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Research Article

Exponential Decay for Nonlinear von Kármán Equations with Memory

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We consider the nonlinear von Kármán equations with memory term. We show the exponential decay result of solutions. Our result is established without imposing the usual relation between g and its derivative. This result improves on earlier ones concerning the exponential decay.

1. Introduction

In this paper we consider the exponential decay rate of solutions for the nonlinear von Kármán equations with memory term:

$$\left|u'\right|^{\rho} u'' - h\Delta u'' + \Delta^2 u - \int_0^t g(t-s)\,\Delta^2 u(s)\,ds = [u,v], \quad \text{in } \Omega\times(0,\infty), \tag{1}$$

$$\Delta^2 v = -[u,u], \quad \text{in } \Omega\times(0,\infty), \tag{2}$$

$$u(x,y,0) = u_0(x,y), \quad u'(x,y,0) = u_1(x,y), \quad \text{in } \Omega, \tag{3}$$

and the boundary conditions

$$\begin{gathered} v = \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \Gamma\times(0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \Gamma_0\times(0,\infty), \\ \mathscr{B}_1 u - \mathscr{B}_1\left\{\int_0^t g(t-s)\,u(s)\,ds\right\} = 0, \quad \text{on } \Gamma_1\times(0,\infty), \\ \mathscr{B}_2 u - h\frac{\partial u''}{\partial \nu} - \mathscr{B}_2\left\{\int_0^t g(t-s)\,u(s)\,ds\right\} = 0, \quad \text{on } \Gamma_1\times(0,\infty), \end{gathered} \tag{4}$$

where Ω is an open bounded set of $\mathbb{R}^2$, with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint. The constants $h, \rho > 0$. Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ and by $\eta = (-\nu_2, \nu_1)$ the corresponding unit tangent vector. The von Kármán bracket is given by

$$[u,v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}. \tag{5}$$

Here, we are denoting by $\mathscr{B}_1$, $\mathscr{B}_2$ the following differential operators:

$$\begin{gathered} \mathscr{B}_1 u = \Delta u + (1-\mu) B_1 u, \\ \mathscr{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu) B_2 u, \end{gathered} \tag{6}$$

where B_1 and B_2 are given by

$$\begin{gathered} B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \\ B_2 u = \frac{\partial}{\partial \eta}\left[\left(\nu_1^2 - \nu_2^2\right) u_{xy} + \nu_1\nu_2\left(u_{yy} - u_{xx}\right)\right], \end{gathered} \tag{7}$$

and the constant μ $(0 < \mu < 1/2)$ represents Poisson's ratio.

This system describes the transversal displacement $u(x,y,t)$ and the Airy stress function $v(x,y,t)$ of a vibrating plate. The dissipation in (1) is due to the term $-g * \Delta^2 u$, where g is positive real function and the convolution product $*$ is given by $(g * u)(t) = \int_0^t g(t-s)u(s)ds$. A material whose contained term is $-g * \Delta^2 u$ is called viscoelastic and is said to be "endowed with long-range memory" since the stress at any instant depends on the complete history of strain that the material has undergone.

Problems related to

$$f\left(u'\right)u'' - \Delta u - \Delta u'' = 0 \tag{8}$$

are interesting not only from the point of view of PDE general theory, but also due to its applications in Mechanics. For instance, when the material density, $f(u')$, is equal to 1, (8) describes the extensional vibrations of thin rods; see Love [1] for the physical details. When the material density $f(u')$ is not constant, we are dealing with a thin rod which possesses a rigid surface and whose interior is somehow permissive to slight deformations such that the material density varies according to the velocity.

On the other hand, the problem of stability of the solutions to the following wave equation with memory was studied by many authors [2–6]:

$$\left|u'\right|^{\rho} u'' - \Delta u - \Delta u'' + \int_0^t g(t-s)\,\Delta u(s)\,ds + F\left(u,u'\right) = 0, \quad x \in \Omega, \quad t > 0. \tag{9}$$

Cavalcanti et al. [2] showed an exponential and polynomial decay for the viscoelastic wave equation (9) with $F(u,u') = -\gamma \Delta u'$ under the usual conditions

$$-c_1 g(t) \le g'(t) \le -c_2 g(t), \qquad 0 \le g''(t) \le c_3 g(t) \tag{10}$$

for some c_i, $i = 1,2,3$. Han and Wang [3] proved the uniform decay for the nonlinear viscoelastic equation under condition

$$g'(t) \le -cg(t), \tag{11}$$

where $c > 0$. Park and Kang [7] studied the uniform decay for a nonlinear viscoelastic problem with damping. They obtained the exponential decay estimate under condition (11). Later, this assumption was relaxed by several authors. Messaoudi and Tatar [6] investigated exponential and polynomial decay for a quasilinear viscoelastic equation under condition on g such as

$$g'(t) \le -\xi g^p(t), \quad \text{for } 1 \le p < \frac{3}{2},\ t \ge 0, \tag{12}$$

where $\xi > 0$, by choosing a suitable perturbed energy. Liu [5] showed exponential and polynomial decay for the system of two coupled quasilinear viscoelastic equation, under condition (12). Messaoudi and Tatar [8] proved the exponential decay rate for a quasilinear viscoelastic equation under the conditions

$$g'(t) \le 0, \qquad \int_0^\infty g(t)\,e^{\alpha t}\,dt < +\infty \quad \text{for some large } \alpha > 0. \tag{13}$$

