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Research Article

A Linear Functional Equation of Third Order Associated with the Fibonacci Numbers

Soon-Mo Jung[1] and Michael Th. Rassias[2]

[1] *Mathematics Section, College of Science and Technology, Hongik University, Sejong 339-701, Republic of Korea*
[2] *Department of Mathematics, ETH-Zürich, Ramistraße 101, 8092 Zürich, Switzerland*

Correspondence should be addressed to Soon-Mo Jung; smjung@hongik.ac.kr

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Given a vector space X, we investigate the solutions $f : \mathbb{R} \to X$ of the linear functional equation of third order $f(x) = pf(x-1) + qf(x-2) + rf(x-3)$, which is strongly associated with a well-known identity for the Fibonacci numbers. Moreover, we prove the Hyers-Ulam stability of that equation.

1. Introduction

The problem of stability of functional equations was motivated by a question of Ulam [1] and a solution to it by Hyers [2]. Since then, numerous papers have been published on that subject and we refer to [3–6] for more details, some discussions, and further references; for examples of very recent results, see, for example, [7].

In this paper, as usual, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ stand for the sets of complex numbers, real numbers, integers, and positive integers, respectively. For a nonempty subset S of a vector space, let $\xi : S \to S$ be a function. Moreover, $\xi^0(x) = x$, $\xi^{n+1}(x) = \xi(\xi^n(x))$, and (only for bijective ξ) $\xi^{-n-1}(x) = \xi^{-1}(\xi^{-n}(x))$ for $x \in S$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Jung has proved in [3] (see also [8]) some results on solutions and Hyers-Ulam stability of the functional equation

$$f(x) = pf(\xi(x)) - qf(\xi^2(x)), \tag{1}$$

in the case where $S = \mathbb{R}$ and $\xi(x) = x - 1$ for $x \in \mathbb{R}$.

If $S := \mathbb{N}_0$ and $p, q \in \mathbb{Z}$, then solutions $x : \mathbb{N}_0 \to \mathbb{Z}$ of the difference equation $f(x) = pf(x-1) - qf(x-2)$ are called the Lucas sequences (see, e.g., [9]). In some special cases they are called with specific names, for example, the Fibonacci numbers ($p = 1$, $q = -1$, $x(0) = 0$, and $x(1) = 1$), the Lucas numbers ($p = 1$, $q = -1$, $x(0) = 2$, and $x(1) = 1$), the Pell numbers ($p = 2$, $q = -1$, $x(0) = 0$, and $x(1) = 1$), the Pell-Lucas (or companion Lucas) numbers ($p = 2$, $q = -1$, $x(0) = 2$, and $x(1) = 2$), and the Jacobsthal numbers ($p = 1$, $q = -2$, $x(0) = 0$, and $x(1) = 1$).

For some information and further references concerning the functional equations in a single variable, we refer to [10–12]. Let us mention yet that the problem of Hyers-Ulam stability of functional equations is connected to the notions of controlled chaos and shadowing (see [13]).

We remark that if $\xi : S \to S$ is bijective, then (1) can be written in the following equivalent form:

$$f(\eta^2(x)) = pf(\eta(x)) - qf(x), \tag{2}$$

where $\eta := \xi^{-1}$.

In view of the last remark, the following Hyers-Ulam stability result concerning (1) can be derived from [14, Theorem 2] (see also [15]).

Theorem 1. *Let $p, q \in \mathbb{R}$ be given with $q \neq 0$ and let S be a nonempty subset of a vector space. Assume that a_1, a_2 are the complex roots of the quadratic equation $x^2 - px + q = 0$ with $|a_i| \neq 1$ for $i \in \{1, 2\}$. Moreover, assume that X is either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$.*

Let $\xi : S \to S$ be bijective. If a function $f : S \to X$ satisfies the inequality

$$\left\| f(x) - pf(\xi(x)) + qf\left(\xi^2(x)\right) \right\| \le \varepsilon \quad (3)$$

for all $x \in S$ and for some $\varepsilon \ge 0$, then there exists a unique solution $F : S \to X$ of (1) *with*

$$\| f(x) - F(x) \| \le \frac{\varepsilon}{\left| \left(|a_1| - 1 \right) \left(|a_2| - 1 \right) \right|} \quad (4)$$

for all $x \in S$.

