quad = d\left(\frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n+1} \lambda_i} w_n(x) \oplus \frac{\lambda_{n+1}}{\sum_{i=1}^{n+1} \lambda_i} T_{n+1}(x), \right. \\ &\qquad \left. \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n+1} \lambda_i} w_n(y) \oplus \frac{\lambda_{n+1}}{\sum_{i=1}^{n+1} \lambda_i} T_{n+1}(y) \right) \\ &\quad \leq \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n+1} \lambda_i} d(w_n(x), w_n(y)) \\ &\qquad + \frac{\lambda_{n+1}}{\sum_{i=1}^{n+1} \lambda_i} d(T_{n+1}(x), T_{n+1}(y)) \\ &\quad \leq d(x, y). \end{aligned} \tag{29}$$

Thus w_{n+1} is nonexpansive.

(iii) In view of that $\lim_{n \to \infty} \lambda_n = 0$, for any $x \in B$, we have

$$\begin{aligned} &d(w_{n+1}(x), w_n(x)) \\ &\quad = d\left(\frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n+1} \lambda_i} w_n(x) \oplus \frac{\lambda_{n+1}}{\sum_{i=1}^{n+1} \lambda_i} T_{n+1}(x), w_n(x) \right) \\ &\quad \leq \frac{\lambda_{n+1}}{\sum_{i=1}^{n+1} \lambda_i} d(T_{n+1}(x), w_n(x)) \\ &\quad \leq \frac{\lambda_{n+1}}{\lambda_1} d(T_{n+1}(x), w_n(x)) \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{30}$$

This implies that the sequence $\{w_n(x)\}$ converges uniformly to an element $T(x) = \bigoplus_{n=1}^{\infty} \lambda_n T_n(x) \in X$. Since C is closed, $T(x) \in C$. □

Lemma 11. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let* $\{T_n : C \to C\}_{n=1}^{\infty}$ *be a family of nonexpansive mappings satisfying* $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Define* $T : C \to C$ *by* $T(x) = \bigoplus_{n=1}^{\infty} \lambda_n T_n(x)$ *for all* $x \in C$, *where* $\{\lambda_n\} \subset (0, 1)$ *with* $\sum_{n=1}^{\infty} \lambda_n = 1$. *Then T is nonexpansive and* $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. For any $x, y \in C$, we have

$$\begin{aligned} &d(T(x), T(y)) \\ &\quad \leq d(T(x), w_n(x)) + d(w_n(x), w_n(y)) \\ &\qquad + d(w_n(y), T(y)) \\ &\quad \leq d(T(x), w_n(x)) + d(x, y) \\ &\qquad + d(w_n(y), T(y)) \longrightarrow d(x, y) \quad (n \longrightarrow \infty). \end{aligned} \tag{31}$$

This implies that T is nonexpansive.

It is easy to see that $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$. We only show that $F(T) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $q \in F(T)$. For given $p \in \bigcap_{n=1}^{\infty} F(T_n)$, from Lemma 10(iii) we have

$$\begin{aligned} d(q, p) &= d(T(q), p) = \lim_{n \to \infty} d(w_n(q), p) \\ &\leq \lim_{n \to \infty} \Big(\lambda_1 d(T_1(q), p) + \lambda_2 d(T_2(q), p) \\ &\qquad + \cdots + \lambda_n d(T_n(q), p) \Big) \\ &= \sum_{n=1}^{\infty} \lambda_n d(T_n(q), p) \leq d(q, p). \end{aligned} \tag{32}$$

In view of that

$$d(T_n(q), p) = d(T_n(q), T_n(p)) \leq d(q, p), \quad \forall n \in \mathbb{N}, \tag{33}$$

we obtain that $d(T_n(q), p) = d(q, p)$ for all $n \in \mathbb{N}$. By condition (A), $T_n(q) = q$ for all $n \in \mathbb{N}$. Thus we complete the proof of Lemma 10. □

Now we are in a position to state and prove our main results.

