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Research Article

Boundedness for Parametrized Littlewood-Paley Operators with Rough Kernels on Weighted Weak Hardy Spaces

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The authors prove that the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ are bounded from the weighted weak Hardy space $WH_w^1(\mathbb{R}^n)$ to the weighted weak Lebesgue space $WL_w^1(\mathbb{R}^n)$ as Ω satisfies a class of the integral Dini condition, respectively.

1. Introduction and Main Results

Suppose that $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$ and satisfies

$$\int_{S^{n-1}} \Omega\left(x'\right) \mathrm{d}\sigma\left(x'\right) = 0, \tag{1}$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(x')$, $x' = x/|x|$, $x \neq 0$. The parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ are defined by

$$\begin{aligned} &\mu_{\Omega,S}^{\rho}(f)(x) \\ &\quad = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\,\mathrm{d}z \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \right)^{1/2}, \\ &\mu_{\lambda}^{*,\rho}(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \right. \\ &\qquad \left. \times \left| \frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\,\mathrm{d}z \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \right)^{1/2}, \end{aligned} \tag{2}$$

respectively, where

$$\begin{gathered} \Gamma(x) = \left\{ (y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t \right\}, \\ \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty), \quad \rho > 0, \quad \lambda > 1. \end{gathered} \tag{3}$$

It is well known that Littlewood-Paley functions are very important tools in harmonic analysis and PDE (see [1–3]). Some well-known results related to the classical Littlewood-Paley operators can be seen in [4–8]. In 1999, inspired by Hörmander's work [9], when Ω satisfies the Lipschitz condition of α, Sakamoto and Yabuta [10] established the L^p $(1 < p < \infty)$ boundedness of the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and the parametrized g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ and gave the boundedness on *BMO* spaces and Campanato spaces. For any $0 < \alpha \leqslant 1$, $1 < q \leqslant \infty$, it is easy to see that the inclusion relationship

$$\begin{gathered} \mathrm{Lip}_{\alpha}\left(S^{n-1}\right) \subsetneq L^q\left(S^{n-1}\right) \subsetneq L\log^+ L\left(S^{n-1}\right) \\ \subsetneq H^1\left(S^{n-1}\right) \subsetneq L^1\left(S^{n-1}\right) \end{gathered} \tag{4}$$

holds. In 2002, Ding et al. [11] extended the previous L^p-boundedness to the case as Ω belongs to $L\log^+ L(S^{n-1})$. In 2007, Ding et al. [12, 13] gave the boundedness of the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ on the Hardy space and weak Hardy space when Ω satisfies a class of the integral Dini conditions. Recently, Wang and Liu [14] obtained the boundedness on the weighted Hardy space for the parametrized Littlewood-Paley operators with Ω satisfying the logarithmic type Lipschitz conditions. On the other hand, the boundedness properties of the intrinsic square functions on weighted weak Hardy spaces were studied by Wang in [15]. Inspired by the results mentioned previously, in this paper, we will study the boundedness of the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ on the weighted weak Hardy spaces.

Before stating our main results, let us recall some definitions. Firstly, let $\Omega(x') \in L^q(S^{n-1})$, $q \geqslant 1$. Then, the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\gamma\|\leqslant\delta}\left(\int_{S^{n-1}}\left|\Omega(\gamma x') - \Omega(x')\right|^q d\sigma(x')\right)^{1/q}, \quad (5)$$

where, γ denotes a rotation on S^{n-1} and $\|\gamma\| = \sup_{x'\in S^{n-1}}|\gamma x' - x'|$. The function Ω is said to satisfy the L^q-Dini condition, if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty. \quad (6)$$

Secondly, given a weight function w on $\mathbb{R}^n$, for $1 \leqslant p < \infty$, the weighted Lebesgue spaces is defined by

$$L_w^p(\mathbb{R}^n) = \left\{f : \|f\|_{p,w} = \left(\int_{\mathbb{R}^n}|f(x)|^p w(x)dx\right)^{1/p} < \infty\right\}. \quad (7)$$

And also, the weighted weak Lebesgue spaces is defined by

$$WL_w^p(\mathbb{R}^n) = \left\{f : \|f\|_{WL_w^p} = \sup_{\lambda>0}\lambda \cdot w\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right)^{1/p} < \infty\right\}. \quad (8)$$

Let us now turn to recall the definition of the weighted weak Hardy spaces. The weak Hardy spaces were first introduced in [16]. The atomic decomposition theory of weak H^1 spaces on $\mathbb{R}^n$ was given by Fefferman and Soria in [17]. Later, Liu established the weak H^p spaces on homogeneous groups in [18]. In 2000, Quek and Yang introduced the weighted weak Hardy spaces $WH_w^p(\mathbb{R}^n)$ in [19] and established their atomic decompositions. Moreover, by using the atomic decomposition theory of $WH_w^p(\mathbb{R}^n)$, Quek and Yang also obtained the boundedness of $C-Z$ operators on these weighted spaces in [19]. Let $w \in A_\infty$, $0 < p \leqslant 1$, and $N = [n(q_w/p - 1)]$. Define

$$\mathscr{A}_{N,w} = \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n}\sup_{|\alpha|\leqslant N+1}(1+|x|)^{N+n+1}\left|D^\alpha\varphi(x)\right| \leqslant 1\right\}, \quad (9)$$

where, $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha\varphi = \partial^{|\alpha|}\varphi/(\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n})$.

For $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of f is defined by

$$G_w f(x) = \sup_{\varphi\in\mathscr{A}_{N,w}}\sup_{|y-x|<t}\left|(\varphi_t * f)(y)\right|. \quad (10)$$

Then, weighted weak Hardy space is defined by $WH_w^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_w f \in WL_w^p(\mathbb{R}^n)\}$. Moreover, we set $\|f\|_{WH_w^p} = \|G_w f\|_{WL_w^p}$.

Our main results are stated as follows.

Theorem 1. *Let $\Omega \in L^2(S^{n-1})$ satisfying (1) and the following condition*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 < \alpha \leqslant 1. \quad (11)$$

Then, for $\rho > n/2$, $w \in A_1$, there exists a constant $C > 0$ such that

$$\left\|\mu_{\Omega,S}^{\rho}(f)\right\|_{WL_w^1} \leqslant C\|f\|_{WH_w^1}. \quad (12)$$

The relationship between condition (11) and $\mathrm{Lip}_\alpha(S^{n-1})$ condition is not clear up to now. We point that the conclusion of Theorem 1 still holds if we replace the condition (11) by the $\mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \leqslant 1)$ condition. In other words, we have the following result.

Theorem 2. *Let $\Omega \in Lip_\alpha(S^{n-1})$, $0 < \alpha \leqslant 1$, satisfying (1). Then, for $\rho > n/2$, $w \in A_1$, there exists a constant $C > 0$ such that*

$$\left\|\mu_{\Omega,S}^{\rho}(f)\right\|_{WL_w^1} \leqslant C\|f\|_{WH_w^1}. \quad (13)$$

Theorem 3. *Let $\Omega \in L^2(S^{n-1})$ satisfying (1) and the following condition*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 < \alpha \leqslant 1. \quad (14)$$

Then, for $\rho > n/2$, $w \in A_1$, $\lambda > 2$, there exists a constant $C > 0$ such that

$$\left\|\mu_{\lambda}^{*,\rho}(f)\right\|_{WL_w^1} \leqslant C\|f\|_{WH_w^1}. \quad (15)$$

2. Notations and Preliminaries

In this section, we will introduce some notations and preliminary lemmas used in the proofs of our main theorems in the next section.

The classical A_p weighted theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [20]. A weight w is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ at almost everywhere. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. We also denote the weighted measure of E by $w(E)$; that is, $w(E) = \int_E w(x)dx$. We say that $w \in A_p$ with $1 < p < \infty$ if there exists a constant $C > 0$, such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1} \leqslant C. \quad (16)$$

We say that $w \in A_1$ if

$$\frac{1}{|B|}\int_B w(x)\,dx \leqslant C \cdot \operatorname{ess}\inf_{x\in B} w(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \tag{17}$$

A weight function $w \in A_\infty$ if it satisfies the A_p condition for some $1 < p < \infty$. It is well known that if $w \in A_p$, $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w.

