Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 638638, 5 pages
http://dx.doi.org/10.1155/2013/638638

Research Article

Discreteness and Convergence of Complex Hyperbolic Isometry Groups

Xi Fu

Department of Mathematics, Shaoxing University, Shaoxing, Zhejiang 312000, China

Correspondence should be addressed to Xi Fu; fuxi1984@hotmail.com

Received 10 June 2013; Accepted 2 October 2013

Academic Editor: Pedro M. Lima

We investigate the discreteness and convergence of complex isometry groups and some discreteness criteria and algebraic convergence theorems for subgroups of $\mathbf{PU}(n,1)$ are obtained. All of the results are generalizations of the corresponding known ones.

1. Introduction

In 1976, Jørgensen obtained a very useful necessary condition for two-generator Kleinian groups of $M(\overline{\mathbb{R}}^2)$, which is known as Jørgensen's inequality. As an application, he obtained the following [1, 2].

Theorem J. *A nonelementary subgroup G of $M(\overline{\mathbb{R}}^2)$ is discrete if and only if each two-generator subgroup in G is discrete.*

Furthermore, Gilman [3] and Isachenko [4] showed that the discreteness of all two-generator subgroups of G, where each generator is loxodromic, is enough to secure the discreteness of G. See [5–8] and so forth for some other discussions along this line.

It is interesting to generalize Theorem J into the higher dimensional case. By adding some conditions, several generalizations of Theorem J into $M(\overline{\mathbb{R}}^n)$ $(n \geq 3)$ have been obtained; see [9–13] and so forth. In 2005, Wang et al. [14] proved the following.

Theorem WLC. *Let $G \subset M(\overline{\mathbb{R}}^n)$ be nonelementary and $f \in G$ loxodromic. Then G is discrete if and only if $WY(G)$ is discrete and each nonelementary subgroup $\langle f, gfg^{-1}\rangle$ is discrete, where $g \in G$.*

Here

$$WY(G) = \{h : h|_{M(G)} = I, h \in G\}, \quad (1)$$

and $M(G)$ is the smallest G-invariant hyperbolic subspace whose boundary contains the limit set $L(G)$ of G (cf. [15]).

Since the real hyperbolic plane can be viewed as a complex hyperbolic 1-space $\mathbb{H}^1_{\mathbb{C}}$, it is natural to generalize these results mentioned above to the setting of complex hyperbolic space. Recently, Qin and Jiang [16] proved the following.

Theorem QJ 1. *Let G be an n-dimensional subgroup of $\mathbf{PU}(n,1)$ and f a nonelliptic element in $\mathbf{PU}(n,1)$. If for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, g\rangle$ is discrete, then G is discrete.*

Theorem QJ 2. *Let G be an n-dimensional subgroup of $\mathbf{PU}(n,1)$ and f a regular elliptic element in $\mathbf{PU}(n,1)$. If for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, g\rangle$ is discrete, then G is discrete.*

Here G is called n-dimensional if it does not leave a point in $\partial\mathbb{H}^n_{\mathbb{C}}$ or a proper totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$ invariant. Obviously, if G is n-dimensional, then G is nonelementary and $M(G) = \mathbb{H}^n_{\mathbb{C}}$.

Motivated by Theorem WLC, a natural question will be asked: can we use the discreteness of subgroups $\langle f, gfg^{-1}\rangle$ to determine the discreteness of G in Theorems QJ1 and QJ2? In this paper, we will give this question a positive answer (see Section 3).

Let $\mathbb{G}$ be the Möbius group $M(\overline{\mathbb{R}}^n)$ or the complex hyperbolic isometry group $\mathbf{PU}(n,1)$.

Definition 1. Let $\{G_{r,i}\}$ be a sequence of subgroups in $\mathbb{G}$ and each $G_{r,i}$ be generated by $g_{1,i}, g_{2,i}, \ldots, g_{r,i}$. If, for each $t \in \{1, 2, \ldots, r\}$ $(r < \infty)$,

$$g_{t,i} \longrightarrow g_t \in \mathbb{G} \quad \text{as } i \longrightarrow \infty, \tag{2}$$

then we say that $\{G_{r,i}\}$ converges algebraically to $G_r = \langle g_1, g_2, \ldots, g_r \rangle$ and G_r is called the *algebraic limit group* of $\{G_{r,i}\}$. If for each i, $G_{r,i}$ is a Kleinian group, then the question when G_r is still a Kleinian group has attracted much attention. Jørgensen and Klein proved that G_r is still a Kleinian group, when $n = 2$. For the higher dimensional case, there are a number of discussions; see [11, 12, 17].

