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Research Article

On the Covariance of Moore-Penrose Inverses in Rings with Involution

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We consider the so-called covariance set of Moore-Penrose inverses in rings with an involution. We deduce some new results concerning covariance set. We will show that if a is a regular element in a C^*-algebra, then the covariance set of a is closed in the set of invertible elements (with relative topology) of C^*-algebra and is a cone in the C^*-algebra.

1. Introduction

Suppose that $\mathfrak{R}$ is a ring with unity $1 \neq 0$. A mapping $* : x \mapsto x^*$ of $\mathfrak{R}$ into itself is called an *involution* if

$$(x^*)^* = x, \qquad (x+y)^* = x^* + y^*, \qquad (xy)^* = y^* x^*, \tag{1}$$

for all x and y in $\mathfrak{R}$. A ring $\mathfrak{R}$ with an involution $*$ is called $*$-*ring*. Throughout this paper $\mathfrak{R}$ is a $*$-*ring*.

An element $a \in \mathfrak{R}$ is called *regular* if it has a generalized inverse (in the sense of von Neumann) in $\mathfrak{R}$; that is, there exists $b \in \mathfrak{R}$ such that

$$aba = a. \tag{2}$$

Note that such b is not unique [1, 2].

Definition 1. Let $\mathfrak{R}$ be a $*$-ring and $a \in \mathfrak{R}$.

(i) a is called Moore-Penrose invertible if there exists $b \in \mathfrak{R}$ such that

$$aba = a, \qquad bab = b, \qquad (ab)^* = ab, \qquad (ba)^* = ba. \tag{3}$$

(ii) a is called Drazin invertible if there exists $b \in \mathfrak{R}$ such that

$$bab = b, \qquad ab = ba, \qquad a^{k+1} b = a^k \tag{4}$$

for some nonnegative integer k. The least such k is the Drazin index of a, denoted by $\operatorname{ind}(a)$.

Obviously, $\operatorname{ind}(a) = 0$ if and only if a is invertible and in this case the Drazin inverses of a and a^{-1} coincide. If $\operatorname{ind}(a) \leq 1$, then the Drazin inverse is known as the *group inverse*.

It is well known that the Moore-Penrose inverse (briefly, MP-inverse) and the Drazin inverse are unique if they exist. We reserve the notations $a^\dagger$ and a^D for the MP-inverse and Drazin inverse of a, respectively. According to the uniqueness of the notion under consideration, if a has a MP-inverse, then a^* and $a^\dagger$ also have MP-inverses. Moreover

$$(a^\dagger)^\dagger = a, \qquad (a^\dagger)^* = (a^*)^\dagger, \qquad a^* = a^\dagger a a^* = a^* a a^\dagger. \tag{5}$$

In what follows, we will denote by $\mathfrak{R}^{-1}$ the subset of invertible elements of $\mathfrak{R}$ and by $\mathfrak{R}^\dagger$ the set of all MP-invertible elements of $\mathfrak{R}$. An element x in $\mathfrak{R}$ is called *idempotent* if $x^2 = x$. A *projection* $p \in \mathfrak{R}$ satisfies $p = p^* = p^2$. Note that if $x \in \mathfrak{R}^\dagger$, then $xx^\dagger$ and $x^\dagger x$ are projections. In addition,

$$(xx^\dagger)^\dagger = xx^\dagger, \qquad (x^\dagger x)^\dagger = x^\dagger x. \tag{6}$$

The *commutator* of a pair of elements x and y in $\mathfrak{R}$ is given by

$$[x, y] = xy - yx. \tag{7}$$

Note that $[x, y] = 0$ if and only if x and y commute. Also, it is well known that if x, y, and z are in $\mathfrak{R}$, then

$$[x, yz] = [x, y] z + y [x, z],$$
$$[xy, z] = x [y, z] + [x, z] y. \tag{8}$$

Let a be an element in $\mathfrak{R}^{-1}$; its inverse a^{-1} is *covariant* with respect to $\mathfrak{R}^{-1}$; that is, for all $b \in \mathfrak{R}^{-1}$, we have

$$\left(bab^{-1}\right)^{-1} = ba^{-1}b^{-1}. \tag{9}$$

In general, the elements of $\mathfrak{R}^{\dagger}$ are not covariant under $\mathfrak{R}^{-1}$ (see [2–4]). For a given element $a \in \mathfrak{R}^{\dagger}$ with MP-inverse $a^{\dagger}$ we define its *covariance set*

$$\mathfrak{C}(a) = \left\{b \in \mathfrak{R}^{-1} : \left(bab^{-1}\right)^{\dagger} = ba^{\dagger}b^{-1}\right\}. \tag{10}$$