In [16, Theorem 1.4], the method presented in [3] was modified so as to prove a theorem which is a complement of Theorem 1. Note that, for bijective ξ, the following theorem improves the estimation (4) in some cases (e.g., $a_1 = 3/2$, $a_2 = -3/2$, or $a_1 = 1/2$, $a_2 = -1/2$). However, in some other situations (e.g., $a_1 = 3$, $a_2 = -3$), the estimation (4) is better than (5). The following theorem also complements Theorem 1, because ξ can be quite arbitrary in the case of (α).

Theorem 2. *Given $p, q \in \mathbb{R}$ with $q \neq 0$, assume that the distinct complex roots a_1, a_2 of the quadratic equation $x^2 - px + q = 0$ satisfy one of the following two conditions:*

(α) $|a_i| < 1$ for $i \in \{1, 2\}$;

(β) $|a_i| \neq 1$ for $i \in \{1, 2\}$ and $\xi : S \to S$ is bijective.

Moreover, assume that X is either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$. If a function $f : S \to X$ satisfies inequality (3)*, then there exists a solution $F : S \to X$ of* (1) *such that*

$$\| f(x) - F(x) \| \le \frac{\varepsilon}{|a_1 - a_2|} \left(\frac{|a_1|}{\left| |a_1| - 1 \right|} + \frac{|a_2|}{\left| |a_2| - 1 \right|} \right) \quad (5)$$

for all $x \in S$. Moreover, if the condition (β) is true, then the F is the unique solution of (1) *satisfying* (5).

In this paper, we investigate the solutions of the functional equation

$$f(x) = pf(x-1) + qf(x-2) + rf(x-3), \quad (6)$$

where p, q, r are real constants. Moreover, we also prove the Hyers-Ulam stability of that equation. Equation (6) is a kind of linear functional equations of third order because it is of the form

$$f(x) = a_1(x) f(\xi(x)) + a_2(x) f\left(\xi^2(x)\right) + a_3(x) f\left(\xi^3(x)\right) \quad (7)$$

for the case of $a_1(x) = p$, $a_2(x) = q$, $a_3(x) = r$, and $\xi(x) = x - 1$.

2. General Solution

In the following theorem, we apply [16, Theorem 1.1] for the investigation of general solutions of the functional equation (6).

Theorem 3. *Let p, q, r be real constants such that the cubic equation*

$$x^3 + px^2 - qx + r = 0 \quad (8)$$

has the following properties:

(i) *α_1 and α_2 are two distinct nonzero real roots of the cubic equation* (8)*;*

(ii) *it holds true that either $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for $i \in \{1, 2\}$ or $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for $i \in \{1, 2\}$.*

Let X be either a real vector space if $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for $i \in \{1, 2\}$ or a complex vector space if $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for $i \in \{1, 2\}$. Then, a function $f : \mathbb{R} \to X$ is a solution of the functional equation (6) *if and only if there exist functions $h_1, h_2 : [-1, 1) \to X$ such that*

$$\begin{aligned} f(x) = {} & \frac{\alpha_1}{\alpha_1 - \alpha_2} V_{[x]+1} h_2(x - [x]) \\ & + \frac{\alpha_1 r}{\alpha_2 (\alpha_1 - \alpha_2)} V_{[x]} h_2(x - [x] - 1) \\ & - \frac{\alpha_2}{\alpha_1 - \alpha_2} U_{[x]+1} h_1(x - [x]) \\ & - \frac{\alpha_2 r}{\alpha_1 (\alpha_1 - \alpha_2)} U_{[x]} h_1(x - [x] - 1), \end{aligned} \quad (9)$$

where $[x]$ denotes the largest integer not exceeding x, and U_n, V_n are defined in (13) *and* (23).