Theorem 12. *Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a family of nonexpansive mappings satisfying $\mathcal{F} := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let f be a contraction on C with coefficient $\alpha \in (0,1)$, $\{w_n\}$ and let $\{\lambda_n\}$ be as in Lemma 10. Suppose the sequence $\{x_n\}$ is given by*

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), \tag{34}$$

for all $n \geq 0$, where $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$. Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{35}$$

Proof. We will divide the proof of Theorem 12 into five steps.

Step 1. The sequence $\{x_n\}$ defined by (34) is well defined for all $n \geq 0$.

In fact, let us define the mapping $G : C \to C$ by

$$G_n(x) := \alpha_n f(x) \oplus (1-\alpha_n) w_n(x), \quad x \in C. \tag{36}$$

For any $x, y \in C$, from Lemma 2, we have

$$\begin{aligned} d(G_n(x), G_n(y)) &= d\Big(\alpha_n f(x) \oplus (1-\alpha_n) w_n(x), \\ &\qquad \alpha_n f(y) \oplus (1-\alpha_n) w_n(y)\Big) \\ &\leq \alpha_n d(f(x), f(y)) + (1-\alpha_n) d(w_n(x), w_n(y)) \\ &\leq \alpha_n \alpha d(x,y) + (1-\alpha_n) d(x,y) \\ &= (1-\alpha_n(1-\alpha)) d(x,y). \end{aligned} \tag{37}$$

This implies that G_n is a contraction mapping. Hence, the sequence $\{x_n\}$ is well defined for all $n \geq 0$.

Step 2. The sequence $\{x_n\}$ is bounded.

For any $p \in \mathcal{F}$, from Lemma 3, we have that

$$\begin{aligned} d(x_n, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(w_n(x_n), p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(x_n, p). \end{aligned} \tag{38}$$

Then

$$\begin{aligned} d(x_n, p) &\leq d(f(x_n), p) \\ &\leq d(f(x_n), f(p)) + d(f(p), p) \\ &\leq \alpha d(x_n, p) + d(f(p), p). \end{aligned} \tag{39}$$

This implies that

$$d(x_n, p) \leq \frac{1}{1-\alpha} d(f(p), p). \tag{40}$$

Hence $\{x_n\}$ is bounded.

Step 3. $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, where $T = \bigoplus_{n=1}^{\infty} \lambda_n T_n$.

From (34) and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\begin{aligned} d(x_n, w_n(x_n)) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), w_n(x_n)) \\ &\leq \alpha_n d(f(x_n), w_n(x_n)) \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{41}$$

From Lemma 10, we get

$$\begin{aligned} d(x_n, T(x_n)) &\leq d(x_n, w_n(x_n)) \\ &\quad + d(w_n(x_n), T(x_n)) \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{42}$$

Step 4. The sequence $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to (35).

Since $\{x_n\}$ is bounded, by Lemma 4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ (without loss of generality we denote it by $\{x_j\}$) which Δ-converges to a point $\tilde{x}$.

First we claim that $\tilde{x} \in \mathcal{F}$. Since every CAT(0) space has Opial's property, if $T(\tilde{x}) \neq \tilde{x}$, we have

$$\begin{aligned} &\limsup_{j\to\infty} d(x_j, T(\tilde{x})) \\ &\quad \leq \limsup_{j\to\infty} \left(d(x_j, T(x_j)) + d(T(x_j), T(\tilde{x}))\right) \\ &\quad \leq \limsup_{j\to\infty} \left(d(x_j, T(x_j)) + d(x_j, \tilde{x})\right) \\ &\quad = \limsup_{j\to\infty} d(x_j, \tilde{x}) < \limsup_{j\to\infty} d(x_j, T(\tilde{x})). \end{aligned} \tag{43}$$

This is a contraction, and hence $\tilde{x} \in \mathcal{F}$.