Lemma 4 (see [21]). *Let $1 \leqslant p < \infty$, $w \in A_p$. Then, for any ball B, there exists an absolute constant $C > 0$, such that*

$$w(2B) \leqslant Cw(B). \tag{18}$$

In general, for any $\lambda > 0$, we have

$$w(\lambda B) \leqslant C\lambda^{np} w(B), \tag{19}$$

where C does not depend on B nor on λ.

Lemma 5 (see [19]). *Let $0 < p \leqslant 1$, $w \in A_\infty$. For every $f(x)$ belongs to $WH^p_w(\mathbb{R}^n)$, there exists a sequence of bounded measurable functions $\{f_k(x)\}_{k=-\infty}^{\infty}$ such that*

(i) *$f(x) = \sum_{k=-\infty}^{\infty} f_k(x)$, in $\mathcal{S}'$,*

(ii) *each f_k can be further decomposed into $f_k = \sum_i b_i^k$, where b_i^k satisfies the following conditions.*

(a) *$\operatorname{supp}(b_i^k) \subset Q_i^k$, where Q_i^k denotes the ball with center x_i^k and radius r_i^k. Moreover,*

$$\sum_i w\left(Q_i^k\right) \leqslant C_1 2^{-kp}, \qquad \sum_i \chi_{Q_i^k} \leqslant C_1, \tag{20}$$

where χ_E denotes the characteristic function of the set E and $C_1 \leqslant \|f\|^p_{WH^p_w}$.

(b) *$\|b_i^k\|_{L^\infty} \leqslant C2^k$, where $C > 0$ is independent of i, k.*

(c) *$\int_{\mathbb{R}^n} b_i^k x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leqslant [n(q_w/p - 1)]$.*

Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ have a decomposition satisfying (i) and (ii), then $f \in WH^p_w(\mathbb{R}^n)$. Moreover, we have $\|f\|^p_{WH^p_w} \sim C$.

In the end of this section, we need the following lemmas used in the next section.

Lemma 6 (see [22]). *Suppose that $\Omega \in L^2(S^{n-1})$ satisfies (1) and the following condition*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}\left(1 + |\log\delta|\right)^\sigma d\delta < \infty, \quad \sigma > 1, \tag{21}$$

$w \in A_p$. Then, for $\rho > n/2$, $\lambda > 2$, and $f \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, there is a constant C independent of f, such that

$$\begin{aligned} \left\|\mu^\rho_{\Omega,S}(f)\right\|_{L^p_w} &\leqslant C\|f\|_{L^p_w}, \\ \left\|\mu^{*,\rho}_\lambda(f)\right\|_{L^p} &\leqslant C\|f\|_{L^p_w}. \end{aligned} \tag{22}$$

Lemma 7 (see [23]). *Suppose that $\rho > 0$, Ω is homogeneous of degree zero and satisfies the L^2-Dini condition. If there exists a constant $0 < \theta < 1/2$ such that $|x| < \theta R$, then we have*

$$\begin{aligned} &\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y)}{|y|^{n-\rho}}\right|^2 dy\right)^{1/2} \\ &\quad \leqslant CR^{n/2-(n-\rho)}\left\{\frac{|x|}{R} + \int_{|x|/2R<\delta<|x|/R}\frac{\omega_2(\delta)}{\delta}d\delta\right\}, \end{aligned} \tag{23}$$

where the constant $C > 0$ is independent of R, x.

3. Proof of Main Results

Proof of Theorem 1. In order to prove Theorem 1, it suffices to show that there exists a constant $C > 0$, for any $f \in WH^1_w(\mathbb{R}^n)$ and $\beta > 0$, such that

$$\beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu^\rho_{\Omega,S}(f)(x)\right| > \beta\right\}\right) \leqslant C\|f\|_{WH^1_w}. \tag{24}$$

Take $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leqslant \beta < 2^{k_0} + 1$; then by Lemma 5 we can write

$$f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k =: F_1 + F_2, \tag{25}$$

where $F_1 = \sum_{k=-\infty}^{k_0}\sum_i b_i^k$, $F_2 = \sum_{k=k_0+1}^{\infty}\sum_i b_i^k$, and $\{b_i^k\}$ satisfies (a)–(c) in Lemma 5. Then, we have

$$\begin{aligned} &\beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu^\rho_{\Omega,S}(f)(x)\right| > \beta\right\}\right) \\ &\quad \leqslant \beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu^\rho_{\Omega,S}(F_1)(x)\right| > \frac{\beta}{2}\right\}\right) \\ &\qquad + \beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu^\rho_{\Omega,S}(F_2)(x)\right| > \frac{\beta}{2}\right\}\right) \\ &\quad =: I_1 + I_2. \end{aligned} \tag{26}$$

First, we claim that the following inequality holds:

$$\|F_1\|_{L^2_w} \leqslant C\beta^{1/2}\|f\|^{1/2}_{WH^1_w}. \tag{27}$$

In fact, since $\operatorname{supp}(b_i^k) \subset Q_i^k = Q(x_i^k, r_i^k)$, $\|b_i^k\|_{L^\infty} \leqslant C2^k$, then it follows from Minkowski's integral inequality that

$$\begin{aligned} \|F_1\|_{L^2_w} &\leqslant C\sum_{k=-\infty}^{k_0}\sum_i \left\|b_i^k\right\|_{L^\infty} w\left(Q_i^k\right)^{1/2} \\ &\leqslant C\sum_{k=-\infty}^{k_0} 2^k\left(\sum_i w\left(Q_i^k\right)\right)^{1/2} \\ &\leqslant C\sum_{k=-\infty}^{k_0} 2^{k/2}\|f\|^{1/2}_{WH^1_w} \\ &\leqslant C\sum_{k=-\infty}^{k_0} 2^{(k-k_0)/2}\beta^{1/2}\|f\|^{1/2}_{WH^1_w} \leqslant C\beta^{1/2}\|f\|^{1/2}_{WH^1_w}. \end{aligned} \tag{28}$$

Let us estimate I_1. By Chebyshev's inequality, Lemma 6 and (27), we have

$$\begin{aligned} I_1 &\leqslant \beta \frac{4}{\beta^2} \int_{\mathbb{R}^n} \left|\mu_{\Omega,S}^{\rho}(F_1)(x)\right|^2 w(x)\,\mathrm{d}x \\ &= C\frac{1}{\beta\left\|\mu_{\Omega,S}^{\rho}(F_1)\right\|_{L_w^2}^2} \\ &\leqslant C\frac{1}{\beta\left\|F_1\right\|_{L_w^2}^2} \\ &\leqslant C\|f\|_{WH_w^1}. \end{aligned} \tag{29}$$

Now we turn our attention to the estimate of I_2. If we set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q}_i^k, \tag{30}$$

where $\widetilde{Q}_i^k = Q(x_i^k, 8\tau^{(k-k_0)/n} r_i^k)$ and τ is a fixed positive number such that $1 < \tau < 2$, therefore,

$$\begin{aligned} I_2 &= \beta w\left(\left\{x \in 2A_{k_0} : \left|\mu_{\Omega,S}^{\rho}(F_2)(x)\right| > \frac{\beta}{2}\right\}\right) \\ &\quad + \beta w\left(\left\{x \in \left(2A_{k_0}\right)^c : \left|\mu_{\Omega,S}^{\rho}(F_2)(x)\right| > \frac{\beta}{2}\right\}\right) \\ &=: I_2' + I_2''. \end{aligned} \tag{31}$$

Since $w \in A_1$, then by Lemma 4 we can get

$$\begin{aligned} I_2' &\leqslant C\beta \sum_{k=k_0+1}^{\infty} \sum_i w\left(\widetilde{Q}_i^k\right) \\ &\leqslant C\beta \sum_{k=k_0+1}^{\infty} \tau^{k-k_0} \sum_i w\left(Q_i^k\right) \\ &\leqslant C\|f\|_{WH_w^1} \sum_{k=k_0+1}^{\infty} \left(\frac{\tau}{2}\right)^{k-k_0} \\ &\leqslant C\|f\|_{WH_w^1}. \end{aligned} \tag{32}$$