When $\mathbb{G} = \mathbf{PU}(n, 1)$, Cao proved [18] the following.

Theorem C 1. *Let $\{G_{r,i}\}$ be a sequence of groups of $\mathbb{G}$. If each $G_{r,i}$ is discrete, then the algebraic limit group G_r of $\{G_{r,i}\}$ is either a complex Kleinian group, or it is elementary, or $W(Gr)$ is not finite.*

Theorem C 2. *Let G_r be the algebraic limit group of complex Kleinian groups $\{G_{r,i}\}$ of $\mathbb{G}$. If $\{G_{r,i}\}$ satisfies IP-condition, then G_r is a complex Kleinian group.*

Here $\{G_{r,i}\}$ satisfies IP-condition means that $\{G_{r,i}\}$ satisfies the following conditions: for any sequence $\{f_i\}$, $f_i \in G_{r,i}$, if $\text{card}[\text{fix}(f_i)] = \infty$ for each i, and $f_i \rightarrow f$ as $i \rightarrow \infty$ with f being the identity or parabolic, then $\{f_i\}$ has uniformly bounded torsion (see [18]).

In this paper, we will discuss the discreteness criteria and algebraic convergence theorems for subgroups of $\mathbf{PU}(n, 1)$ further. The rest of this paper is organized as follows: in Section 2, we introduce some preliminary results that we need in the sequel; in Section 3, we show three discreteness criteria for subgroups of $\mathbf{PU}(n, 1)$; finally Section 4 is dedicated to three algebraic convergence theorems for complex Kleinian groups.

2. Preliminaries

Let $\mathbb{C}^{n,1}$ be the complex vector space of dimension $n + 1$ with the Hermitian form

$$\langle z, w \rangle = z_1 \overline{w}_1 + z_2 \overline{w}_2 + \cdots + z_n \overline{w}_n - z_{n+1} \overline{w}_{n+1}, \tag{3}$$

where z, w are the column vectors in $\mathbb{C}^{n+1}$. Consider the following subspaces of $\mathbb{C}^{n,1}$:

$$\begin{gathered} V_0 = \left\{ z \in \mathbb{C}^{n,1} - \{0\} : \langle z, z \rangle = 0 \right\}, \\ V_- = \left\{ z \in \mathbb{C}^{n,1} : \langle z, z \rangle < 0 \right\}. \end{gathered} \tag{4}$$

Let $P : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{CP}^n$ be the canonical projection from $\mathbb{C}^{n+1} - \{0\}$ onto the complex hyperbolic space $\mathbb{CP}^n$. The complex hyperbolic space is defined to be $\mathbb{H}^n_{\mathbb{C}} = PV_-$ and $\partial \mathbb{H}^n_{\mathbb{C}} = PV_0$ is its boundary. The biholomorphic isometry group of $\mathbb{H}^n_{\mathbb{C}}$ is given by the projective unitary group $\mathbf{PU}(n, 1)$. For a nontrivial element g of $\mathbf{PU}(n, 1)$, we say that g is *elliptic* if it has a fixed point in $\mathbb{H}^n_{\mathbb{C}}$, g is *parabolic* if it has only one fixed point in $\partial \mathbb{H}^n_{\mathbb{C}}$, and g is *loxodromic* if it has exactly two different fixed points in $\partial \mathbb{H}^n_{\mathbb{C}}$.