Schwerdtfeger [4] described the class $\mathfrak{C}(a)$ for the matrices of rank 1 or 2. The characterization of the covariance set $\mathfrak{C}(a)$ for an algebra of matrices was studied by Robinson [2] and some interesting results of $\mathfrak{C}(a)$ were presented by Meenakshi and Chinnadurai [3].

The paper is organized as follows. The endeavour in Section 2 is to show how the results of [3] can be extended to MP-inverses in $*$-rings. Moreover, we show that Drazin inverses are covariant under the group of invertible elements of $*$-rings. In Section 3 we prove that the covariance set is a *closed* set in $\mathfrak{A}^{-1}$ and is a *cone* in $\mathfrak{A}$. Furthermore, we show that if $\{a_n\}$ is a sequence of MP-invertible elements of a C^*-algebra such that their MP-inverses norm is bounded and a_n converges to a, then there is some kind of convergence of $\mathfrak{C}(a_n)$ to $\mathfrak{C}(a)$.

2. Covariance Set of Moore-Penrose Inverses in $*$-Rings

Many of the results of this section are essentially due to [3], withthe main difference being that in [3] one considers covariance set for matrices. In this section we generalized these results to any $*$-ring.

The next proposition describes a relation between the covariance set $\mathfrak{C}(a)$ and commutators. It was also shown in [2–4] in the special case of matrices. Here, we include a shorter proof for the sake of completeness.

Proposition 2. *Let $\mathfrak{R}$ be $*$-ring and $a \in \mathfrak{R}^{\dagger}$ with MP-inverse $a^{\dagger}$. Then the following statements are equivalent:*

(i) $b \in \mathfrak{C}(a)$;

(ii) $[b^*b, aa^{\dagger}] = 0$ *and* $[b^*b, a^{\dagger}a] = 0$.

Proof. (i)⇒(ii) Suppose that $b \in \mathfrak{C}(a)$. Then $(bab^{-1})^{\dagger} = ba^{\dagger}b^{-1}$. Set $p = (bab^{-1})(bab^{-1})^{\dagger}$. Then p is projection, so $p = p^*$ and $p = baa^{\dagger}b^{-1}$. From here we get $baa^{\dagger}b^{-1} = (b^{-1})^* aa^{\dagger}b^*$. This implies that $[b^*b, aa^{\dagger}] = 0$. Similarly by putting $q = (bab^{-1})^{\dagger}(bab^{-1})$, we conclude that $[b^*b, a^{\dagger}a] = 0$.

(ii)⇒(i) From the assumptions it is not hard to see that $ba^{\dagger}b^{-1}$ is the MP-inverse of bab^{-1}. By the uniqueness of Moore-Penrose inverse we get $(bab^{-1})^{\dagger} = ba^{\dagger}b^{-1}$; that is, $b \in \mathfrak{C}(a)$. □

From Proposition 2 we deduce the following result.

Corollary 3. *Let $\mathfrak{R}$ be $*$-ring and $a \in \mathfrak{R}^{\dagger}$ with MP-inverse $a^{\dagger}$. Then*

$$b^{-1} \in \mathfrak{C}(a) \quad \textit{iff} \; \left[bb^*, aa^{\dagger}\right] = 0,$$
$$\left[bb^*, a^{\dagger}a\right] = 0. \tag{11}$$

Combining the above corollary and Proposition 2, we get the following corollary.

Corollary 4. *If b is normal, then*

$$b \in \mathfrak{C}(a) \quad \textit{iff} \; b^{-1} \in \mathfrak{C}(a). \tag{12}$$

We now have some equalities for the covariance sets. See also [3].