Next we prove that $\{x_j\}$ converges strongly to $\tilde{x}$. Indeed, it follows from Lemma 8 that

$$\begin{aligned} d^2(x_j, \tilde{x}) &= \left\langle \overrightarrow{x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \\ &\leq \alpha_j \left\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle + (1-\alpha_j) \left\langle \overrightarrow{w_j(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \\ &\leq \alpha_j \left\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \\ &\quad + (1-\alpha_j) d(w_j(x_j), \tilde{x}) d(x_j, \tilde{x}) \\ &\leq \alpha_j \left\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle + (1-\alpha_j) d^2(x_j, \tilde{x}). \end{aligned} \tag{44}$$

It follows that

$$\begin{aligned} d^2\left(x_j,\tilde{x}\right) &\le \left\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \\ &= \left\langle \overrightarrow{f(x_j)f(\tilde{x})}, \overrightarrow{x_j\tilde{x}} \right\rangle + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \\ &\le d\left(f\left(x_j\right), f\left(\tilde{x}\right)\right) d\left(x_j,\tilde{x}\right) + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \\ &\le \alpha d^2\left(x_j,\tilde{x}\right) + \left\langle \overrightarrow{f\left(\tilde{x}\right)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle, \end{aligned} \tag{45}$$

and thus

$$d^2\left(x_j,\tilde{x}\right) \le \frac{1}{1-\alpha}\left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle. \tag{46}$$

Since $\{x_j\}$ Δ-converges to $\tilde{x}$, by Lemma 6 we have

$$\limsup_{n\to\infty}\left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \right\rangle \le 0. \tag{47}$$

It follows from (46) that $\{x_j\}$ converges strongly to $\tilde{x}$.

Next we show that $\tilde{x}$ solves the variational inequality (35). Applying Lemma 1, for any $q \in \mathscr{F}$, we have

$$\begin{aligned} d^2\left(x_j,q\right) &= d^2\left(\alpha_j f\left(x_j\right) \oplus \left(1-\alpha_j\right) w_j\left(x_j\right), q\right) \\ &\le \alpha_j d^2\left(f\left(x_j\right),q\right) + \left(1-\alpha_j\right) d^2\left(w_j\left(x_j\right),q\right) \\ &\quad - \alpha_j\left(1-\alpha_j\right) d^2\left(f\left(x_j\right), w_j\left(x_j\right)\right). \end{aligned} \tag{48}$$

This together with Lemma 10(ii) implies that

$$\begin{aligned} d^2\left(x_j,q\right) &\le d^2\left(f\left(x_j\right),q\right) \\ &\quad - \left(1-\alpha_j\right)\left(d(f(x_j),x_j) + d\left(x_j, w_j(x_j)\right)\right)^2. \end{aligned} \tag{49}$$

Taking the limit through $j \to \infty$, we can obtain

$$d^2\left(\tilde{x},q\right) \le d^2\left(f\left(\tilde{x}\right),q\right) - d^2\left(f\left(\tilde{x}\right),\tilde{x}\right). \tag{50}$$

On the other hand, from (20) we have

$$\begin{aligned} \left\langle \overrightarrow{\tilde{x}f\left(\tilde{x}\right)}, \overrightarrow{q\tilde{x}} \right\rangle = \frac{1}{2}\Big[& d^2\left(\tilde{x},\tilde{x}\right) + d^2\left(f\left(\tilde{x}\right),q\right) \\ & - d^2\left(\tilde{x},q\right) - d^2\left(f\left(\tilde{x}\right),\tilde{x}\right)\Big]. \end{aligned} \tag{51}$$

From (50) and (51) we have

$$\left\langle \overrightarrow{\tilde{x}f\left(\tilde{x}\right)}, \overrightarrow{q\tilde{x}} \right\rangle \ge 0, \quad \forall q \in \mathscr{F}. \tag{52}$$

That is, $\tilde{x}$ solves the inequality (35).

Step 5. The sequence $\{x_n\}$ converges strongly to $\tilde{x}$.