An application of Chebyshev's inequality and Minkowski integral inequality gives us that

$$\begin{aligned} I_2'' &\leqslant \beta \frac{2}{\beta} \int_{(2A_{k_0})^c} \mu_{\Omega,S}^{\rho}(F_2)(x)\, w(x)\,\mathrm{d}x \\ &= C\int_{(2A_{k_0})^c} \left(\iint_{|y-x|<t} \left|\sum_{k=k_0+1}^{\infty}\sum_i \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\,\mathrm{d}z\right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}}\right)^{1/2} w(x)\,\mathrm{d}x \\ &\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i \int_{(2A_{k_0})^c} \left(\iint_{|y-x|<t} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\,\mathrm{d}z\right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}}\right)^{1/2} w(x)\,\mathrm{d}x \\ &\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i \left[\int_{(2A_{k_0})^c} \left(\iint_{\substack{y\in 4Q_i^k\\ |y-x|<t}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\,\mathrm{d}z\right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}}\right)^{1/2} w(x)\,\mathrm{d}x \right. \\ &\qquad \left. + \int_{(2A_{k_0})^c} \left(\iint_{\substack{y\in (4Q_i^k)^c\\ |y-x|<t}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\,\mathrm{d}z\right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}}\right)^{1/2} w(x)\,\mathrm{d}x\right] \\ &=: C\sum_{k=k_0+1}^{\infty}\sum_i \left[I_3 + I_4\right]. \end{aligned} \tag{33}$$

Firstly, let us estimate I_3. As $y \in 4Q_i^k$, $x \in (2A_{k_0})^c$, $z \in Q_i^k$, it is easy to see that

$$\begin{gathered} |y-x| \geqslant \left|x - x_i^k\right| - \left|y - x_i^k\right| \geqslant \frac{\left|x - x_i^k\right|}{2}, \\ |y-z| < 8r_i^k, \end{gathered}$$

$$
\begin{aligned}
I_3 &\leqslant C2^k \int_{(2A_{k_0})^c} \left(\iint_{\substack{y\in 4Q_i^k \\ |x-x_i^k|/2\leqslant |y-x|<t}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \mathrm{d}z \right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}} \right)^{1/2} w(x)\,\mathrm{d}x \\
&\leqslant C2^k \int_{(2A_{k_0})^c} \left(\int_{y\in 4Q_i^k} \int_{|x-x_i^k|/2}^{\infty} \left| \int_{S^{n-1}} \int_0^{8r_i^k} \frac{|\Omega(z')|}{s^{n-\rho}} s^{n-1}\,\mathrm{d}s\,\mathrm{d}\sigma(z') \right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}} \right)^{1/2} w(x)\,\mathrm{d}x \\
&\leqslant C2^k \int_{(2A_{k_0})^c} \left(\int_{y\in 4Q_i^k} \int_{|x-x_i^k|/2}^{\infty} (8r_i^k)^{2\rho} \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}} \right)^{1/2} w(x)\,\mathrm{d}x \\
&\leqslant C2^k (8r_i^k)^{\rho} \int_{(2A_{k_0})^c} \left(\int_{y\in 4Q_i^k} \frac{\mathrm{d}y}{|x-x_i^k|^{n+2\rho}} \right)^{1/2} w(x)\,\mathrm{d}x \\
&\leqslant C2^k (8r_i^k)^{\rho} |4Q_i^k|^{1/2} \int_{(2A_{k_0})^c} \frac{w(x)}{|x-x_i^k|^{n/2+\rho}} \mathrm{d}x.
\end{aligned}
\tag{34}
$$

Since $x \in (2A_{k_0})^c$, $\widetilde{Q}_i^k = Q(x_i^k, 8\tau^{(k-k_0)/n} r_i^k)$, then $|x - x_i^k| > 8\tau^{(k-k_0)/n} r_i^k > 8r_i^k$. By Lemma 4, we obtain that

$$
\begin{aligned}
I_3 &\leqslant C2^k (r_i^k)^{\rho+n/2} \\
&\quad \times \sum_{j=3}^{\infty} \int_{2^j \tau^{(k-k_0)/n} r_i^k < |x-x_i^k| < 2^{j+1}\tau^{(k-k_0)/n} r_i^k} \frac{w(x)}{|x-x_i^k|^{n/2+\rho}} \mathrm{d}x \\
&\leqslant C2^k (r_i^k)^{\rho+n/2} \\
&\quad \times \sum_{j=3}^{\infty} \left(2^{j+1}\tau^{(k-k_0)/n} r_i^k\right)^{-n/2-\rho} \\
&\quad \times \int_{|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n} r_i^k} w(x)\,\mathrm{d}x \\
&\leqslant C2^k \sum_{j=3}^{\infty} 2^{-(\rho-n/2)j} \frac{1}{2^{(j+1)n}} \left(2^{j+1}\tau^{(k-k_0)/n}\right)^n w(Q_i^k) \\
&\leqslant C2^k \sum_{j=3}^{\infty} 2^{-(\rho-n/2)j} \left(\tau^{(k-k_0)/n}\right)^{n/2-\rho} w(Q_i^k) \\
&\leqslant C2^k \left(\tau^{(k-k_0)/n}\right)^{n/2-\rho} w(Q_i^k).
\end{aligned}
\tag{35}
$$

Noticing that $\rho > n/2$, $1 < \tau < 2$, we have

$$
\begin{aligned}
\sum_{k=k_0+1}^{\infty} \sum_i I_3 &\leqslant C \sum_{k=k_0+1}^{\infty} 2^k \left(\tau^{(k-k_0)/n}\right)^{n/2-\rho} \sum_i w(Q_i^k) \\
&\leqslant C \sum_{k=k_0+1}^{\infty} \left(\tau^{(k-k_0)/n}\right)^{n/2-\rho} \|f\|_{WH_w^1} \\
&\leqslant C \|f\|_{WH_w^1}.
\end{aligned}
\tag{36}
$$

Now we consider I_4. Write

$$
\begin{aligned}
I_4 &\leqslant C \int_{(2A_{k_0})^c} \left(\iint_{\substack{y\in (4Q_i^k)^c \\ |y-x|<t \\ t\leqslant |y-x_i^k|+2r_i^k}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\,\mathrm{d}z \right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}} \right)^{1/2} w(x)\,\mathrm{d}x \\
&\quad + C \int_{(2A_{k_0})^c} \left(\iint_{\substack{y\in (4Q_i^k)^c \\ |y-x|<t \\ |y-x_i^k|+2r_i^k<t}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\,\mathrm{d}z \right|^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}} \right)^{1/2} w(x)\,\mathrm{d}x \\
&=: C(I_{41} + I_{42}).
\end{aligned}
\tag{37}
$$

Take $0 < \varepsilon < \min\{1/2, \alpha, \rho - n/2\}$. First we deal with I_{41}. As $|y-z| < t$, we have

$$\begin{aligned} t &> |y-z| \geqslant \left|y - x_i^k\right| - \left|x_i^k - z\right| \\ &\geqslant \left|y - x_i^k\right| - 2r_i^k, \\ \left|x - x_i^k\right| &\leqslant |x-y| + \left|y - x_i^k\right| \leqslant t + \left|y - x_i^k\right| \\ &\leqslant 2\left|y - x_i^k\right| + 2r_i^k \leqslant 3\left|y - x_i^k\right|, \\ &\left|\frac{1}{\left(\left|y - x_i^k\right| - 2r_i^k\right)^{n+2\rho}} - \frac{1}{\left(\left|y - x_i^k\right| + 2r_i^k\right)^{n+2\rho}}\right| \\ &\leqslant C\frac{r_i^k}{\left|y - x_i^k\right|^{n+2\rho+1}}. \end{aligned} \tag{38}$$