For elliptic element $g \in \mathbf{PU}(n, 1)$, let Λ_0 and Λ_i $(i = 1, 2, \ldots, n)$ be its negative and positive eigenvalues, respectively. Then the fixed point set of g in $\mathbb{H}^n_{\mathbb{C}}$ contains only one point if $\Lambda_0 \neq \Lambda_i$ and is a totally geodesic submanifold, which is equivalent to $\mathbb{H}^m_{\mathbb{C}}$ if Λ_0 coincides with exact m of class Λ_i $(m < n)$. We call g *regular elliptic* if $\Lambda_s \neq \Lambda_t$, where $s, t \in \{0, 1, \ldots, n\}$ and $s \neq t$. Obviously, if g is regular elliptic, then g has only one fixed point in $\mathbb{H}^n_{\mathbb{C}}$. The following proposition follows directly from [19].

Proposition 2. *The regular elliptic (resp., loxodromic) elements of $\mathbf{PU}(n, 1)$ form an open set.*

Let G be a subgroup of $\mathbf{PU}(n, 1)$. The limit set $L(G)$ of G is defined as

$$L(G) = \overline{G(p)} \cap \partial \mathbb{H}^n_{\mathbb{C}}, \quad p \in \mathbb{H}^n_{\mathbb{C}}. \tag{5}$$

G is called nonelementary if $L(G)$ contains more than two points; otherwise, it is called elementary. We call a subgroup G of $\mathbf{PU}(n, 1)$ complex Kleinian group if it is discrete and nonelementary. For a nonelementary subgroup G of $\mathbf{PU}(n, 1)$, we denote by $M(G)$ the smallest totally geodesic submanifold of G whose boundary contains the limit set $L(G)$. It is easy to see that $M(G)$ is G-invariant since $L(G)$ is G-invariant. As in [18], the subgroup $W(G)$ of G is defined as

$$W(G) = \{ g : g|_{M(G)} = I, g \in G \}. \tag{6}$$

For an element $f \in \mathbf{PU}(n, 1)$, we denote $N(f) = \| f - I \|$, where $\| \cdot \|$ is the Hilbert-Schmidt norm. Then we have the following.

Lemma 3 (see [18, 20]). *Suppose that two elements $f, g \in \mathbf{PU}(n, 1)$ generate a complex Kleinian group.*

(1) *If f is parabolic or loxodromic, then*

$$\max \{ N(f), N([f, g]) \} \geq 2 - \sqrt{3}, \tag{7}$$

where $[f, g] = fgf^{-1}g^{-1}$ is the commutator of f and g.

(2) *If f is elliptic, then*

$$\max \{ N(f), N([f, g^q]) : q = 1, 2, 3, \ldots, n + 1 \} \geq 2 - \sqrt{3}. \tag{8}$$

3. Discreteness Criteria

In this section, we prove the following theorems.

Theorem 4. *Let G be an n-dimensional subgroup of $\mathbf{PU}(n, 1)$ and f a nonelliptic element in $\mathbf{PU}(n, 1)$. If for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, gfg^{-1} \rangle$ is discrete, then G is discrete.*

Theorem 5. *Let G be an n-dimensional subgroup of $\mathbf{PU}(n, 1)$ and f a regular elliptic element with finite order k $(k \geq 3)$*

in $\mathbf{PU}(n,1)$*. If for each loxodromic (resp., regular elliptic) element* $g \in G$ *the two-generator group* $\langle f, gfg^{-1}\rangle$ *is discrete, then* G *is discrete.*

When f is elliptic (may not be regular), we have the following.

Theorem 6. *Let* G *be an* n*-dimensional subgroup of* $\mathbf{PU}(n,1)$ *and* f *an elliptic element with finite order* k $(k \geq 3)$ *in* $\mathbf{PU}(n,1)$*. If, for each loxodromic (resp., regular elliptic) element* $g \in G$ *the two-generator group* $\langle f, g\rangle$ *is discrete, then* G *is discrete.*

In order to prove the above theorems, we need the following lemma which is a classification of elementary subgroups of $\mathbf{PU}(n,1)$.