Assume that $x_{n_i} \to \hat{x}$ as $n \to \infty$. By the same argument, we get that $\hat{x} \in \mathscr{F}$ which solves the variational inequality (35); that is,

$$\left\langle \overrightarrow{\tilde{x}f\left(\tilde{x}\right)}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle \le 0, \tag{53}$$

$$\left\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \right\rangle \le 0. \tag{54}$$

Adding up (53) and (54), we get that

$$\begin{aligned} 0 &\ge \left\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle - \left\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle \\ &= \left\langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle + \left\langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle \\ &\quad - \left\langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle - \left\langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle \\ &= \left\langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle - \left\langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle \\ &\ge \left\langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \right\rangle - d\left(f\left(\hat{x}\right), f\left(\tilde{x}\right)\right) d\left(\hat{x},\tilde{x}\right) \\ &\ge d^2\left(\tilde{x},\hat{x}\right) - \alpha d^2\left(\hat{x},\tilde{x}\right) \\ &= \left(1-\alpha\right) d^2\left(\tilde{x},\hat{x}\right). \end{aligned} \tag{55}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x},\hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the sequence $\{x_n\}$ converges strongly to $\tilde{x}$, which is the unique solution to the variational inequality (35).

This completes the proof. □

Theorem 13. *Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a family of nonexpansive mappings satisfying $\mathscr{F} := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let f be a contraction on C with coefficient $\alpha \in (0,1)$ and let $\{w_n\}$ be as in Lemma 10. Suppose $x_0 \in C$ and the sequence $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f\left(x_n\right) \oplus \left(1-\alpha_n\right) w_n\left(x_n\right), \tag{56}$$

such that $d(w_n(x_n), w_{n+1}(x_{n+1})) \le d(x_n, x_{n+1}) + \varepsilon_n$ for all $n \in \mathbb{N}$, where $\sum_{n=1}^{\infty}\varepsilon_n < \infty$ and $\{\alpha_n\} \subset (0,1)$ satisfies

(i) $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\mathscr{F}}f(\tilde{x})$, which is equivalent to the variational inequality (35).

Proof. We first show that the sequence $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), p) \\ &\le \alpha_n d(f(x_n), p) + (1-\alpha_n) d(w_n(x_n), p) \\ &\le \alpha_n (d(f(x_n), f(p)) + d(f(p), p)) \\ &\quad + (1-\alpha_n) d(w_n(x_n), p) \\ &\le \alpha_n \alpha d(x_n, p) + \alpha_n d(f(p), p) \\ &\quad + (1-\alpha_n) d(x_n, p) \\ &= (1-\alpha_n(1-\alpha)) d(x_n, p) \\ &\quad + \alpha_n (1-\alpha) \cdot \frac{1}{1-\alpha} d(f(p), p) \\ &\le \max\left\{ d(x_n, p), \frac{1}{1-\alpha} d(f(p), p) \right\}. \end{aligned} \tag{57}$$

By induction, we have

$$d(x_n, p) \le \max\left\{ d(x_0, p), \frac{1}{1-\alpha} d(f(p), p) \right\}, \tag{58}$$

for all $n \ge 0$. Hence $\{x_n\}$ is bounded, so are $\{w_n(x_n)\}$ and $\{f(x_n)\}$.

From (56), we have

$$\begin{aligned} &d(x_{n+1}, x_n) \\ &= d\Big(\alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), \\ &\qquad \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1}) w_{n-1}(x_{n-1})\Big) \\ &\le d\Big(\alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), \\ &\qquad \alpha_n f(x_{n-1}) \oplus (1-\alpha_n) w_{n-1}(x_{n-1})\Big) \\ &\quad + d\Big(\alpha_n f(x_{n-1}) \oplus (1-\alpha_n) w_{n-1}(x_{n-1}), \\ &\qquad \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1}) w_{n-1}(x_{n-1})\Big) \\ &\le \alpha_n d(f(x_n), f(x_{n-1})) \\ &\quad + (1-\alpha_n) d(w_n(x_n), w_{n-1}(x_{n-1})) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(f(x_{n-1}), w_{n-1}(x_{n-1})) \\ &\le \alpha_n \alpha d(x_n, x_{n-1}) + (1-\alpha_n) d(x_n, x_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(f(x_{n-1}), w_{n-1}(x_{n-1})) + \varepsilon_n \\ &\le (1-(1-\alpha)\alpha_n) d(x_n, x_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(f(x_{n-1}), w_{n-1}(x_{n-1})) + \varepsilon_n. \end{aligned} \tag{59}$$