Using the Minkowski inequality, we get that

$$\begin{aligned} I_{41} &\leqslant \int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\iint_{\substack{y\in(4Q_i^k)^c\\ |y-x|<t\\ t\leqslant|y-x_i^k|+2r_i^k\\ |y-z|<t}}\frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}t\,\mathrm{d}y}{t^{n+2\rho+1}}\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant \int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\int_{\substack{y\in(4Q_i^k)^c\\ |x-x_i^k|\leqslant3|y-x_i^k|}}\frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}}\left(\int_{|y-x_i^k|-2r_i^k}^{|y-x_i^k|+2r_i^k}\frac{\mathrm{d}t}{t^{n+2\rho+1}}\right)\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right| \\ &\qquad\times\left(\int_{\substack{y\in(4Q_i^k)^c\\ |x-x_i^k|\leqslant3|y-x_i^k|}}\frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}}\left|\frac{1}{\left(\left|y-x_i^k\right|-2r_i^k\right)^{n+2\rho}}-\frac{1}{\left(\left|y-x_i^k\right|+2r_i^k\right)^{n+2\rho}}\right|\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\int_{\substack{y\in(4Q_i^k)^c\\ |x-x_i^k|\leqslant3|y-x_i^k|}}\frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}}\frac{r_i^k}{\left|y-x_i^k\right|^{n+2\rho+1}}\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\int_{y\in(4Q_i^k)^c}\frac{\left|\Omega(y-z)\right|^2}{|y-z|^{n-2\varepsilon+1}}\frac{r_i^k}{\left|x-x_i^k\right|^{2n+2\varepsilon}}\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant C\left(r_i^k\right)^{1/2}\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left|x-x_i^k\right|^{n+\varepsilon}}\left(\int_{y\in(4Q_i^k)^c}\frac{\left|\Omega(y-z)\right|^2}{|y-z|^{n-2\varepsilon+1}}\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant C\left(r_i^k\right)^{1/2}\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left|x-x_i^k\right|^{n+\varepsilon}}\left(\int_{S^{n-1}}\int_{2r_i^k}^{\infty}\frac{\left|\Omega(z')\right|^2}{s^{2-2\varepsilon}}\mathrm{d}s\,\mathrm{d}\sigma(z')\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\ &\leqslant C\left(r_i^k\right)^{\varepsilon}\int_{(2A_{k_0})^c}\frac{w(x)}{\left|x-x_i^k\right|^{n+\varepsilon}}\mathrm{d}x\int_{Q_i^k}\left|b_i^k(z)\right|\mathrm{d}z \\ &\leqslant C2^k\left(r_i^k\right)^{\varepsilon+n}\sum_{j=3}^{\infty}\int_{2^j\tau^{(k-k_0)/n}r_i^k<|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n}r_i^k}\frac{w(x)}{\left|x-x_i^k\right|^{n+\varepsilon}}\mathrm{d}x \\ &\leqslant C2^k\left(r_i^k\right)^{\varepsilon+n}\sum_{j=3}^{\infty}\left(2^{j+1}r_i^k\right)^{-n-\varepsilon}\tau^{(k-k_0)(-n-\varepsilon)/n}\int_{|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n}r_i^k}w(x)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leqslant C2^k\sum_{j=3}^{\infty}\left(2^{j+1}\right)^{-n-\varepsilon}\tau^{(k-k_0)(-n-\varepsilon)/n}2^{(j+1)n}\tau^{k-k_0}w\left(Q_i^k\right)\\
&\leqslant C2^k\left[\tau^{(k-k_0)/n}\right]^{-\varepsilon}\sum_{j=3}^{\infty}2^{-\varepsilon(j+1)}w\left(Q_i^k\right).
\end{aligned}
\tag{39}
$$

By Lemma 5, we have

$$
\begin{aligned}
&\sum_{k=k_0+1}^{\infty}\sum_i I_{41}\\
&\qquad\leqslant C\sum_{k=k_0+1}^{\infty}2^k\left(\tau^{(k-k_0)/n}\right)^{-\varepsilon}\sum_i w\left(Q_i^k\right)\\
&\qquad\leqslant C\sum_{k=k_0+1}^{\infty}\left(\tau^{(k-k_0)/n}\right)^{-\varepsilon}\|f\|_{WH_w^1}\\
&\qquad\leqslant C\|f\|_{WH_w^1}.
\end{aligned}
\tag{40}
$$

Now let us consider I_{42}. It is easy to check that $Q_i^k\subset\{z:|y-z|<t\}$ as $y\in(4Q_i^k)^c$, $t>|y-x_i^k|+2r_i^k$. Thus, we can obtain by the condition (c) of b_i^k in Lemma 5

$$
\begin{aligned}
I_{42}&=\int_{(2A_{k_0})^c}\left(\iint_{\substack{y\in(4Q_i^k)^c\\|y-x|<t\\|y-x_i^k|+2r_i^k<t}}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}}b_i^k(z)\,dz\right|^2\frac{dt\,dy}{t^{n+2\rho+1}}\right)^{1/2}w(x)\,dx\\
&\leqslant\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\iint_{\substack{y\in(4Q_i^k)^c\\\max\{|y-x|,|y-x_i^k|+2r_i^k,|y-z|\}<t\\|x-x_i^k|\leqslant2|y-x_i^k|}}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}}\right|^2\frac{dt\,dy}{t^{n+2\rho+1}}\right)^{1/2}dz\,w(x)\,dx\\
&\quad+\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\iint_{\substack{y\in(4Q_i^k)^c\\\max\{|y-x|,|y-x_i^k|+2r_i^k,|y-z|\}<t\\|x-x_i^k|>2|y-x_i^k|}}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}}\right|^2\frac{dt\,dy}{t^{n+2\rho+1}}\right)^{1/2}dz\,w(x)\,dx\\
&=I_{42}'+I_{42}''.\\
I_{42}'&\leqslant\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\int_{\substack{y\in(4Q_i^k)^c\\|x-x_i^k|\leqslant2|y-x_i^k|}}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}}\right|^2\left(\int_{|y-x_i^k|+2r_i^k}^{\infty}\frac{dt}{t^{n+2\rho+1}}\right)dy\right)^{1/2}dz\,w(x)\,dx\\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\left|b_i^k(z)\right|\left(\int_{\substack{y\in(4Q_i^k)^c\\|x-x_i^k|\leqslant2|y-x_i^k|}}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}}\right|^2\frac{1}{\left(|y-x_i^k|+2r_i^k\right)^{n+2\rho}}dy\right)^{1/2}dz\,w(x)\,dx\\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(|x-x_i^k|+2r_i^k\right)^{n+\varepsilon}}\left(\int_{y\in(4Q_i^k)^c}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}}\right|^2\frac{1}{\left(|y-x_i^k|+2r_i^k\right)^{2\rho-n-2\varepsilon}}dy\right)^{1/2}dz\,w(x)\,dx.
\end{aligned}
\tag{41}
$$

By using Lemma 7, we can get

$$\begin{aligned}
I'_{42} &\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}} \\
&\quad\times\sum_{j=1}^{\infty}\left(\int_{2^j r_i^k<|y-x_i^k|<2^{j+1}r_i^k}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega\left(y-x_i^k\right)}{\left|y-x_i^k\right|^{n-\rho}}\right|^2\frac{1}{\left(\left|y-x_i^k\right|+2r_i^k\right)^{2\rho-n-2\varepsilon}}\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\sum_{j=1}^{\infty}\frac{1}{\left(2^j r_i^k+2r_i^k\right)^{\rho-n/2-\varepsilon}} \\
&\qquad\times\left(\int_{2^j r_i^k<|y-x_i^k|<2^{j+1}r_i^k}\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega\left(y-x_i^k\right)}{\left|y-x_i^k\right|^{n-\rho}}\right|^2\mathrm{d}y\right)^{1/2}\mathrm{d}z w(x)\,\mathrm{d}x \\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\sum_{j=1}^{\infty}\frac{1}{\left(2^j r_i^k+2r_i^k\right)^{\rho-n/2-\varepsilon}}\left(2^j r_i^k\right)^{n/2-(n-\rho)}\left\{\frac{\left|z-x_i^k\right|}{2^j r_i^k}+\int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k}\frac{w_2(\delta)}{\delta}\mathrm{d}\delta\right\}\mathrm{d}z w(x)\,\mathrm{d}x \\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\sum_{j=1}^{\infty}\left(2^j r_i^k\right)^{\varepsilon}\left\{\frac{1}{2^j}+\int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k}\frac{w_2(\delta)}{\delta}\mathrm{d}\delta\right\}\mathrm{d}z w(x)\,\mathrm{d}x \\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\sum_{j=1}^{\infty}\left(2^j r_i^k\right)^{\varepsilon}\frac{1}{2^j}\mathrm{d}z w(x)\,\mathrm{d}x \\
&\quad+C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\sum_{j=1}^{\infty}\left(2^j r_i^k\right)^{\varepsilon}\int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k}\frac{w_2(\delta)}{\delta}\mathrm{d}\delta\,\mathrm{d}z w(x)\,\mathrm{d}x \\
&=: C\left(U_1+U_2\right).
\end{aligned} \tag{42}$$