Proposition 5. *Let $\mathfrak{R}$ be $*$-ring and $a \in \mathfrak{R}^{\dagger}$ with MP-inverse $a^{\dagger}$. Then*

$$\mathfrak{C}(a) = \mathfrak{C}\left(a^{\dagger}\right) = \mathfrak{C}\left(a^*\right) = \mathfrak{C}\left(aa^{\dagger}\right) \cap \mathfrak{C}\left(a^{\dagger}a\right). \tag{13}$$

Proof. By replacing a with $a^{\dagger}$, part (ii) of Proposition 2 does not change so the first equality holds. Since $(a^*)^{\dagger}a^* = aa^{\dagger}$ and $a^*(a^*)^{\dagger} = a^{\dagger}a$, Proposition 2 yields the second equality. Also $a = aa^{\dagger}a$ and $a^{\dagger}aa^{\dagger} = a^{\dagger}$, again from Proposition 2 we get the last equality. □

Note that if u is any unitary element in $\mathfrak{R}^{-1}$, the $u^*u = uu^* = 1$; thus $u \in \mathfrak{C}(a)$ for every $a \in \mathfrak{R}^{\dagger}$. This implies that $\mathfrak{C}(a) \neq \emptyset$ for each $a \in \mathfrak{R}^{\dagger}$.

In the next proposition, we will show that if $a \in \mathfrak{R}$ is Drazin invertible with Drazin inverse a^D, then $\{b \in \mathfrak{R}^{-1} : (bab)^D = ba^Db^{-1}\} = \mathfrak{R}^{-1}$. For this reason, the notion of covariance sets is not studied to Drazin inverses.

Proposition 6. *Suppose that $\mathfrak{R}$ is a $*$-ring and a is a Drazin invertible element in $\mathfrak{R}$. Then a^D is covariant under $\mathfrak{R}^{-1}$; that is,*

$$\left(bab^{-1}\right)^D = ba^Db^{-1}, \quad \forall b \textit{ in } \mathfrak{R}^{-1}. \tag{14}$$

Proof. Suppose that a^D is the Drazin inverse of a and b is an arbitrary element in $\mathfrak{R}^{-1}$. For simplicity of calculations, set $X = bab^{-1}$ and $Y = ba^Db^{-1}$. By hypothesis, $a^Daa^D = a^D$, $a^Da = aa^D$, and $a^{k+1}a^D = a^k$; thus

$$YXY = \left(ba^Db^{-1}\right)\left(bab^{-1}\right)\left(ba^Db^{-1}\right)$$
$$= ba^Daa^Db^{-1} = ba^Db^{-1} = Y;$$

$$\begin{aligned} YX &= \left(ba^Db^{-1}\right)\left(bab^{-1}\right) = ba^Dab^{-1} \\ &= baa^Db^{-1} = XY; \\ X^{k+1}Y &= ba^{k+1}a^Db^{-1} = ba^kb^{-1} \\ &= \left(bab^{-1}\right)^k = X^k. \end{aligned} \tag{15}$$

Now the uniqueness of the Drazin inverse implies that $Y = X^D$; that is, a^D is covariant under $\mathfrak{R}^{-1}$. □

In particular, by applying the above proposition, if a is group invertible with the group inverse $a^\sharp \in \mathfrak{R}$, then $a^\sharp$ is also covariant under $\mathfrak{R}^{-1}$.

We reproduce the following definition from [5].

Definition 7. Let $\mathfrak{R}$ be a ring; $a \in \mathfrak{R}$ is called simply polar if it has a commuting generalized inverse (in the sense of von Neumann); that is, if b is any generalized inverse of a, then $[a, b] = 0$.

Some authors used the expression EP instead of simply polar. Indeed, they called $a \in \mathfrak{R}^\dagger$ with MP-inverse $a^\dagger$ is EP if and only if $aa^\dagger = a^\dagger a$.

The next remark provides a large class of simply polar elements and some related properties.

Remark 8. Let $a \in \mathfrak{R}^\dagger$ with MP-inverse $a^\dagger$.

(i) If a is self-adjoint, then it is simply polar, since

$$aa^\dagger = \left(aa^\dagger\right)^* = \left(a^\dagger\right)^* a^* = a^\dagger a. \tag{16}$$

(ii) If a is normal, then it is simply polar, since

$$\begin{aligned} a &= a\left(a^\dagger a\right)^* = aa^*\left(a^\dagger\right)^* = a^*a\left(a^\dagger\right)^* \\ &= \left(a^\dagger a\right)^* a^*a\left(a^\dagger\right)^* = \left(a^\dagger a\right)\left(a^*a\right)^*\left(a^\dagger\right)^* \\ &= \left(a^\dagger a\right)\left(a^\dagger aa^*\right)^* = \left(a^\dagger a\right)\left(a^*\right)^* = a^\dagger a^2; \end{aligned} \tag{17}$$

thus $a = a^\dagger a^2$. In a similar manner we get $a = a^2 a^\dagger$. Therefore

$$aa^\dagger = a^\dagger a^2 a^\dagger = a^\dagger a. \tag{18}$$

(iii) It is easy to check that simply polar properties of a, a^* and $a^\dagger$ are equivalent; that is, if one of them is simply polar, then two others are also simply polar.