Proof. Assume that $f : \mathbb{R} \to X$ is a solution of (6). If we define an auxiliary function $g_1 : \mathbb{R} \to X$ by

$$g_1(x) := f(x) + \alpha_1 f(x-1), \quad (10)$$

then it follows from (6) that g_1 satisfies

$$g_1(x) = (\alpha_1 + p) g_1(x-1) + \frac{r}{\alpha_1} g_1(x-2) \quad (11)$$

for any $x \in \mathbb{R}$. According to [16, Theorem 1.1] or [3, Theorem 2.1], there exists a function $h_1 : [-1, 1) \to X$ such that

$$\begin{aligned} g_1(x) &= f(x) + \alpha_1 f(x-1) \\ &= U_{[x]+1} h_1(x - [x]) + \frac{r}{\alpha_1} U_{[x]} h_1(x - [x] - 1) \end{aligned} \quad (12)$$

for all $x \in \mathbb{R}$, where

$$U_n = \frac{a^n - b^n}{a - b} \quad (n \in \mathbb{Z}) \quad (13)$$

and a, b are the distinct roots of the quadratic equation

$$x^2 - (\alpha_1 + p) x - \frac{r}{\alpha_1} = 0, \quad (14)$$

that is,

$$\begin{aligned} a &= \frac{\alpha_1 + p}{2} + \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}, \\ b &= \frac{\alpha_1 + p}{2} - \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}. \end{aligned} \quad (15)$$

Since a is a root of the quadratic equation (14), we have

$$a^2 = (\alpha_1 + p)\, a + \frac{r}{\alpha_1}. \tag{16}$$

We multiply both sides of (16) with a and make use of (16) and (i) to get

$$\begin{aligned} a^3 &= pa^2 + \alpha_1 a^2 + \frac{r}{\alpha_1} a \\ &= pa^2 + \alpha_1 \left((\alpha_1 + p)\, a + \frac{r}{\alpha_1} \right) + \frac{r}{\alpha_1} a \\ &= pa^2 + \frac{a}{\alpha_1} \left(\alpha_1^3 + p\alpha_1^2 + r \right) + r \\ &= pa^2 + qa + r. \end{aligned} \tag{17}$$

Similarly, we also obtain

$$b^3 = pb^2 + qb + r. \tag{18}$$

Using (13), (17), and (18), we have

$$\begin{aligned} & pU_{n-1} + qU_{n-2} + rU_{n-3} \\ &\quad = \frac{\left(pa^2 + qa + r\right) a^{n-3} - \left(pb^2 + qb + r\right) b^{n-3}}{a - b} \\ &\quad = \frac{a^n - b^n}{a - b} = U_n \end{aligned} \tag{19}$$

for all $n \in \mathbb{Z}$.

If we define an auxiliary function $g_2 : \mathbb{R} \to X$ by

$$g_2(x) := f(x) + \alpha_2 f(x-1), \tag{20}$$

then it follows from (6) that g_2 satisfies

$$g_2(x) = (\alpha_2 + p)\, g_2(x-1) + \frac{r}{\alpha_2} g_2(x-2) \tag{21}$$

for any $x \in \mathbb{R}$. According to [16, Theorem 1.1] or [3, Theorem 2.1], there exists a function $h_2 : [-1, 1) \to X$ such that

$$\begin{aligned} g_2(x) &= f(x) + \alpha_2 f(x-1) \\ &= V_{[x]+1} h_2(x - [x]) + \frac{r}{\alpha_2} V_{[x]} h_2(x - [x] - 1) \end{aligned} \tag{22}$$

for all $x \in \mathbb{R}$, where

$$V_n = \frac{c^n - d^n}{c - d} \quad (n \in \mathbb{Z}) \tag{23}$$

and c, d are the distinct roots of the quadratic equation

$$x^2 - (\alpha_2 + p)\, x - \frac{r}{\alpha_2} = 0, \tag{24}$$

that is,

$$\begin{aligned} c &= \frac{\alpha_2 + p}{2} + \sqrt{\left(\frac{\alpha_2 + p}{2}\right)^2 + \frac{r}{\alpha_2}}, \\ d &= \frac{\alpha_2 + p}{2} - \sqrt{\left(\frac{\alpha_2 + p}{2}\right)^2 + \frac{r}{\alpha_2}}. \end{aligned} \tag{25}$$

As in the first part, we verify that

$$V_n = pV_{n-1} + qV_{n-2} + rV_{n-3} \tag{26}$$

for all $n \in \mathbb{Z}$.