Lemma 7. *Let* G *be a subgroup of* $\mathbf{PU}(n,1)$*.*

(1) *If* G *contains a loxodromic element, then* G *is elementary if and only if it fixes a point in* $\partial\mathbb{H}^n_{\mathbb{C}}$ *or a point-pair* $\{x, y\}$ *in* $\partial\mathbb{H}^n_{\mathbb{C}}$*.*

(2) *If* G *contains a parabolic element but no loxodromic element, then* G *is elementary if and only if it fixes a point in* $\partial\mathbb{H}^n_{\mathbb{C}}$*.*

(3) *If* G *is purely elliptic, then* G *fixes a point in* $\overline{\mathbb{H}}^n_{\mathbb{C}}$*.*

Proof of Theorem 4. Firstly, we prove the case when each g is loxodromic. Suppose not. Then G is dense in $\mathbf{PU}(n,1)$ according to Corollary 4.5.1 of [15]. By Proposition 2, there exists a sequence $\{g_i\}$ in G such that each g_i is loxodromic and $g_i \to I$ as $i \to \infty$. Then, for large enough i, we have

$$N\left(g_i f g_i^{-1} f^{-1}\right) + \sum_{q=1}^{n+1} N\left(\left[g_i f g_i^{-1} f^{-1}, f^q\right]\right) < 2 - \sqrt{3}. \quad (9)$$

Since f is nonelliptic and $\langle f, g_i f g_i^{-1} f^{-1}\rangle = \langle f, g_i f g_i^{-1}\rangle$, by Lemma 3, we know that, for all large enough i, $\langle f, g_i f g_i^{-1}\rangle$ are elementary. This implies that

$$\operatorname{fix}(f) \cap \operatorname{fix}(g_i) \neq \emptyset. \quad (10)$$

Since G is nonelementary, we can find three loxodromic elements h_s $(s = 1,2,3)$ in G such that

$$\operatorname{fix}(f) \cap \operatorname{fix}(h_s) = \emptyset, \qquad \operatorname{fix}(h_j) \cap \operatorname{fix}(h_k) = \emptyset, \quad (11)$$

where $i, k \in \{1,2,3\}$ and $j \neq k$. It follows from a discussion similar to the above that we can obtain that, for large enough i,

$$\operatorname{fix}(f) \cap \operatorname{fix}\left(h_s g_i h_s^{-1}\right) \neq \emptyset, \quad s = 1,2,3. \quad (12)$$

Since f is nonelliptic, that is, $\operatorname{fix}(f)$ contains less than three points; it is a contradiction.

Now, we come to prove the case when each g is regular elliptic. Suppose that G is nondiscrete. Similarly, by Proposition 2, we can find a sequence $\{g_i\}$ in G such that each g_i is regular elliptic and $g_i \to I$ as $i \to \infty$. This implies that, for sufficiently large i, the subgroups $\langle f, g_i f g_i^{-1}\rangle$ are elementary. It follows that

$$\operatorname{fix}(f) = \operatorname{fix}(g_i). \quad (13)$$

It is a contradiction since f is nonelliptic and g_i is regular elliptic.

This completes the proof. □

Proof of Theorem 5. The proof of Theorem 5 follows from a discussion similar to that in the proof of Theorem 4. □

Proof of Theorem 6. We only prove the case when g is loxodromic; similar arguments can be applied to the case when g is regular elliptic. Suppose that G is nondiscrete. Then there exists a sequence $\{g_i\} \subset G$ such that, for each i, g_i is loxodromic and

$$g_i \longrightarrow I \quad \text{as } i \longrightarrow \infty. \quad (14)$$

Since G is n-dimensional, we can find finitely many loxodromic elements $h_1, h_2, \ldots, h_t$ in G such that the set $S = \{A_{\operatorname{fix}(h_1)}, A_{\operatorname{fix}(h_2)}, \ldots, A_{\operatorname{fix}(h_t)}\}$ can span the whole complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, where $A_{\operatorname{fix}(h)}$ is the attractive fixed point of h. For each k $(k = 1,2,\ldots,t)$, let $U_{A_{\operatorname{fix}(h_k)}}$ be a small neighbourhood of $A_{\operatorname{fix}(h_k)}$ in $\overline{\mathbb{H}}^n_{\mathbb{C}}$; then there exists an integer N such that

$$h_k^N(\operatorname{fix}(f)) \subset U_{A_{\operatorname{fix}(h_k)}}. \quad (15)$$