(iv) If a is simply polar, then

$$\left(aa^\dagger\right)^2 = a^2\left(a^\dagger\right)^2 = \left(a^\dagger\right)^2 a^2. \tag{19}$$

(v) If a is simply polar, then Proposition 5 implies that $\mathfrak{C}(a) = \mathfrak{C}(aa^\dagger)$.

For finding more equivalent statements about the simply polar elements see [1, Theorem 2.3 and final remark].

Proposition 9. *Let $a, b \in \mathfrak{R}^\dagger$ with MP-inverses $a^\dagger$ and $b^\dagger$, respectively. If $a^\dagger b = 0 = ab^\dagger$ and $ba^\dagger = 0 = b^\dagger a$, then $\mathfrak{C}(a) \cap \mathfrak{C}(b) \subset \mathfrak{C}(a + b)$.*

Proof. The assumptions, after some easy calculations, imply that $a^\dagger + b^\dagger$ is the MP-inverse of $a + b$. Thus $(a + b)^\dagger = a^\dagger + b^\dagger$. Suppose that $x \in \mathfrak{C}(a) \cap \mathfrak{C}(b)$. Then Proposition 2 implies that

$$\begin{aligned} &\left[x^*x, aa^\dagger\right] = 0, \qquad \left[x^*x, a^\dagger a\right] = 0, \\ &\left[x^*x, bb^\dagger\right] = 0, \qquad \left[x^*x, b^\dagger b\right] = 0. \end{aligned} \tag{20}$$

Since $a^\dagger b = 0 = ab^\dagger$ and $ba^\dagger = 0 = b^\dagger a$, we have $(a + b)(a^\dagger + b^\dagger) = aa^\dagger + bb^\dagger$ and $(a^\dagger + b^\dagger)(a + b) = a^\dagger a + b^\dagger b$. From the linearity of commutator we obtain

$$\begin{aligned} &\left[x^*x, (a + b)\left(a^\dagger + b^\dagger\right)\right] = 0, \\ &\left[x^*x, \left(a^\dagger + b^\dagger\right)(a + b)\right] = 0. \end{aligned} \tag{21}$$

Again by applying Proposition 2, we get $x \in \mathfrak{C}(a + b)$. □

Corollary 10. *Let $a, b \in \mathfrak{R}^\dagger$ with MP-inverses $a^\dagger$ and $b^\dagger$, respectively. If a and b are self adjoint and $ba^\dagger = 0 = b^\dagger a$, then $\mathfrak{C}(a) \cap \mathfrak{C}(b) \subset \mathfrak{C}(a + b)$.*

Proof. By assumption a and b are self adjoint. Thus $ba^\dagger = 0 = b^\dagger a$ implies that $a^\dagger b = 0 = ab^\dagger$. The result now follows from Proposition 9. □

The next example shows that in Proposition 9 inclusion can be proper.

Example 11. Set $a = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $b = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then $a^\dagger = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] = b$, $b^\dagger = a$, and $a^\dagger b = 0 = ab^\dagger$, and $a+b = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is invertible; thus $\mathfrak{C}(a + b) = \mathfrak{R}^{-1}$. Now if we set $y = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ then y is invertible:

$$y^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad yy^* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{22}$$

On the other hand $aa^\dagger = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$; therefore

$$aa^\dagger yy^* = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{but} \quad yy^* aa^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{23}$$

From here we conclude that $[aa^\dagger, yy^*] \neq 0$. Thus $y \notin \mathfrak{C}(a)$.

Let X and Y be two subsets of $\mathfrak{R}$. We recall that

$$\begin{aligned} X + Y &= \{x + y : x \in X, y \in Y\}, \\ XY &= \{xy : x \in X, y \in Y\}. \end{aligned} \tag{24}$$

Note that the reverse order rule for the MP-inverse, that is, $(ab)^\dagger = b^\dagger a^\dagger$, is valid under certain conditions on MP-invertible elements; see [6].