They improved some earlier results concerning the exponential decay. Han and Wang [4] studied the general decay rate for the nonlinear viscoelastic equations under the more general conditions on g such as

$$g'(t) \le -\xi(t)\,g(t), \qquad \frac{\left|\xi'(t)\right|}{\left|\xi(t)\right|} \le k, \qquad \xi(t) > 0, \quad \xi'(t) \le 0, \quad \forall t > 0. \tag{14}$$

When $\rho = 0$, the problem of stability of the solutions to the viscoelastic system with memory has been studied by many authors. In [9, 10], the authors proved exponential and polynomial decay for the viscoelastic wave equation under conditions (10). Berrimi and Messaoudi [11] studied exponential and polynomial decay rates under condition (12). Messaoudi [12] investigated the general decay rate for the viscoelastic equations under general conditions (14). Guesmia and Messaoudi [13] obtained general stability for the Timoshenko system under weaker condition on g such as

$$g'(t) \le -\xi(t)\,g(t), \tag{15}$$

where ξ is a nonincreasing and positive function. As for problem of stability of the solutions to a viscoelastic system under condition (15), we also refer the reader to [14–16] and references therein. These general decay estimates extended and improved on some earlier results—exponential or polynomial decay rates.

The problem of stability of the solutions to a von Kármán system with dissipative effects has been studied by several authors. For example, in [17, 18] the authors studied the von Kármán equation in the presence of thermal effects. In [19–23] the authors considered the von Kármán system with frictional dissipations effective in the boundary. It is shown in these works that these dissipations produce uniform rate of decay of the solution when t goes to infinity. Rivera and Menzala [24] and Rivera et al. [25] studied the stability of the solutions to a von Kármán system for viscoelastic plates with memory and boundary memory conditions. They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function. Later, Santos and Soufyane [26] generalized the decay result of [24]. Raposo and Santos [27] considered the general decay of the solutions to a von Kármán plate model (1)–(4) for $\rho = 0$. They showed that the energy decays with a similar rate of decay of the relaxation function, which is not necessarily decaying in a polynomial or exponential fashion. Kang [28] investigated the general decay of the solution to a von Kármán system with memory and boundary damping. Recently, Kang [29] proved that solutions for a von Kármán plate with memory decay exponentially to zero as time goes to infinity in case $g'(t) + \gamma g(t) \ge 0$ for all $t \ge 0$ provided that $[g'(t) + \gamma g(t)]e^{\alpha t} \in L^1(0,\infty)$ for some $\alpha > 0$.

In this paper, we establish an exponential decay of the solutions to the nonlinear von Kármán plate model (1)–(4) without assumption (15), which is the usual relation between g and its derivative. Instead of (15), we require the function $e^{\alpha t} g(t)$ to have sufficiently small L^1-norms on $(0,\infty)$ for some $\alpha > 0$. This result improves on earlier ones concerning the exponential decay of the solutions to the von Kármán equations.

The organization of this paper is as follows. In Section 2, we give some notations and introduce the relative results of Airy stress function and von Kármán bracket. In Section 3, we prove that the energy decreases exponentially. The construction of the Lyapunov function is inspired in multiplier techniques that was used in [8].

2. Preliminaries

In this section, we present some material needed in the proof of our result and state the main result. Throughout this paper we denote

$$(u, v) = \int_\Omega u(x, y)\, v(x, y)\, d\Omega \tag{16}$$

and define

$$V = \left\{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\right\},$$
$$U = \left\{u \in H^2(\Omega) \mid u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}. \tag{17}$$

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$. We define for all $1 \le p < \infty$

$$\|u\|_p^p = \int_\Omega |u(x, y)|^p d\Omega. \tag{18}$$

A simple calculation, based on the integration by parts formula, yields

$$\left(\Delta^2 u, v\right) = a(u, v) + (\mathscr{B}_2 u, v)_\Gamma - \left(\mathscr{B}_1 u, \frac{\partial v}{\partial \nu}\right)_\Gamma, \tag{19}$$

where the bilinear symmetric form $a(u, v)$ is given by

$$a(u, v) = \int_\Omega \left\{u_{xx}v_{xx} + u_{yy}v_{yy} + \mu\left(u_{xx}v_{yy} + u_{yy}v_{xx}\right) + 2(1-\mu)\, u_{xy}v_{xy}\right\} d\Omega, \tag{20}$$

where $d\Omega = dx\,dy$. Since $\Gamma_0 \neq \emptyset$, we know that $\sqrt{a(u,u)}$ is equivalent to the $H^2(\Omega)$ norm; that is,

$$c_0\|u\|^2_{H^2(\Omega)} \le a(u, u) \le c_1\|u\|^2_{H^2(\Omega)}, \tag{21}$$

where c_0 and c_1 are generic positive constants. This and Sobolev embedding theorem imply that for some positive constants C_p and C_s

$$\|u\|^2 \le C_p a(u, u), \quad \|\nabla u\|^2 \le C_s a(u, u), \quad \forall u \in U. \tag{22}$$