Since

$$\begin{gathered}\left\langle h_k^N f h_k^{-N}, g_i\right\rangle = h_k^N \left\langle f, h_k^{-N} g_i h_k^N\right\rangle h_k^{-N}, \\ \max\left\{N\left(h_k^{-N} g_i h_k^N\right), N\left(\left[h_k^{-N} g_i h_k^N, f\right]\right)\right\} < 2 - \sqrt{3},\end{gathered} \quad (16)$$

for large enough i, we can see that the subgroups $\langle h_k^N f^2 h_k^{-N}, g_i\rangle$ are elementary. By Lemma 7, we know that, for each k, $(k = 1,2,\ldots,t)$,

$$\operatorname{fix}(g_i) \cap U_{A_{\operatorname{fix}(h_k)}} \neq \emptyset. \quad (17)$$

Obviously, it is a contradiction. □

4. Algebraic Convergence

In this section, we discuss the algebraic convergence of complex hyperbolic Kleinian groups. Firstly, we generalize Theorem C1 into the following form.

Theorem 8. *Let* $\{G_{r,i}\}$ *be a sequence of groups of* $\mathbf{PU}(n,1)$ *and* G_r *be its algebraic limit group. Then we have the following.*

(1) *If, for each* i, $G_{r,i}$ *is a complex Kleinian group, then* G_r *is nonelementary and* G_r *is discrete if and only if each one-generator subgroup of* $W(G_r)$ *is discrete.*

(2) *If, for each* i, $G_{r,i}$ *is discrete, then* G_r *is elementary if and only if for large enough* i*, all* $G_{r,i}$ *are elementary.*

Proof. The proof of (1). The nonelementariness of G_r follows from [21, Theorem 1.4]. Now, we come to prove that if G_r is nondiscrete, then there is an element $f \in W(G_r)$ such that the subgroup $\langle f \rangle$ is nondiscrete. Suppose that G_r is nondiscrete. Since $r < \infty$ (that is, G_r is finitely generated), by Selberg's Lemma we know that G_r contains a torsion free subgroup G_{r_1} with finite index which is nonelementary and nondiscrete either. Then there exists a sequence $\{f_j\}$ in G_{r_1} such that

$$f_j \longrightarrow I \quad \text{as } j \longrightarrow \infty. \tag{18}$$

As G_{r_1} is nonelementary, we can find finitely many loxodromic elements $g_1, g_2, \ldots, g_k$ in G_{r_1} such that the set $\{\text{fix}(g_1), \text{fix}(g_2), \ldots, \text{fix}(g_k)\}$ spans $\partial M(G_{r_1})$, the boundary of $M(G_{r_1})$. Then, for large enough j, we have

$$N\left(f_j\right) + \sum_{q=1}^{n+1} N\left(\left[f_j, g_s^q\right]\right) < 2 - \sqrt{3}, \quad s \in \{1, 2, \ldots, k\}. \tag{19}$$

Let $f_{i,j}$ and $g_{i,s}$ be the corresponding elements of f_j and g_s in $G_{r,i}$, respectively. Then, for large enough i and j,

$$N\left(f_{i,j}\right) + \sum_{q=1}^{n+1} N\left(\left[f_{i,j}, g_{i,s}^q\right]\right) < 2 - \sqrt{3}. \tag{20}$$

Lemma 3 implies that, for large enough i and j, the subgroups $\langle f_{i,j}, g_{i,s} \rangle$ are elementary. Since the loxodromic elements of $\mathbf{PU}(n, 1)$ form an open set, we know that, for sufficiently large i, $g_{i,s}$ are loxodromic as well. It follows that

$$\text{fix}\left(g_{i,s}\right) \subset \text{fix}\left(f_{i,j}\right), \tag{21}$$

which shows that, for $s \in \{1, 2, \ldots, k\}$ and all sufficiently large j,

$$\text{fix}\left(g_s\right) \subset \text{fix}\left(f_j\right). \tag{22}$$

Thus, for all sufficiently large j,

$$f_j \in W\left(G_{r_1}\right). \tag{23}$$

Since G_{r_1} is torsion free, we know that there exists an element $f \in W(G_{r_1})$ such that $\langle f \rangle$ is nondiscrete. Note that $M(G_r) = M(G_{r_1})$, so $f \in W(G_r)$. Hence, the conclusion of (1) follows.