Remark 12. Let $a, b \in \mathfrak{R}^\dagger$ with MP-inverses $a^\dagger$ and $b^\dagger$, respectively. One can easily check the following.

(i) If $a^\dagger b = 0 = ab^\dagger$ and $ba^\dagger = 0 = b^\dagger a$, then $\mathfrak{C}(a) \cap \mathfrak{C}(b) \cap (\mathfrak{C}(a) + \mathfrak{C}(b)) = \mathfrak{C}(a) \cap \mathfrak{C}(b) \cap \mathfrak{C}(a+b)$.

(ii) If $(ab)^\dagger = b^\dagger a^\dagger$, then $\mathfrak{C}(a) \cap \mathfrak{C}(b) \cap (\mathfrak{C}(b)\mathfrak{C}(a)) = \mathfrak{C}(a) \cap \mathfrak{C}(b) \cap \mathfrak{C}(ab)$.

(iii) Generally, there is no subset relation between $\mathfrak{C}(a+b)$ and $\mathfrak{C}(a) + \mathfrak{C}(b)$. For instance, if we put $b = -a$, then $0 \in \mathfrak{C}(a) + \mathfrak{C}(-a)$ which is not a subset of $\mathfrak{R}^{-1}$ but $\mathfrak{C}(a+b) = \mathfrak{C}(0) = \mathfrak{R}^{-1}$.

(iv) Generally, there is no subset relation between $\mathfrak{C}(ab)$ and $\mathfrak{C}(a)\mathfrak{C}(b)$. Set $a = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ as Example 11. Then $a^2 = 0$, and so $\mathfrak{C}(a^2) = \mathfrak{R}^{-1} \neq \mathfrak{C}(a)\mathfrak{C}(a)$.

Proposition 13. *Let $a, b \in \mathfrak{R}^\dagger$ with MP-inverses $a^\dagger$ and $b^\dagger$, respectively. If $a\mathfrak{R} = b\mathfrak{R}$, then $aa^\dagger = bb^\dagger$, where $a\mathfrak{R} = \{ax : x \in \mathfrak{R}\}$.*

Proof. By assumption $a\mathfrak{R} = b\mathfrak{R}$, so there exists x in $\mathfrak{R}$ such that $a = bx = bb^\dagger bx$. Therefore $a = bb^\dagger a$, and so $aa^\dagger = bb^\dagger aa^\dagger$. In a similar manner we get $bb^\dagger = aa^\dagger bb^\dagger$. Since $aa^\dagger$ is projection, $aa^\dagger = bb^\dagger$. □

Corollary 14. *Let $a, b \in \mathfrak{R}^\dagger$ with MP-inverses $a^\dagger$ and $b^\dagger$, respectively. If $a\mathfrak{R} = b\mathfrak{R}$ and $a^\dagger\mathfrak{R} = b^\dagger\mathfrak{R}$, then $\mathfrak{C}(a) = \mathfrak{C}(b)$.*

Proof. The proof is an immediate consequence of Propositions 5 and 13. □

The following corollary was also proved for matrices in [3].

Corollary 15. *Let $a, b \in \mathfrak{R}^\dagger$ be simply polar and $a\mathfrak{R} = b\mathfrak{R}$. Then $\mathfrak{C}(a) = \mathfrak{C}(b)$.*

According to the above corollary and Remark 8, we have the following.

Corollary 16. *If $a \in \mathfrak{R}^\dagger$ and a is simply polar, then $\mathfrak{C}(a) = \mathfrak{C}(a^2) = \mathfrak{C}(a^4) = \cdots = \mathfrak{C}(a^{2n})$ for each $n \in \mathbb{N}$.*

Corollary 17. *If $a \in \mathfrak{R}^\dagger$ and a is normal, then $\mathfrak{C}(a) = \mathfrak{C}(a^2) = \mathfrak{C}(a^4) = \cdots = \mathfrak{C}(a^{2n})$ for each $n \in \mathbb{N}$.*

Note that Example 11 shows that the converses of the two last corollaries do not hold. Indeed, if we set $a = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, then a is neither simply polar nor normal and $y \notin \mathfrak{C}(a)$ but $y \in \mathfrak{C}(a^2) = \mathfrak{C}(0) = \mathfrak{R}^{-1}$.