We now multiply (12) with α_2 and (22) with α_1, we subtract the former from the latter, and we then divide the resulting equation by $(\alpha_1 - \alpha_2)$ to get (9).

We assume that a function $f : \mathbb{R} \to X$ is given by (9), where $h_1, h_2 : [-1, 1) \to X$ are arbitrarily given functions and U_n, V_n are given by (13) and (23), respectively. Then, by (9), (19), and (26), we have

$$\begin{aligned} & pf(x-1) + qf(x-2) + rf(x-3) \\ &\quad = \frac{\alpha_1}{\alpha_1 - \alpha_2} \left(pV_{[x]} + qV_{[x]-1} + rV_{[x]-2} \right) h_2(x - [x]) \\ &\qquad + \frac{\alpha_1 r}{\alpha_2 (\alpha_1 - \alpha_2)} \left(pV_{[x]-1} + qV_{[x]-2} + rV_{[x]-3} \right) \\ &\qquad \times h_2(x - [x] - 1) \\ &\qquad - \frac{\alpha_2}{\alpha_1 - \alpha_2} \left(pU_{[x]} + qU_{[x]-1} + rU_{[x]-2} \right) h_1(x - [x]) \\ &\qquad - \frac{\alpha_2 r}{\alpha_1 (\alpha_1 - \alpha_2)} \left(pU_{[x]-1} + qU_{[x]-2} + rU_{[x]-3} \right) \\ &\qquad \times h_1(x - [x] - 1) \\ &\quad = \frac{\alpha_1}{\alpha_1 - \alpha_2} V_{[x]+1} h_2(x - [x]) \\ &\qquad + \frac{\alpha_1 r}{\alpha_2 (\alpha_1 - \alpha_2)} V_{[x]} h_2(x - [x] - 1) \\ &\qquad - \frac{\alpha_2}{\alpha_1 - \alpha_2} U_{[x]+1} h_1(x - [x]) \\ &\qquad - \frac{\alpha_2 r}{\alpha_1 (\alpha_1 - \alpha_2)} U_{[x]} h_1(x - [x] - 1) = f(x) \end{aligned} \tag{27}$$

for all $x \in \mathbb{R}$, which implies that f is a solution of (6). □

According to [17, p. 92], the Fibonacci numbers F_n satisfy the identity

$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2 \tag{28}$$

for all integers $n > 3$. We can easily notice that the linear equation of third order

$$f(x) = 2f(x-1) + 2f(x-2) - f(x-3) \tag{29}$$

is strongly related to identity (28).

Corollary 4. *Let X be a real vector space. A function $f : \mathbb{R} \to X$ is a solution of the functional equation (29) if and only if there exist functions $h_1, h_2 : [-1, 1) \to X$ such that*

$$\begin{aligned} f(x) = {} & \frac{5+3\sqrt{5}}{10} U_{[x]+1} h_1 (x - [x]) \\ & + \frac{15+7\sqrt{5}}{10} U_{[x]} h_1 (x - [x] - 1) \\ & + \frac{5-3\sqrt{5}}{10} V_{[x]+1} h_2 (x - [x]) \\ & + \frac{15-7\sqrt{5}}{10} V_{[x]} h_2 (x - [x] - 1), \end{aligned} \tag{30}$$

where U_n and V_n are defined in (33).

Proof. If we set $p = 2$, $q = 2$, and $r = -1$ in (8), then the cubic equation

$$x^3 + 2x^2 - 2x - 1 = 0 \tag{31}$$

has three distinct nonzero roots including

$$\alpha_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}, \qquad \alpha_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}. \tag{32}$$

Moreover, it holds that $(\alpha_1 + p)^2 + 4r/\alpha_1 > 0$ and $(\alpha_2 + p)^2 + 4r/\alpha_2 > 0$. By (13), (15), (23), and (25), we have

$$U_n = \frac{a^n - b^n}{a - b}, \qquad V_n = \frac{c^n - d^n}{c - d}, \tag{33}$$

where we make use of (15) and (25) to calculate

$$a = \frac{3+\sqrt{5}}{2}, \qquad b = -1, \qquad c = \frac{3-\sqrt{5}}{2}, \qquad d = -1. \tag{34}$$