It is easy to see that

$$\begin{aligned}
U_1 &\leqslant C2^k\left(r_i^k\right)^{\varepsilon+n} \\
&\quad\times\int_{(A_{k_0})^c}\frac{w(x)}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\mathrm{d}x\sum_{j=1}^{\infty}2^{j(\varepsilon-1)} \\
&\leqslant C2^k\left(r_i^k\right)^{\varepsilon+n}\int_{(A_{k_0})^c}\frac{w(x)}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\mathrm{d}x.
\end{aligned} \tag{43}$$

Using the same method as what used to deal with the inequality (39), we can obtain that

$$U_1 \leqslant C2^k\left[\tau^{(k-k_0)/n}\right]^{-\varepsilon}w\left(Q_i^k\right). \tag{44}$$

For U_2, we have

$$\begin{aligned}
U_2 &\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}} \\
&\qquad\times\sum_{j=1}^{\infty}\frac{\left(2^j r_i^k\right)^{\varepsilon}}{2^{j\alpha}}\int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k}\frac{w_2(\delta)}{\delta^{1+\alpha}}\mathrm{d}\delta\,\mathrm{d}z w(x)\,\mathrm{d}x \\
&\leqslant C\int_{(2A_{k_0})^c}\int_{\mathbb{R}^n}\frac{\left(r_i^k\right)^{\varepsilon}\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\mathrm{d}z w(x)\,\mathrm{d}x \\
&\qquad\times\sum_{j=1}^{\infty}2^{j(\varepsilon-\alpha)}\int_0^1\frac{w_2(\delta)}{\delta^{1+\alpha}}\mathrm{d}\delta \\
&\leqslant C2^k\left(r_i^k\right)^{\varepsilon+n}\int_{(2A_{k_0})^c}\frac{w(x)}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\mathrm{d}x \\
&\leqslant C2^k\left[\tau^{(k-k_0)/n}\right]^{-\varepsilon}w\left(Q_i^k\right).
\end{aligned} \tag{45}$$

Hence, by the inequalities (44) and (45), we have

$$I'_{42} \leqslant C\left(U_1 + U_2\right) \leqslant C2^k\left[\tau^{(k-k_0)/n}\right]^{-\varepsilon} w\left(Q_i^k\right). \quad (46)$$

Now we give the estimate for I''_{42}. Since $|x - x_i^k| > 2|y - x_i^k|$, $t > |y - x_i^k| + 2r_i^k$, then

$$t > |x - y| \geqslant \left|x - x_i^k\right| - \left|y - x_i^k\right| \geqslant \frac{\left|x - x_i^k\right|}{2}. \quad (47)$$

Thus,

$$\begin{aligned}
I''_{42} &\leqslant \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \left|b_i^k(z)\right| \\
&\quad \times \left(\iint_{\substack{y \in (4Q_i^k)^c \\ \max\{|y-x|,|y-x_i^k|+2r_i^k,|y-z|\}<t}} \frac{1}{\left(\left|y - x_i^k\right| + 2r_i^k\right)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega\left(y - x_i^k\right)}{\left|y - x_i^k\right|^{n-\rho}} \right|^2 \frac{dt\,dy}{t^{2n+2\varepsilon+1}} \right)^{1/2} dz\,w(x)\,dx \\
&\leqslant \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \left|b_i^k(z)\right| \left(\int_{y \in (4Q_i^k)^c} \frac{1}{\left(\left|y - x_i^k\right| + 2r_i^k\right)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega\left(y - x_i^k\right)}{\left|y - x_i^k\right|^{n-\rho}} \right|^2 \left(\int_{|x-x_i^k|/2}^{\infty} \frac{dt}{t^{2n+2\varepsilon+1}} \right) dy \right)^{1/2} dz\,w(x)\,dx \\
&\leqslant \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \left|b_i^k(z)\right| \left(\int_{y \in (4Q_i^k)^c} \frac{1}{\left(\left|y - x_i^k\right| + 2r_i^k\right)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega\left(y - x_i^k\right)}{\left|y - x_i^k\right|^{n-\rho}} \right|^2 \frac{1}{\left|x - x_i^k\right|^{2n+2\varepsilon}} dy \right)^{1/2} dz\,w(x)\,dx \\
&\leqslant \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{\left|b_i^k(z)\right|}{\left|x - x_i^k\right|^{n+\varepsilon}} \left(\int_{y \in (4Q_i^k)^c} \frac{1}{\left(\left|y - x_i^k\right| + 2r_i^k\right)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega\left(y - x_i^k\right)}{\left|y - x_i^k\right|^{n-\rho}} \right|^2 dy \right)^{1/2} dz\,w(x)\,dx.
\end{aligned} \quad (48)$$

Repeating this process which is similar to the one of estimating I'_{42} (from (42) to (46)), we may have

$$I''_{42} \leqslant C2^k\left[\tau^{(k-k_0)/n}\right]^{-\varepsilon} w\left(Q_i^k\right). \quad (49)$$

Thus by (36) and (40), we get

$$\begin{aligned}
I_2 &= I'_2 + I''_2 \\
&\leqslant I'_2 + C \sum_{k=k_0+1}^{\infty} \sum_i \left(I_3 + I_4\right) \\
&\leqslant I'_2 + C \sum_{k=k_0+1}^{\infty} \sum_i \left(I_3 + I_{41} + I_{42}\right) \\
&\leqslant I'_2 + C \sum_{k=k_0+1}^{\infty} \sum_i \left(I_3 + I_{41} + I'_{42} + I''_{42}\right) \\
&\leqslant C\|f\|_{WH_w^1}.
\end{aligned} \quad (50)$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Combining the idea of proving Theorem 1 with the similar steps as in [12] and the following inequalities

$$\begin{aligned}
\left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z)}{\left|y - x_i^k\right|^{n-\rho}} \right| &\leqslant C \left| \frac{1}{|y-z|^{n-\rho}} - \frac{1}{\left|y - x_i^k\right|^{n-\rho}} \right| \\
&\leqslant C \frac{r_i^k}{\left|y - x_i^k\right|^{n-\rho+1}}, \\
\left|\Omega(y-z) - \Omega\left(y - x_i^k\right)\right| &\leqslant C \left| \frac{y-z}{|y-z|} - \frac{y - x_i^k}{\left|y - x_i^k\right|} \right|^{\alpha} \\
&\leqslant C \frac{\left(r_i^k\right)^{\alpha}}{\left|y - x_i^k\right|^{\alpha}},
\end{aligned} \quad (51)$$

it is not difficult to get the proof of Theorem 2. We omit the details here. □

Proof of Theorem 3. We follow the strategy of the proof of Theorem 1. It suffices to show that there exists a constant $C > 0$, such that, for any $f \in WH_w^1(\mathbb{R}^n)$, $\beta > 0$,

$$\beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu_{\lambda}^{*,\rho}(f)(x)\right| > \beta\right\}\right) \leqslant C\|f\|_{WH_w^1}. \quad (52)$$