We know that if either $a = 0$ or $a \in \mathfrak{R}^{-1}$, then $\mathfrak{C}(a) = \mathfrak{R}^{-1}$. One can easily check that if $\mathfrak{R}$ is a $*$-ring with no nonzero nilpotent element, then $\mathfrak{C}(p) = \mathfrak{R}^{-1}$ where $p \in \mathfrak{R}^\dagger$ and it is an idempotent element of ring. In all cases, we consider that $\mathfrak{C}(a)$ has a group structure. But in general $\mathfrak{C}(a)$ is not a group; see for instance [3]. Our purpose is to find a subset of $\mathfrak{C}(a)$ which has mathematical (group) structure. For this purpose, let a be an element in $\mathfrak{R}^\dagger$, with MP-inverse $a^\dagger$. We define $H(a)$ (as it is defined in [3] for matrices) by

$$H(a) = \left\{x \in \mathfrak{R}^{-1} : \left[x, aa^\dagger\right] = 0,\ \left[x, a^\dagger a\right] = 0\right\}. \tag{25}$$

In the next proposition we collect some interesting properties of $H(a)$.

Proposition 18. *Let a be an element in $\mathfrak{R}^\dagger$ with MP-inverse $a^\dagger$. Then*

(i) *if $b \in H(a)$, then $b^* \in H(a)$;*

(ii) *$H(a) \subset \mathfrak{C}(a)$;*

(iii) *$H(a)$ is a group;*

(iv) *$a^\dagger$ is covariant under $H(a)$;*

(v) *if $b, c \in H(a)$ such that $b + c \in \mathfrak{R}^{-1}$, then $b + c \in H(a)$;*

(vi) *if $b \in H(a)$, then $P(b) \in H(a)$, where $P(b)$ is a polynomial in b;*

(vii) *if $b \in \mathfrak{C}(a)$ and $c \in H(a)$, then $bc \in \mathfrak{C}(a)$.*

Proof. (i) Assume that $b \in H(a)$. Then $[b, aa^\dagger] = 0$ and so $baa^\dagger = aa^\dagger b$. By taking the adjoint it follows that $aa^\dagger b^* = b^* aa^\dagger$. Thus $[b^*, aa^\dagger] = 0$. In a similar manner, from $[b, a^\dagger a] = 0$, we obtain $[b^*, a^\dagger a] = 0$. Therefore $b^* \in H(a)$.

(ii) Let $b \in H(a)$ by part (i) and definition of $H(a)$; we have

$$\begin{aligned} \left[b, aa^\dagger\right] &= 0, & \left[b^*, aa^\dagger\right] &= 0, \\ \left[b, a^\dagger a\right] &= 0, & \left[b^*, a^\dagger a\right] &= 0. \end{aligned} \tag{26}$$

From (8) and (26) we conclude that

$$\left[b^* b, a^\dagger a\right] = 0, \qquad \left[b^* b, aa^\dagger\right] = 0. \tag{27}$$

Therefore $b \in \mathfrak{C}(a)$.

(iii) Suppose that $b, c \in H(a)$. Then

$$\begin{aligned} \left[b, aa^\dagger\right] &= 0, & \left[b, a^\dagger a\right] &= 0, \\ \left[c, aa^\dagger\right] &= 0, & \left[c, a^\dagger a\right] &= 0. \end{aligned} \tag{28}$$

From (8) and (28) we get

$$\left[bc, aa^\dagger\right] = 0, \qquad \left[bc, a^\dagger a\right] = 0. \tag{29}$$

This means that $bc \in H(a)$. If $b \in H(a)$. Then $[b, aa^\dagger] = 0$ and so $baa^\dagger = aa^\dagger b$. Multiply this from left and right to b^{-1}; we obtain $[b^{-1}, aa^\dagger] = 0$. Similarly we have $[b^{-1}, a^\dagger a] = 0$. This means that $b^{-1} \in H(a)$. Therefore, $H(a)$ is subgroup of $\mathfrak{R}^{-1}$.