Finally, in view of Theorem 3, we conclude that the assertion of our corollary is true. □

Corollary 5. *If a function $f : \mathbb{R} \to \mathbb{R}$ is a solution of functional equation* (29), *then there exist real constants μ_1, μ_2, ν_1, and ν_2 such that*

$$\begin{aligned} f(n) = {} & \frac{5+3\sqrt{5}}{10} \mu_1 U_{n+1} + \frac{15+7\sqrt{5}}{10} \mu_2 U_n \\ & + \frac{5-3\sqrt{5}}{10} \nu_1 V_{n+1} + \frac{15-7\sqrt{5}}{10} \nu_2 V_n \end{aligned} \tag{35}$$

for all $n \in \mathbb{Z}$, where U_n and V_n are defined in (33).

3. Hyers-Ulam Stability

We apply the classical direct method to the proof of the following theorem. The classical direct method was first proposed by Hyers [2].

Theorem 6. *Let p, q, r be real constants with $r \neq 0$, let α be a nonzero root of the cubic equation* (8), *and let a, b be the roots of the quadratic equation $x^2 - (\alpha + p)x - r/\alpha = 0$ with $|a| > 1$ and $0 < |b| < 1$. Let X be either a real Banach space if $(\alpha + p)^2 + 4r/\alpha > 0$ or a complex Banach space if $(\alpha + p)^2 + 4r/\alpha < 0$. If a function $f : \mathbb{R} \to X$ satisfies the inequality*

$$\| f(x) - pf(x-1) - qf(x-2) - rf(x-3) \| \leq \varepsilon \tag{36}$$

for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a solution $G : \mathbb{R} \to X$ of (6) *such that*

$$\| f(x) + \alpha f(x-1) - G(x) \| \leq \frac{|a| - |b|}{|a - b|} \frac{\varepsilon}{(|a| - 1)(1 - |b|)} \tag{37}$$

for all $x \in \mathbb{R}$.

Proof. If we define an auxiliary function $g : \mathbb{R} \to X$ by

$$g(x) := f(x) + \alpha f(x-1), \tag{38}$$

then, as we did in (11), it follows from (36) that g satisfies the inequality

$$\left\| g(x) - (\alpha + p) g(x-1) - \frac{r}{\alpha} g(x-2) \right\| \leq \varepsilon \tag{39}$$

or

$$\| g(x) - ag(x-1) - b[g(x-1) - ag(x-2)] \| \leq \varepsilon \tag{40}$$

for any $x \in \mathbb{R}$.

If we replace x with $x - k$ in the last inequality, then we have

$$\begin{aligned} & \| g(x-k) - ag(x-k-1) \\ & \quad - b[g(x-k-1) - ag(x-k-2)] \| \leq \varepsilon \end{aligned} \tag{41}$$

for all $x \in \mathbb{R}$. Furthermore, we get

$$\begin{aligned} & \Big\| b^k [g(x-k) - ag(x-k-1)] \\ & \quad - b^{k+1} [g(x-k-1) - ag(x-k-2)] \Big\| \leq |b|^k \varepsilon \end{aligned} \tag{42}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (42), we obviously have

$$\begin{aligned} & \| g(x) - ag(x-1) - b^n [g(x-n) - ag(x-n-1)] \| \\ & \leq \sum_{k=0}^{n-1} \Big\| b^k [g(x-k) - ag(x-k-1)] \\ & \qquad - b^{k+1} [g(x-k-1) - ag(x-k-2)] \Big\| \\ & \leq \sum_{k=0}^{n-1} |b|^k \varepsilon \end{aligned} \tag{43}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (42) implies that the sequence $\{b^n[g(x-n) - ag(x-n-1)]\}$ is a Cauchy sequence (note that $0 < |b| < 1$). Therefore, we can define a function $G_1 : \mathbb{R} \to X$ by

$$G_1(x) := \lim_{n \to \infty} b^n [g(x-n) - ag(x-n-1)], \tag{44}$$

since X is complete. In view of the definition of G_1 and using the relations, $a+b=\alpha+p$ and $ab=-r/\alpha$, we obtain