The proof of (2). We only need to prove that if, for large enough i, all $G_{r,i}$ are elementary, then is G_r since the converse is trivial by (1). Suppose that G_r is nonelementary. Then we can find two loxodromic elements f and g in G_r such that

$$\text{fix}\left(f\right) \cap \text{fix}\left(g\right) = \emptyset. \tag{24}$$

Let f_i and g_i be the corresponding elements of f and g in $G_{r,i}$, respectively. Then, for large enough i, we have

$$\text{fix}\left(f_i\right) \cap \text{fix}\left(g_i\right) = \emptyset. \tag{25}$$

It follows a discussion similar to that in the proof of (1) that, for large enough i, both f_i and g_i are loxodromic. This shows that, for large enough i, all $G_{r,i}$ are nonelementary. It is a contradiction. □

Definition 9. Let $\{G_i\}$ be a sequence of complex Kleinian groups of $\mathbf{PU}(n, 1)$. We say that $\{G_i\}$ satisfies E-condition if there is no sequence $\{f_i\}$, $f_i \in W(G_i)$ such that $f_i \to f$ as $i \to \infty$, where f is an elliptic element with infinite order.

In the following, we give an example which shows that, if the sequence $\{G_i\}$ does not satisfy IP-condition but E-condition, then the limit group G_r is still a complex Kleinian groups.

Example 10. Suppose that H is a purely loxodromic nonelementary subgroup of $\mathbf{PU}(1, 1)$ and, for each j,

$$f_j = \begin{bmatrix} e^{i(1/2^j)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

Let $\widetilde{H}$ be the Poincaré extension of H in $\mathbf{PU}(2, 1)$ and $G_j = \langle \widetilde{H}, f_j \rangle$. Then it is easy to see that the algebraic limit group G of $\{G_j\}$ is a complex Kleinian group. Note that $f_j \to I$ as $j \to \infty$; we know that $\{G_j\}$ does not satisfy IP-condition but E-condition.

As applications of Theorem 8 and E-condition, we have the following.

Theorem 11. *Let G_r be the algebraic limit group of complex Kleinian groups $\{G_{r,i}\}$ of $\mathbf{PU}(n, 1)$. If $\{G_{r,i}\}$ satisfies E-condition, then G_r is a complex Kleinian group.*

Proof. By Theorem 8(1), we know that G_r is nonelementary. Suppose that G_r is nondiscrete. Then there exist an elliptic element $f \in W(G_r)$ and an integer sequence $\{n_j\}$ such that $\text{ord}(f) = \infty$ and

$$f^{n_j} \longrightarrow I \quad \text{as } n_j \longrightarrow \infty. \tag{27}$$

For each n_j, let $f_i^{n_j}$ be the corresponding element of f^{n_j} in $G_{r,i}$. By [21, Lemma 4.2], we know that $f_i^{n_j} \in W(G_{r,i})$. It follows from the hypothesis that $\{G_{r,i}\}$ satisfies E-condition; we have $f_i^{n_j} = I$ for large enough i. This implies that $f^{n_j} = I$. It is a contradiction.

The proof is completed. □

When $r \leq \infty$, Wang [17] proved the following.

Theorem W. *Let $r \leq \infty$. If the generator system $\{g_{t,i}\}_{t=1}^r$ of $G_{r,i}$ satisfies that none are elliptic and no two have any fixed point in common, and, if all $G_{r,i}$ are Kleinian groups, then*

(1) *all the generators $g_t = \lim_{i \to \infty} g_{t,i}$ are neither elliptic nor identity;*

(2) *if $G_r = \langle g_1, g_2, \ldots, g_r \rangle$ is nonelementary and $W(G_r)$ is discrete, then G_r is discrete.*

It easily follows a similar argument as in the proof of Theorem 8 and we can obtain the following.

Theorem 12. *Let $r \leq \infty$. If the generator system $\{g_{t,i}\}_{t=1}^r$ of $G_{r,i}$ satisfies that none are elliptic and no two have any fixed point in common, and, if all $G_{r,i}$ are discrete, then*

(1) $G_r = \langle g_1, g_2, \ldots, g_r \rangle$ *is nonelementary;*

(2) G_r *is discrete if and only if* $W(G_r)$ *is discrete.*

Acknowledgment

The research was partly supported by Tian-Yuan Foundation (no. 11226096).