(iv) It is easy to check that if $a \in \mathfrak{R}^\dagger$, then for every $b \in H(a)$, we have

$$\left(bab^{-1}\right)^\dagger = ba^\dagger b^{-1}. \tag{30}$$

(v) If $b, c \in H(a)$, by linearity of the commutator we get $[b + c, aa^\dagger] = 0$ and $[b + c, a^\dagger a] = 0$. That is, $b + c \in H(a)$.

(vi) It follows from (ii) and (iv).

(vii) Using (8) and part (i), we see that $[(bc)^* bc, aa^\dagger] = 0$ and $[(bc)^* bc, a^\dagger a] = 0$; that is, $bc \in \mathfrak{C}(a)$. □

Let $\mathfrak{R}$ be the set of all $n \times n$ matrices. It was shown that in [3] $H(a)$ is a nonabelian subgroup of $\mathfrak{R}^{-1}$ if and only if $n > 2$.

Proposition 19. *Assume that a is an element in $\mathfrak{R}^{\dagger}$ with MP-inverse $a^{\dagger}$. If $b \in \mathfrak{C}(a)$ is normal, then $\langle b \rangle \subset \mathfrak{C}(a)$ where $\langle b \rangle$ is the cyclic group generated by b.*

Proof. Using Proposition 2, Corollary 4, and induction, we can show that for all integer n, $b^n \in \mathfrak{C}(a)$. □

Note that, in fact if $b \in \mathfrak{C}(a)$ is normal, then $P(b) \in \mathfrak{C}(a)$, *where $P(b)$ is a polynomial in b.*

3. Covariance Set in C^*-Algebras

Given unital C^*-algebras $\mathfrak{A}$ with the nonzero element $1_{\mathfrak{A}}$. We will denote by $\mathfrak{A}^{-1}$ and $\mathfrak{A}^{\dagger}$ the subset of invertible elements and MP-invertible elements of $\mathfrak{A}$, respectively.

In this section, we find some topological properties for $\mathfrak{C}(a)$; for instance, we will show that $\mathfrak{C}(a)$ is a closed set in $\mathfrak{A}^{-1}$ with respect to the relative topology.

Theorem 20. *Suppose that $\mathfrak{A}$ is a C^*-algebra and $a \in \mathfrak{A}^{\dagger}$. Then $\mathfrak{C}(a)$ is closed in $\mathfrak{A}^{-1}$ with respect to the relative topology.*

Proof. Suppose that b belongs to the closure of $\mathfrak{C}(a)$ in $\mathfrak{A}^{-1}$. Then there exists a sequence $b_n \in \mathfrak{C}(a)$ such that $b_n \to b$, from which it follows that $b_n^* \to b^*$. Thus

$$\left[b_n^* b_n, aa^{\dagger}\right] = 0, \quad \left[b_n^* b_n, a^{\dagger} a\right] = 0 \quad \forall n \in \mathbb{N} \tag{31}$$

by Proposition 2. Therefore

$$b_n^* b_n aa^{\dagger} = aa^{\dagger} b_n^* b_n, \quad b_n^* b_n a^{\dagger} a = a^{\dagger} a b_n^* b_n \quad \forall n \in \mathbb{N}. \tag{32}$$

By taking limits in (32) as $n \to \infty$, we get

$$b^* b aa^{\dagger} = aa^{\dagger} b^* b, \qquad b^* b a^{\dagger} a = a^{\dagger} a b^* b. \tag{33}$$

Since b and b^* are in $\mathfrak{A}^{-1}$, again Proposition 2 implies that $b \in \mathfrak{C}(a)$. This means that $\mathfrak{C}(a)$ is closed in $\mathfrak{A}^{-1}$ with respect to the relative topology. □

Note that generally $\mathfrak{C}(a)$ is not a closed set in $\mathfrak{A}$. For example, if we set $a = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $b_n = \left[\begin{smallmatrix} 1/n & 0 \\ 0 & 1/n \end{smallmatrix}\right]$, then $b_n \in \mathfrak{C}(a)$ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} b_n = 0 \notin \mathfrak{C}(a)$.

We will now reproduce an important theorem of [7] that will be crucial to prove the next result.

Theorem 21 ([see [7]). *Let a_n, a be nonzero elements of $\mathfrak{A}$ such that $a_n \to a$ in $\mathfrak{A}$. Then the following conditions are equivalent:*

(i) $a_n^{\dagger} \to a^{\dagger}$;

(ii) $a_n^{\dagger} a_n \to a^{\dagger} a$;

(iii) $a_n a_n^{\dagger} \to aa^{\dagger}$;

(iv) $\sup_n \|a_n^{\dagger}\| < \infty$.