Take $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leqslant \beta < 2^{k_0} + 1$, we have

$$\begin{aligned}
&\beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu_\lambda^{*,\rho}(f)(x)\right| > \beta\right\}\right) \\
&\quad \leqslant \beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu_\lambda^{*,\rho}(F_1)(x)\right| > \frac{\beta}{2}\right\}\right) \\
&\qquad + \beta w\left(\left\{x \in \mathbb{R}^n : \left|\mu_\lambda^{*,\rho}(F_2)(x)\right| > \frac{\beta}{2}\right\}\right) \\
&\quad =: J_1 + J_2,
\end{aligned} \tag{53}$$

where the notations F_1, F_2 are the same as in the proof of Theorem 1. Using the same method of the proof of Theorem 1, we can get

$$\begin{aligned}
J_1 &\leqslant 4\beta^{-1} \int_{\mathbb{R}^n} \left|\mu_\lambda^{*,\rho}(F_1)(x)\right| w(x)\, dx \\
&\leqslant C\beta^{-1} \left\|\mu_\lambda^{*,\rho}(F_1)\right\|_{L_w^2}^2 \\
&\leqslant C\beta^{-1} \left\|F_1\right\|_{L_w^2}^2 \\
&\leqslant C\|f\|_{WH_w^1}.
\end{aligned} \tag{54}$$

Below, we will give the estimate of J_2. If we set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q}_i^k, \tag{55}$$

where $\widetilde{Q}_i^k = Q(x_i^k, 8\tau^{(k-k_0)/n} r_i^k)$, τ is a fixed positive number such that $1 < \tau < 2$; thus,

$$\begin{aligned}
J_2 &= \beta w\left(\left\{x \in 2A_{k_0} : \left|\mu_\lambda^{*,\rho}(F_2)(x)\right| > \frac{\beta}{2}\right\}\right) \\
&\quad + \beta w\left(\left\{x \in \left(2A_{k_0}\right)^c : \left|\mu_\lambda^{*,\rho}(F_2)(x)\right| > \frac{\beta}{2}\right\}\right) \\
&=: J_2' + J_2''.
\end{aligned} \tag{56}$$

Noting that $w \in A_2$, then by Lemmas 4 and 5, we have

$$\begin{aligned}
J_2' &\leqslant 2\beta \sum_{k=k_0+1}^{\infty} \sum_i w\left(\widetilde{Q}_i^k\right) \leqslant C\beta \sum_{k=k_0+1}^{\infty} \tau^{k-k_0} \sum_i w\left(Q_i^k\right) \leqslant C \sum_{k=k_0+1}^{\infty} \left(\frac{\tau}{2}\right)^{k-k_0} \|f\|_{WH_w^1} \leqslant C\|f\|_{WH_w^1}. \\
J_2'' &\leqslant \beta \frac{2}{\beta} \int_{(2A_{k_0})^c} \mu_\lambda^{*,\rho}(F_2)(x)\, w(x)\, dx \\
&= C \int_{(2A_{k_0})^c} \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \sum_{k=k_0+1}^{\infty} \sum_i \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\, dz \right|^2 \frac{dt\, dy}{t^{n+2\rho+1}} \right)^{1/2} w(x)\, dx \\
&\leqslant C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\, dz \right|^2 \frac{dt\, dy}{t^{n+2\rho+1}} \right)^{1/2} w(x)\, dx \\
&\leqslant C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} (J_3 + J_4)\, w(x)\, dx,
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
J_3 &= \left(\iint_{|y-x|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\, dz \right|^2 \frac{dt\, dy}{t^{n+2\rho+1}} \right)^{1/2}, \\
J_4 &= \left(\iint_{|y-x|\geqslant t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z)\, dz \right|^2 \frac{dt\, dy}{t^{n+2\rho+1}} \right)^{1/2}.
\end{aligned} \tag{58}$$

Similarly as in the proof of Theorem 1, for $x \in (2A_{k_0})^c$, if the integration domains of J_4 is $y \in 4Q_i^k$, $|y-x| \geqslant t$, then we denote it by J_{41}. If $y \in (4Q_i^k)^c$, $|y-x| \geqslant t$, we denote J_4 by J_{42}. Moreover, if $t \leqslant |y - x_i^k| + 2r_i^k$ in the integration domains of J_{42}, we denote it by J_5, otherwise denote it by J_6. Further, we divide J_5 again by the integration domains; namely, if $|x-x_i^k| \leqslant 2|y-x_i^k|$, we denote it by J_{51}, otherwise denote it by J_{52}. Now, we are in a position to give the estimates of J_3, J_{41}, J_{51}, J_{52}, J_6, respectively. First, we take $0 < \varepsilon < \min\{1/2, \rho - n/2, \alpha, (\lambda-2)n/2\}$ in the whole proof of Theorem 2. Obviously,

$$\begin{aligned}\int_{(2A_{k_0})^c} & J_3 w(x)\,dx \\ &\leqslant \int_{(2A_{k_0})^c}\left(\int\int_{|y-x|<t}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\right.\right. \\ &\qquad\left.\left.\times b_i^k(z)\,dz\right|^2\frac{dt\,dy}{t^{n+2\rho+1}}\right)^{1/2} w(x)\,dx. \end{aligned} \tag{59}$$

By the proof of Theorem 1, we have

$$\sum_{k=k_0+1}^{\infty}\sum_i\int_{(2A_{k_0})^c} J_3 w(x)\,dx \leqslant C\|f\|_{WH_w^1}. \tag{60}$$

Notice that if $y \in 4Q_i^k$, $x \in (2A_{k_0})^c$, $|y-x| \geqslant t$, $z \in Q_i^k$, it is easy to check that

(a′) $|y-x| \geqslant |x-x_i^k| - |y-x_i^k| \geqslant |x-x_i^k|/2$, $|y-z| < 8r_i^k$.

If $y \in (4Q_i^k)^c$, $x \in (2A_{k_0})^c$, $|y-x| \geqslant t$, $z \in Q_i^k$, then

(b′) $|y-z| \sim |y-x_i^k|$, $|y-x_i^k| - 2r_i^k \leqslant t \leqslant |y-x_i^k| + 2r_i^k$, for $t \leqslant |y-x_i^k| + 2r_i^k$;

(c′) $|1/(|y-x_i^k| - 2r_i^k)^{n+2\rho} - 1/(|y-x_i^k| + 2r_i^k)^{n+2\rho}| \leqslant C(r_i^k/|y-x_i^k|^{n+2\rho+1})$;

(d′) $|y-x| \geqslant |x-x_i^k| - |y-x_i^k| \geqslant |x-x_i^k|/2$, for $|x-x_i^k| \geqslant 2|y-x_i^k|$;

(e′) $t + |x-y| \geqslant t + |x-x_i^k| - |y-x_i^k| \geqslant |x-x_i^k| + 2r_i^k$, for $t > |y-x_i^k| + 2r_i^k$.

Now, let us estimate J_{41}. By the fact (a′) and the Minkowski inequality, we have