$$\begin{aligned}
&(\alpha+p)G_1(x-1)+\frac{r}{\alpha}G_1(x-2)\\
&\quad=(a+b)G_1(x-1)-abG_1(x-2)\\
&\quad=\frac{a+b}{b}\lim_{n\to\infty}b^{n+1}\left[g(x-(n+1))-ag(x-(n+1)-1)\right]\\
&\qquad-\frac{ab}{b^2}\lim_{n\to\infty}b^{n+2}\left[g(x-(n+2))-ag(x-(n+2)-1)\right]\\
&\quad=\frac{a+b}{b}G_1(x)-\frac{a}{b}G_1(x)=G_1(x)
\end{aligned}\tag{45}$$

for all $x\in\mathbb{R}$. Since α is a nonzero root of the cubic equation (8), it follows from (45) that

$$\begin{aligned}
&G_1(x)-pG_1(x-1)-qG_1(x-2)-rG_1(x-3)\\
&\quad=(\alpha+p)G_1(x-1)+\frac{r}{\alpha}G_1(x-2)-pG_1(x-1)\\
&\qquad-qG_1(x-2)-rG_1(x-3)\\
&\quad=\alpha G_1(x-1)+\left(-q+\frac{r}{\alpha}\right)G_1(x-2)-rG_1(x-3)\\
&\quad=\alpha G_1(x-1)+\left(-\alpha^2-p\alpha\right)G_1(x-2)-rG_1(x-3)\\
&\quad=\alpha\left((\alpha+p)G_1(x-2)+\frac{r}{\alpha}G_1(x-3)\right)\\
&\qquad-\alpha(\alpha+p)G_1(x-2)-rG_1(x-3)=0
\end{aligned}\tag{46}$$

for all $x\in\mathbb{R}$. Hence, we conclude that G_1 is a solution of (6).

If n tends to infinity, then (43) yields that

$$\left\|g(x)-ag(x-1)-G_1(x)\right\|\le\frac{\varepsilon}{1-|b|}\tag{47}$$

for every $x\in\mathbb{R}$.

On the other hand, it also follows from (36) that

$$\left\|g(x)-bg(x-1)-a\left[g(x-1)-bg(x-2)\right]\right\|\le\varepsilon\tag{48}$$

for all $x\in\mathbb{R}$. Analogously to (42), replacing x by $x+k$ in the last inequality and then dividing by $|a|^k$ both sides of the resulting inequality, then we have

$$\begin{aligned}
&\left\|a^{-k}\left[g(x+k)-bg(x+k-1)\right]\right.\\
&\quad\left.-a^{-k+1}\left[g(x+k-1)-bg(x+k-2)\right]\right\|\le|a|^{-k}\varepsilon
\end{aligned}\tag{49}$$

for all $x\in\mathbb{R}$ and $k\in\mathbb{Z}$. By using (49), we further obtain

$$\begin{aligned}
&\left\|a^{-n}\left[g(x+n)-bg(x+n-1)\right]-\left[g(x)-bg(x-1)\right]\right\|\\
&\quad\le\sum_{k=1}^{n}\left\|a^{-k}\left[g(x+k)-bg(x+k-1)\right]\right.\\
&\qquad\quad\left.-a^{-k+1}\left[g(x+k-1)-bg(x+k-2)\right]\right\|\\
&\quad\le\sum_{k=1}^{n}|a|^{-k}\varepsilon
\end{aligned}\tag{50}$$

for $x\in\mathbb{R}$ and $n\in\mathbb{N}$.

On account of (49), we see that the sequence $\{a^{-n}[g(x+n)-bg(x+n-1)]\}$ is a Cauchy sequence for any fixed $x\in\mathbb{R}$ (note that $|a|>1$). Hence, we can define a function $G_2:\mathbb{R}\to X$ by

$$G_2(x):=\lim_{n\to\infty}a^{-n}\left[g(x+n)-bg(x+n-1)\right].\tag{51}$$

Due to the definition of G_2 and the relations, $a+b=\alpha+p$ and $ab=-r/\alpha$, we get

$$\begin{aligned}
&(\alpha+p)G_2(x-1)+\frac{r}{\alpha}G_2(x-2)\\
&\quad=(a+b)G_2(x-1)-abG_2(x-2)\\
&\quad=\frac{a+b}{a}\lim_{n\to\infty}a^{-(n-1)}\left[g(x+n-1)-bg(x+n-2)\right]\\
&\qquad-\frac{ab}{a^2}\lim_{n\to\infty}a^{-(n-2)}\left[g(x+n-2)-bg(x+n-3)\right]\\
&\quad=\frac{a+b}{a}G_2(x)-\frac{b}{a}G_2(x)=G_2(x)
\end{aligned}\tag{52}$$

for any $x\in\mathbb{R}$. Similarly as in the first part, we can show that G_2 is a solution of (6).