References

[1] T. Jørgensen, "On discrete groups of Möbius transformation," *American Journal of Mathematics*, vol. 98, pp. 739–749, 1976.

[2] T. Jørgensen, "A note on subgroups of $SL(2, \mathbb{C})$," *The Quarterly Journal of Mathematics*, vol. 28, no. 110, pp. 209–211, 1977.

[3] J. Gilman, "Inequalities in discrete subgroups of $PSL(2, \mathbb{R})$," *Canadian Journal of Mathematics*, vol. 40, no. 1, pp. 115–130, 1988.

[4] N. A. Isachenko, "Systems of generators of subgroups of $PSL(2, \mathbb{C})$," *Siberian Mathematical Journal*, vol. 31, no. 1, pp. 162–165, 1990.

[5] A. F. Beardon, *The Geometry of Discrete Groups*, vol. 91 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1983.

[6] P. Tukia and X. Wang, "Discreteness of subgroups of $SL(2, \mathbb{C})$ containing elliptic elements," *Mathematica Scandinavica*, vol. 91, no. 2, pp. 214–220, 2002.

[7] X. Wang and W. Yang, "Discreteness criterions for subgroups in $SL(2, \mathbb{C})$," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 124, no. 1, pp. 51–55, 1998.

[8] S. Yang, "Test maps and discrete groups in $SL(2, \mathbb{C})$," *Osaka Journal of Mathematics*, vol. 46, no. 2, pp. 403–409, 2009.

[9] W. Abikoff and A. Haas, "Nondiscrete groups of hyperbolic motions," *The Bulletin of the London Mathematical Society*, vol. 22, no. 3, pp. 233–238, 1990.

[10] M. Chen, "Discreteness and convergence of Möbius groups," *Geometriae Dedicata*, vol. 104, pp. 61–69, 2004.

[11] A. Fang and B. Nai, "On the discreteness and convergence in n-dimensional Möbius groups," *Journal of the London Mathematical Society*, vol. 61, no. 3, pp. 761–773, 2000.

[12] G. J. Martin, "On discrete Möbius groups in all dimensions: a generalization of Jørgensen's inequality," *Acta Mathematica*, vol. 163, no. 3-4, pp. 253–289, 1989.

[13] X. Wang and W. Yang, "Discreteness criteria of Möbius groups of high dimensions and convergence theorems of Kleinian groups," *Advances in Mathematics*, vol. 159, no. 1, pp. 68–82, 2001.

[14] X. Wang, L. Li, and W. Cao, "Discreteness criteria for Möbius groups acting on $\overline{\mathbb{R}^n}$," *Israel Journal of Mathematics*, vol. 150, pp. 357–368, 2005.

[15] S. S. Chen and L. Greenberg, "Hyperbolic spaces," in *Contributions to Analysis a Collection of Papers Dedicated to Lipman Bers*, pp. 49–87, Academic Press, New York, NY, USA, 1974.

[16] H. Qin and Y. Jiang, "Discreteness criteria based on a test map in $PU(n, 1)$," *Proceedings—Mathematical Sciences*, vol. 122, no. 4, pp. 519–524, 2012.

[17] X. Wang, "Algebraic convergence theorems of n-dimensional Kleinian groups," *Israel Journal of Mathematics*, vol. 162, pp. 221–233, 2007.

[18] W. Cao, "Algebraic convergence theorems of complex Kleinian groups," *Glasgow Mathematical Journal*, vol. 55, no. 1, pp. 1–8, 2013.

[19] W. M. Goldman, *Complex Hyperbolic Geometry*, Oxford University Press, Oxford, UK, 1999.

[20] B. Dai, A. Fang, and B. Nai, "Discreteness criteria for subgroups in complex hyperbolic space," *Proceedings of the Japan Academy*, vol. 77, no. 10, pp. 168–172, 2001.

[21] M. Kapovich, "On sequences of finitely generated discrete groups," in *In the Tradition of Ahlfors-Bers. V*, vol. 510 of *Contemporary Mathematics*, pp. 165–184, American Mathematical Society, Providence, RI, USA, 2010.