$$\begin{aligned} J_{41} &\leqslant \left(\int\int_{\substack{y\in 4Q_i^k\\ |y-x|>|x-x_i^k|/2\\ |y-x|\geqslant t}}\left(\frac{t}{t+|x-y|}\right)^{2n+2\varepsilon}\left|\int_{\substack{|y-z|<8r_i^k\\ |y-z|<t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_i^k(z)\,dz\right|^2\frac{dt\,dy}{t^{n+2\rho+1}}\right)^{1/2} \\ &\leqslant C2^k\int_{Q_i^k}\left(\int\int_{\substack{y\in 4Q_i^k\\ |y-x|>|x-x_i^k|/2\\ |y-z|<8r_i^k,|y-z|<t\\ |y-x|\geqslant t}}\left(\frac{t}{t+|x-x_i^k|/2}\right)^{2n+2\varepsilon}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dt\,dy}{t^{n+2\rho+1}}\right)^{1/2} dz \\ &\leqslant C2^k\left(\frac{1}{|x-x_i^k|/2}\right)^{n+\varepsilon/2}\int_{Q_i^k}\left(\int_{\substack{y\in 4Q_i^k\\ |y-x|>|x-x_i^k|/2\\ |y-z|<8r_i^k}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_0^{|y-x|}\frac{t^{2n+2\varepsilon}}{|x-x_i^k|^{\varepsilon}|y-z|^{2\rho-n-\varepsilon}t^{2n+\varepsilon+1}}\,dt\,dy\right)^{1/2} dz \\ &\leqslant C2^k\frac{1}{|x-x_i^k|^{n+\varepsilon/2}}\int_{Q_i^k}\left(\int_{\substack{y\in 4Q_i^k\\ |y-x|>|x-x_i^k|/2\\ |y-z|<8r_i^k}}\frac{|\Omega(y-z)|^2|y-x|^{\varepsilon}}{|y-z|^{n-\varepsilon}|x-x_i^k|^{\varepsilon}}\,dy\right)^{1/2} dz \\ &\leqslant C2^k\frac{1}{|x-x_i^k|^{n+\varepsilon/2}}\int_{Q_i^k}\left(\int_{|y-z|<8r_i^k}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}}\,dy\right)^{1/2} dz \\ &\leqslant C2^k\frac{1}{|x-x_i^k|^{n+\varepsilon/2}}\int_{Q_i^k}\left(\int_{S^{n-1}}\int_0^{8r_i^k}\frac{|\Omega(y')|^2}{|s|^{n-\varepsilon}}s^{n-1}\,ds\,d\sigma(y')\right)^{1/2} dz \end{aligned}$$

$$
\begin{aligned}
&\leqslant C2^k \frac{1}{\left|x-x_i^k\right|^{n+\varepsilon/2}} \int_{Q_i^k} \left(8r_i^k\right)^{\varepsilon/2} \mathrm{d}z \\
&\leqslant C2^k \frac{\left(r_i^k\right)^{n+\varepsilon/2}}{\left|x-x_i^k\right|^{n+\varepsilon/2}}.
\end{aligned} \tag{61}
$$

As for J_{51}, notice that $y \in (4Q_i^k)^c$, $x \in (2A_{k_0})^c$, $|y-x| \geqslant t$, $z \in Q_i^k$, using the Minkowski inequality and the previous facts (b′) and (c′), we have

$$
\begin{aligned}
J_{51} &\leqslant \int_{Q_i^k} \left|b_i^k(z)\right| \\
&\quad \times \left[\int_{\substack{y\in(4Q_i^k)^c \\ |x-x_i^k|\leqslant 2|y-x_i^k|}} \left(\int_{|y-x_i^k|-2r_i^k}^{|y-x_i^k|+2r_i^k} \frac{\mathrm{d}t}{t^{n+2\rho+1}} \right) \right. \\
&\quad \left. \times \frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}} \mathrm{d}y \right]^{1/2} \mathrm{d}z \\
&\leqslant C2^k \int_{Q_i^k} \left(\int_{\substack{y\in(4Q_i^k)^c \\ |x-x_i^k|\leqslant 2|y-x_i^k|}} \frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}} \right. \\
&\quad \left. \times \frac{r_i^k}{\left|y-x_i^k\right|^{n+2\rho+1}} \mathrm{d}y \right)^{1/2} \mathrm{d}z \\
&\leqslant C2^k \\
&\quad \times \int_{Q_i^k} \left(\int_{\substack{y\in(4Q_i^k)^c \\ |x-x_i^k|\leqslant 2|y-x_i^k|}} \frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}} \right. \\
&\quad \left. \times \frac{r_i^k}{\left|y-x_i^k\right|^{2\rho-n+1-2\varepsilon} \left|y-x_i^k\right|^{2n+2\varepsilon}} \mathrm{d}y \right)^{1/2} \mathrm{d}z \\
&\leqslant C2^k \frac{\left(r_i^k\right)^{1/2}}{\left|x-x_i^k\right|^{n+\varepsilon}} \\
&\quad \times \int_{Q_i^k} \left(\int_{y\in(4Q_i^k)^c} \frac{\left|\Omega(y-z)\right|^2}{|y-z|^{n+1-2\varepsilon}} \mathrm{d}y \right)^{1/2} \mathrm{d}z \\
&\leqslant C2^k \frac{\left(r_i^k\right)^{n+1/2}}{\left|x-x_i^k\right|^{n+\varepsilon}} \\
&\quad \times \left(\int_{S^{n-1}} \int_{4r_i^k}^{\infty} \frac{\left|\Omega(y')\right|^2}{s^{2-2\varepsilon}} \mathrm{d}s\, \mathrm{d}\sigma(y') \right)^{1/2} \\
&\leqslant C2^k \frac{\left(r_i^k\right)^{n+\varepsilon}}{\left|x-x_i^k\right|^{n+\varepsilon}}.
\end{aligned} \tag{62}
$$

Now we consider J_{52}. By the fact (d′) and $t/(t+|x-y|) < t$, we have

$$
\begin{aligned}
J_{52} &\leqslant \int_{Q_i^k} \left|b_i^k(z)\right| \left[\int_{\substack{y\in(4Q_i^k)^c \\ |x-x_i^k|\geqslant 2|y-x_i^k|}} \left(\int_{|y-x_i^k|-2r_i^k}^{|y-x_i^k|+2r_i^k} \left(\frac{t}{t+|x-y|} \right)^{2n+2\varepsilon} \frac{\mathrm{d}t}{t^{n+2\rho+1}} \right) \frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}} \mathrm{d}y \right]^{1/2} \mathrm{d}z \\
&\leqslant C2^k \int_{Q_i^k} \left(\int_{\substack{y\in(4Q_i^k)^c \\ |x-x_i^k|\geqslant 2|y-x_i^k|}} \left(\int_{|y-x_i^k|-2r_i^k}^{|y-x_i^k|+2r_i^k} \frac{t^{n+2\varepsilon-2\rho-1}}{|x-y|^{2n+2\varepsilon}} \mathrm{d}t \right) \frac{\left|\Omega(y-z)\right|^2}{|y-z|^{2n-2\rho}} \mathrm{d}y \right)^{1/2} \mathrm{d}z \\
&\leqslant C2^k \int_{Q_i^k} \left(\int_{\substack{y\in(4Q_i^k)^c \\ |x-x_i^k|\geqslant 2|y-x_i^k|}} \frac{\left|\Omega(y-z)\right|^2}{|x-y|^{2n+2\varepsilon} |y-z|^{2n-2\rho}} \frac{r_i^k}{\left|y-x_i^k\right|^{2\rho-n-2\varepsilon+1}} \mathrm{d}y \right)^{1/2} \mathrm{d}z
\end{aligned}
$$

$$\leqslant C2^k \int_{Q_i^k} \left(\int_{y\in(4Q_i^k)^c} \frac{\left|\Omega(y-z)\right|^2}{\left|y-z\right|^{n-2\varepsilon+1}} \frac{r_i^k}{\left|x-x_i^k\right|^{2n+2\varepsilon}} \mathrm{d}y \right)^{1/2} \mathrm{d}z$$
$$\leqslant C2^k \frac{\left(r_i^k\right)^{n+\varepsilon}}{\left|x-x_i^k\right|^{n+\varepsilon}}. \tag{63}$$