If we let n tend to infinity, then it follows from (50) that

$$\left\|G_2(x)-g(x)+bg(x-1)\right\|\le\frac{\varepsilon}{|a|-1}\tag{53}$$

for $x\in\mathbb{R}$.

It follows from (47) and (53) that

$$\begin{aligned}
&\left\|g(x-1)-\frac{1}{a-b}G_2(x)+\frac{1}{a-b}G_1(x)\right\|\\
&\quad\le\left\|\frac{1}{a-b}G_1(x)-\frac{1}{a-b}g(x)+\frac{a}{a-b}g(x-1)\right\|\\
&\qquad+\left\|\frac{1}{a-b}g(x)-\frac{b}{a-b}g(x-1)-\frac{1}{a-b}G_2(x)\right\|\\
&\quad\le\frac{|a|-|b|}{|a-b|}\frac{\varepsilon}{(|a|-1)(1-|b|)}
\end{aligned}\tag{54}$$

for any $x\in\mathbb{R}$.

Finally, if we define a function $G:\mathbb{R}\to X$ by

$$G(x):=\frac{1}{a-b}G_2(x+1)-\frac{1}{a-b}G_1(x+1)\tag{55}$$

for all $x \in \mathbb{R}$, then G is also a solution of (6). Moreover, the validity of (37) follows from the last inequality. □

The following theorem is the main theorem of this paper.

Theorem 7. *Given real constants p, q, r with $r \neq 0$, let α_1 and α_2 be distinct nonzero roots of cubic equation* (8) *and let a_i, b_i be the roots of the quadratic equation $x^2 - (\alpha_i + p)x - r/\alpha_i = 0$ with $|a_i| > 1$ and $0 < |b_i| < 1$ for $i \in \{1, 2\}$. Assume that either $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for all $i \in \{1, 2\}$ or $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for all $i \in \{1, 2\}$. Let X be either a real Banach space if $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ or a complex Banach space if $(\alpha_i + p)^2 + 4r/\alpha_i < 0$. If a function $f : \mathbb{R} \to X$ satisfies inequality* (36) *for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a solution $F : \mathbb{R} \to X$ of* (6) *such that*

$$\|f(x) - F(x)\| \leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_2|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_1| - 1)(1 - |b_1|)} + \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_2| - 1)(1 - |b_2|)} \quad (56)$$

for all $x \in \mathbb{R}$.

Proof. According to Theorem 6, there exists a solution $F_i : \mathbb{R} \to X$ of (6) such that

$$\|f(x) + \alpha_i f(x-1) - F_i(x)\| \leq \frac{|a_i| - |b_i|}{|a_i - b_i|} \frac{\varepsilon}{(|a_i| - 1)(1 - |b_i|)} \quad (57)$$

for any $x \in \mathbb{R}$ and $i \in \{1, 2\}$. In view of the last inequalities, we have

$$\begin{aligned} &\left\| f(x) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) + \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \right\| \\ &\leq \left\| \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} f(x) - \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x-1) \right\| \\ &\quad + \left\| \frac{\alpha_1}{\alpha_1 - \alpha_2} f(x) + \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x-1) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) \right\| \\ &\leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_2|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_1| - 1)(1 - |b_1|)} \\ &\quad + \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_2| - 1)(1 - |b_2|)} \end{aligned} \quad (58)$$

for all $x \in \mathbb{R}$.

If we define a function $F : \mathbb{R} \to X$ by

$$F(x) := \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \quad (59)$$

for each $x \in \mathbb{R}$, then F is also a solution of (6), and inequality (56) follows from the last inequality. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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