From Lemma 9 and conditions (ii) and (iii) we obtain

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{60}$$

From condition (i), we have

$$\begin{aligned} &d(x_n, w_n(x_n)) \\ &\quad = d(x_n, x_{n+1}) + d(x_{n+1}, w_n(x_n)) \\ &\quad = d(x_n, x_{n+1}) \\ &\qquad + d(\alpha_n f(x_n) \oplus (1-\alpha_n) w_n(x_n), w_n(x_n)) \\ &\quad \le d(x_n, x_{n+1}) \\ &\qquad + \alpha_n d(f(x_n), w_n(x_n)) \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{61}$$

From Lemma 10(iii) we can obtain

$$\begin{aligned} &d(w_m(x_{n+1}), x_{n+1}) \\ &\quad \le d(w_m(x_{n+1}), w_{n+1}(x_{n+1})) \\ &\qquad + d(w_{n+1}(x_{n+1}), x_{n+1}) \longrightarrow 0 \quad (m \longrightarrow \infty,\ n \longrightarrow \infty). \end{aligned} \tag{62}$$

Without loss of generality, we can choose the sequence $\{\alpha_m\}$ such that

$$d(w_m(x_{n+1}), x_{n+1}) = o(\alpha_m) \quad (m \longrightarrow \infty,\ n \longrightarrow \infty). \tag{63}$$

Let $\{z_m\}$ be a sequence in C such that

$$z_m = \alpha_m f(z_m) \oplus (1-\alpha_m) w_m(z_m). \tag{64}$$

It follows from Theorem 12 that $\{z_m\}$ converges strongly to a fixed point $\tilde{x} \in \mathcal{F}$, which solves the variational inequality (35).

Now we claim that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \le 0. \tag{65}$$

Indeed, it follows from Lemma 8 that

$$\begin{aligned} &d^2(z_m, x_{n+1}) \\ &\quad = \left\langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\ &\quad \le \alpha_m \left\langle \overrightarrow{f(z_m) x_{n+1}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\ &\qquad + (1-\alpha_m) \left\langle \overrightarrow{w_m(z_m) x_{n+1}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\ &\quad = \alpha_m \left\langle \overrightarrow{f(z_m) f(\tilde{x})}, \overrightarrow{z_m x_{n+1}} \right\rangle + \alpha_m \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\ &\qquad + \alpha_m \left\langle \overrightarrow{\tilde{x} z_m}, \overrightarrow{z_m x_{n+1}} \right\rangle + \alpha_m \left\langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+(1-\alpha_m)\left\langle \overrightarrow{w_m(z_m)\,w_m(x_{n+1})}, \overrightarrow{z_m x_{n+1}} \right\rangle \\
&+(1-\alpha_m)\left\langle \overrightarrow{w_m(x_{n+1})\,x_{n+1}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\
\le{}& \alpha_m \alpha d(z_m,\tilde{x})\,d(z_m,x_{n+1}) + \alpha_m \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\
&+\alpha_m d(\tilde{x},z_m)\,d(z_m,x_{n+1}) + \alpha_m d^2(z_m,x_{n+1}) \\
&+(1-\alpha_m)\,d^2(z_m,x_{n+1}) \\
&+(1-\alpha_m)\,d(w_m(x_{n+1}),x_{n+1})\,d(z_m,x_{n+1}) \\
\le{}& \alpha_m \alpha d(z_m,\tilde{x})\,M + \alpha_m \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{z_m x_{n+1}} \right\rangle \\
&+\alpha_m d(\tilde{x},z_m)\,M + d^2(z_m,x_{n+1}) \\
&+(1-\alpha_m)\,d(w_m(x_{n+1}),x_{n+1})\,M,
\end{aligned}
\tag{66}
$$

where

$$
M \ge \sup_{m,n\ge 1}\{d(z_m,x_n)\}. \tag{67}
$$

This implies that

$$
\left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1} z_m} \right\rangle \le (1+\alpha)\,Md(z_m,\tilde{x}) + \frac{d(w_m(x_{n+1}),x_{n+1})}{\alpha_m}M. \tag{68}
$$