For $y \in (4Q_i^k)^c$, since $t > |y - x_i^k| + 2r_i^k$, then we have $Q_i^k \subset \{z : |y-z| < t\}$. Thus, by the cancellation of b_i^k, we have $\int_{|y-z|<t} b_i^k(z)\mathrm{d}z = 0$. By using the fact (e′), we get

$$J_6 = \left(\iint_{\substack{y\in(4Q_i^k)^c \\ |y-x_i^k|+2r_i^k<t \\ |y-x|\geqslant t}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \int_{|y-z|<t} \left(\frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right) b_i^k(z)\,\mathrm{d}z \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+2\rho+1}} \right)^{1/2}$$
$$\leqslant \int_{\mathbb{R}^n} \left|b_i^k(z)\right| \left(\iint_{\substack{y\in(4Q_i^k)^c \\ |y-x_i^k|+2r_i^k<t \\ |y-x|\geqslant t \\ |y-z|<t}} \frac{t^{\lambda n}}{(t+|x-y|)^{2n+2\varepsilon}} \frac{1}{(t+|x-y|)^{\lambda n-2n-2\varepsilon}} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i)^k|}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+2\rho+1}} \right)^{1/2} \mathrm{d}z$$
$$\leqslant C \int_{Q_i^k} \frac{|b_i^k(z)|}{\left(|x-x_i^k|+2r_i^k\right)^{n+\varepsilon}} \left(\iint_{\substack{y\in(4Q_i^k)^c \\ |y-x_i^k|+2r_i^k<t \\ |y-x|\geqslant t \\ |y-z|<t}} \frac{t^{\lambda n}}{(t+|x-y|)^{\lambda n-2n-2\varepsilon}} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+2\rho+1}} \right)^{1/2} \mathrm{d}z$$
$$\leqslant C \int_{Q_i^k} \frac{|b_i^k(z)|}{\left(|x-x_i^k|+2r_i^k\right)^{n+\varepsilon}}$$
$$\times \left[\int_{\substack{y\in(4Q_i^k)^c \\ |y-x|>|y-x_i^k|+2r_i^k}} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \left(\int_{|y-x_i^k|+2r_i^k}^{|y-x|} \frac{t^{\lambda n-2n-2\varepsilon}}{(t+|x-y|)^{\lambda n-2n-2\varepsilon} t^{2\rho-n+1-2\varepsilon}} \mathrm{d}t \right) \mathrm{d}y \right]^{1/2} \mathrm{d}z. \tag{64}$$

Noting that $0 < \varepsilon < \rho - n/2$, we have $2\rho - n - 2\varepsilon > 0$. Hence

$$\int_{|y-x_i^k|+2r_i^k}^{|y-x|} \frac{1}{t^{2\rho-n+1-2\varepsilon}} \mathrm{d}t$$
$$\leqslant \int_{|y-x_i^k|+2r_i^k}^{\infty} \frac{1}{t^{2\rho-n+1-2\varepsilon}} \mathrm{d}t \tag{65}$$
$$\leqslant C \frac{1}{\left(|y-x_i^k|+2r_i^k\right)^{2\rho-n-2\varepsilon}}.$$

Thus,

$$J_6 \leqslant C \int_{Q_i^k} \frac{|b_i^k(z)|}{\left(|x-x_i^k|+2r_i^k\right)^{n+\varepsilon}} \times \left(\int_{y\in(4Q_i^k)^c} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \times \frac{1}{|y-x_i^k|^{2\rho-n-2\varepsilon}} \mathrm{d}y \right)^{1/2} \mathrm{d}z$$

$$\begin{aligned}
&\leqslant C\int_{Q_i^k}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\\
&\quad\times\left(\sum_{j=2}^{\infty}\int_{2^j r_i^k\leqslant|y-x_i^k|<2^{j+1}r_i^k}\left|\frac{|\Omega(y-z)|}{|y-z|^{n-\rho}}-\frac{\left|\Omega\left(y-x_i^k\right)\right|}{|y-x_i^k|^{n-\rho}}\right|^2\right.\\
&\qquad\left.\times\frac{1}{\left|y-x_i^k\right|^{2\rho-n-2\varepsilon}}\mathrm{d}y\right)^{1/2}\mathrm{d}z\\
&\leqslant C\int_{Q_i^k}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\\
&\quad\times\sum_{j=2}^{\infty}\left(2^j r_i^k\right)^{\rho-n/2-\varepsilon}\\
&\quad\times\left(\int_{2^j r_i^k\leqslant|y-x_i^k|<2^{j+1}r_i^k}\left|\frac{|\Omega(y-z)|}{|y-z|^{n-\rho}}-\frac{\left|\Omega\left(y-x_i^k\right)\right|}{|y-x_i^k|^{n-\rho}}\right|^2\mathrm{d}y\right)^{1/2}\mathrm{d}z\\
&\leqslant C\int_{Q_i^k}\frac{\left|b_i^k(z)\right|}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\sum_{j=2}^{\infty}\left(2^j r_i^k\right)^{\varepsilon}\\
&\quad\times\left\{\frac{1}{2^j}+\frac{1}{2^{j\alpha}}\int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k}\frac{w_2(\delta)}{\delta^{1+\alpha}}\mathrm{d}\delta\right\}\mathrm{d}z\\
&\leqslant C\int_{Q_i^k}\frac{\left|b_i^k(z)\right|\left(r_i^k\right)^{\varepsilon}}{\left(\left|x-x_i^k\right|+2r_i^k\right)^{n+\varepsilon}}\mathrm{d}z\\
&\leqslant C2^k\frac{\left(r_i^k\right)^{n+\varepsilon}}{\left|x-x_i^k\right|^{n+\varepsilon}}.
\end{aligned}\tag{66}$$

From (61) to (66), we can obtain

$$\begin{aligned}
&\sum_{k=k_0+1}^{\infty}\sum_i\int_{(2A_{k_0})^c}J_4w(x)\,\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i\int_{(2A_{k_0})^c}\left(J_{41}+J_{42}\right)w(x)\,\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i\int_{(2A_{k_0})^c}\left(J_{41}+J_5+J_6\right)w(x)\,\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i\int_{(2A_{k_0})^c}\left(J_{41}+J_{51}+J_{52}+J_6\right)w(x)\,\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i\int_{(2A_{k_0})^c}2^k\\
&\qquad\times\left(\frac{\left(r_i^k\right)^{n+\varepsilon/2}}{\left|x-x_i^k\right|^{n+\varepsilon/2}}+\frac{\left(r_i^k\right)^{n+\varepsilon}}{\left|x-x_i^k\right|^{n+\varepsilon}}\right)w(x)\,\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}2^k\sum_i\left(r_i^k\right)^{\varepsilon/2+n}\\
&\qquad\times\sum_{j=3}^{\infty}\int_{2^j\tau^{(k-k_0)/n}r_i^k<|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n}r_i^k}\frac{w(x)}{\left|x-x_i^k\right|^{n+\varepsilon/2}}\mathrm{d}x\\
&\qquad+C\sum_{k=k_0+1}^{\infty}2^k\sum_i\left(r_i^k\right)^{\varepsilon+n}\\
&\qquad\times\sum_{j=3}^{\infty}\int_{2^j\tau^{(k-k_0)/n}r_i^k<|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n}r_i^k}\frac{w(x)}{\left|x-x_i^k\right|^{n+\varepsilon}}\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}2^k\sum_i\left(r_i^k\right)^{\varepsilon/2+n}\\
&\qquad\times\sum_{j=3}^{\infty}\frac{1}{\left(2^jr_i^k\right)^{\varepsilon/2+n}}\int_{|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n}r_i^k}w(x)\,\mathrm{d}x\\
&\qquad+\sum_{k=k_0+1}^{\infty}2^k\sum_i\left(r_i^k\right)^{\varepsilon+n}\\
&\qquad\times\sum_{j=3}^{\infty}\frac{1}{\left(2^jr_i^k\right)^{\varepsilon+n}}\int_{|x-x_i^k|<2^{j+1}\tau^{(k-k_0)/n}r_i^k}w(x)\,\mathrm{d}x\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}2^k\sum_i\sum_{j=3}^{\infty}2^{-(j+1)(\varepsilon/2)}\tau^{-\varepsilon(k-k_0)/2n}w\left(Q_i^k\right)\\
&\qquad+C\sum_{k=k_0+1}^{\infty}2^k\sum_i\sum_{j=3}^{\infty}2^{-(j+1)\varepsilon}\tau^{-\varepsilon(k-k_0)/n}w\left(Q_i^k\right)\\
&\quad\leqslant C\sum_{k=k_0+1}^{\infty}\tau^{-\varepsilon(k-k_0)/2n}\|f\|_{WH_w^1}\\
&\qquad+C\sum_{k=k_0+1}^{\infty}\tau^{-\varepsilon(k-k_0)/n}\|f\|_{WH_w^1}\\
&\quad\leqslant C\|f\|_{WH_w^1}.
\end{aligned}\tag{67}$$

We conclude the proof of Theorem 3. □

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