The next theorem shows that the covariance set, seen as a multivalued map, has some kind of continuity.

Theorem 22. *Let $\{a_n\}$ be a sequence of MP-invertible elements in the C^*-algebra $\mathfrak{A}$ such that $a_n \to a$ and the norms $\|a_n^{\dagger}\|$ are bounded. If $b_n \in \mathfrak{C}(a_n)$ and $b_n \to b \in \mathfrak{R}^{-1}$ as $n \to \infty$, then $b \in \mathfrak{C}(a)$.*

Proof. By hypothesis, a_n's are MP-invertible, $a_n \to a$, and $\|a_n^{\dagger}\| < \infty$. By Theorem 21, a is MP-invertible and $a_n^{\dagger} \to a^{\dagger}$. Thus

$$a_n^{\dagger} a_n \longrightarrow a^{\dagger} a, \qquad a_n a_n^{\dagger} \longrightarrow aa^{\dagger}. \tag{34}$$

Therefore by Proposition 2

$$\begin{gathered} b_n \in \mathfrak{C}(a_n) \Longleftrightarrow b_n b_n^* a_n^{\dagger} a_n = a_n^{\dagger} a_n b_n b_n^*, \\ b_n b_n^* a_n a_n^{\dagger} = a_n a_n^{\dagger} b_n b_n^*. \end{gathered} \tag{35}$$

Now, letting $n \to \infty$ in (35) we get

$$bb^* a^{\dagger} a = a^{\dagger} a bb^*, \qquad bb^* aa^{\dagger} = aa^{\dagger} bb^*. \tag{36}$$

Again by applying Proposition 2 we conclude that $b \in \mathfrak{C}(a)$. □

We recall that a set $K \subset \mathfrak{A}$ is called a *cone* $\lambda x \in K$ whenever $x \in K$ and $\lambda > 0$.

Proposition 23. *Suppose that a is a regular element in $\mathfrak{A}$ and λ is any nonzero scalar. Then $b \in \mathfrak{C}(a)$ if and only if $\lambda b \in \mathfrak{C}(a)$.*

Proof. Assume that $b \in \mathfrak{C}(a)$. Then by Proposition 2,

$$\left[b^* b, aa^{\dagger}\right] = 0, \qquad \left[b^* b, a^{\dagger} a\right] = 0. \tag{37}$$

This is true if and only if

$$|\lambda|^2 \left[b^* b, aa^{\dagger}\right] = 0, \qquad |\lambda|^2 \left[b^* b, a^{\dagger} a\right] = 0, \tag{38}$$

which is equivalent to

$$\left[(\lambda b)^* (\lambda b), aa^{\dagger}\right] = 0, \qquad \left[(\lambda b)^* (\lambda b), a^{\dagger} a\right] = 0. \tag{39}$$

Again by Proposition 2, these hold if and only if $\lambda b \in \mathfrak{C}(a)$. □

Corollary 24. *If a is regular in $\mathfrak{A}$, then $\mathfrak{C}(a)$ is a cone.*

Proof. The proof is an immediate consequence of the above proposition. □

Proposition 25. *Suppose that a is a regular element in $\mathfrak{A}$ and λ is any nonzero scalar. Then $\mathfrak{C}(a) = \mathfrak{C}(\lambda a)$.*

Proof. By assumption $\lambda \neq 0$, thus $(\lambda a)^{\dagger} = (1/\lambda) a^{\dagger}$ and so

$$(\lambda a)^{\dagger} (\lambda a) = a^{\dagger} a, \qquad (\lambda a)(\lambda a)^{\dagger} = aa^{\dagger}. \tag{40}$$

By applying Proposition 5 we get

$$\begin{aligned} \mathfrak{C}(a) &= \mathfrak{C}\left(aa^{\dagger}\right) \cap \mathfrak{C}\left(a^{\dagger} a\right) \\ &= \mathfrak{C}\left((\lambda a)(\lambda a)^{\dagger}\right) \cap \mathfrak{C}\left((\lambda a)^{\dagger}(\lambda a)\right) = \mathfrak{C}(\lambda a). \end{aligned} \tag{41}$$

□

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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