Taking the upper limit as $m \to \infty$ and $n \to \infty$, from (63) we get

$$
\limsup_{m,n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1} z_m} \rangle \le 0. \tag{69}
$$

Furthermore, we have

$$
\begin{aligned}
\left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle &= \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1} z_m} \right\rangle + \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{z_m \tilde{x}} \right\rangle \\
&\le \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1} z_m} \right\rangle \\
&\quad + d(f(\tilde{x}),\tilde{x})\,d(z_m,\tilde{x}).
\end{aligned}
\tag{70}
$$

Thus, by taking the upper limit as $n \to \infty$ first and then $m \to \infty$, it follows from $z_m \to \tilde{x}$ and (69) that

$$
\limsup_{n\to\infty} \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \le 0. \tag{71}
$$

Finally, we prove that $x_n \to \tilde{x}$ as $n \to \infty$. In fact, for any $n \ge 0$, let

$$
y_n = \alpha_n \tilde{x} \oplus (1-\alpha_n)\,w_n(x_n). \tag{72}
$$

From Lemmas 7 and 8 we have that

$$
\begin{aligned}
&d^2(x_{n+1},\tilde{x}) \\
&\le d^2(y_n,\tilde{x}) + 2\left\langle \overrightarrow{x_{n+1} y_n}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\le (1-\alpha_n)^2 d^2(w_n(x_n),\tilde{x}) \\
&\quad + 2\Big[\alpha_n \left\langle \overrightarrow{f(x_n)\,y_n}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\qquad + (1-\alpha_n)\left\langle \overrightarrow{w_n(x_n)\,y_n}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle\Big] \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) \\
&\quad + 2\Big[\alpha_n^2 \left\langle \overrightarrow{f(x_n)\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\qquad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{f(x_n)\,w_n(x_n)}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\qquad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{w_n(x_n)\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\qquad + (1-\alpha_n)^2\left\langle \overrightarrow{w_n(x_n)\,w_n(x_n)}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle\Big] \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) \\
&\quad + 2\Big[\alpha_n^2 \left\langle \overrightarrow{f(x_n)\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\qquad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{f(x_n)\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle\Big] \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \left\langle \overrightarrow{f(x_n)\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \left\langle \overrightarrow{f(x_n)\,f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\quad + 2\alpha_n \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \alpha d(x_n,\tilde{x})\,d(x_{n+1},\tilde{x}) \\
&\quad + 2\alpha_n \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) \\
&\quad + \alpha_n\alpha\left(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\right) \\
&\quad + 2\alpha_n \left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle.
\end{aligned}
\tag{73}
$$

This implies that

$$
d^2(x_{n+1},\tilde{x}) \le \frac{1-(2-\alpha)\alpha_n + \alpha_n^2}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\left\langle \overrightarrow{f(\tilde{x})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle
$$

$$= \left(1 - \frac{\alpha_n (2 - 2\alpha - \alpha_n)}{1 - \alpha\alpha_n}\right) d^2 (x_n, \tilde{x}) + \frac{2\alpha_n}{1 - \alpha\alpha_n} \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle. \tag{74}$$

Then it follows that

$$d^2 (x_{n+1}, \tilde{x}) \leq (1 - \alpha'_n) d^2 (x_n, \tilde{x}) + \alpha'_n \beta'_n, \tag{75}$$

where

$$\alpha'_n = \frac{\alpha_n (2 - 2\alpha - \alpha_n)}{1 - \alpha\alpha_n}, \quad \beta'_n = \frac{2}{2 - 2\alpha - \alpha_n} \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \right\rangle. \tag{76}$$

Applying Lemma 9, we can conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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