We establish the following hypotheses on the relaxation function g (see [8]). The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing C^1 function satisfying

$$g(0) > 0, \qquad l := \int_0^\infty g(s)\, ds < 1, \tag{23}$$

$$g'(t) \le 0, \qquad \int_0^\infty e^{\alpha t} g(t)\, dt < +\infty, \quad \text{for some } \alpha > 0. \tag{24}$$

To simplify calculation in our analysis, we introduce the following notation:

$$g\Box u := \int_0^t g(t-s)\, \|u(\cdot, t) - u(\cdot, s)\|^2 ds,$$
$$g\Box\partial^2 u := \int_0^t g(t-s)\, a\left(u(\cdot, t) - u(\cdot, s), u(\cdot, t) - u(\cdot, s)\right) ds. \tag{25}$$

From the symmetry of $a(\cdot,\cdot)$, we have that, for any $v \in C^1(0, T; H^2(\Omega))$,

$$a\left(g * v, v'\right) = -\frac{1}{2}g(t)\, a(v, v) + \frac{1}{2}g'\Box\partial^2 v - \frac{1}{2}\frac{d}{dt} \times \left\{g\Box\partial^2 v - \left(\int_0^t g(s)\, ds\right) a(v, v)\right\}. \tag{26}$$

Now, we introduce the relative results of the Airy stress function and von Kármán bracket $[\cdot,\cdot]$.

Lemma 1 (see [30]). *Let u, w be functions in $H^2(\Omega)$ and v in $H_0^2(\Omega)$, where Ω is an open bounded and connected set of $\mathbb{R}^2$ with regular boundary. Then,*

$$\int_\Omega w[v, u]\, d\Omega = \int_\Omega v[w, u]\, d\Omega. \tag{27}$$

Lemma 2 (see [20, 31]). *If $u, v \in H^2(\Omega)$, then $[u, v] \in L^2(\Omega)$ and satisfies*

$$\|[u, v]\| \le c\|u\|_{H^2(\Omega)}\|v\|_{W^{2,\infty}(\Omega)},$$
$$\|v\|_{W^{2,\infty}(\Omega)} \le c\|u\|^2_{H^2(\Omega)}. \tag{28}$$

The energy of problem (1)–(4) is given by

$$E(t) = \frac{1}{\rho+2}\left\|u'(t)\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}a(u, u) + \frac{h}{2}\left\|\nabla u'(t)\right\|^2 + \frac{1}{4}\|\Delta v(t)\|^2. \tag{29}$$

The existence of solutions can be proved by the Faedo-Galerkin method; see [2, 9].

Theorem 3. *Assume that the kernel g is a positive continuous function satisfying (23). Let $(u_0, u_1) \in H^4(\Omega) \times H^2(\Omega)$. Then, the system (1)–(4) has a unique weak solution u such that*

$$u \in L^\infty\left(0, \infty; U \cap H^4(\Omega)\right),$$
$$u' \in L^\infty\left(0, \infty; V \cap H^2(\Omega)\right),$$
$$u'' \in L^2\left(0, \infty; H_0^1(\Omega)\right). \tag{30}$$

3. Exponential Decay of the Energy

In this section we will prove the exponential decay rates. To demonstrate the stability of the system (1)–(4), the lemmas below are essential. The following result shows the dissipative property of the system (1)–(4). Multiplying (1) by $u'(t)$, we get the identity

$$E'(t) = a\left(g * u, u'\right). \tag{31}$$

Define the modified energy by

$$\begin{aligned} F(t) = {} & \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} + \frac{h}{2}\left\|\nabla u'\right\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right) a(u,u) \\ & + \frac{1}{2} g \square \partial^2 u + \frac{1}{4}\|\Delta v\|^2, \end{aligned} \tag{32}$$

and applying (26) to (31), we have

$$F'(t) = -\frac{1}{2} g(t)\, a(u,u) + \frac{1}{2} g' \square \partial^2 u. \tag{33}$$

This implies that $F(t)$ is nonincreasing, and one easily sees that

$$E(t) \le \frac{1}{1-l} F(t), \quad \forall t \ge 0. \tag{34}$$

Therefore, it is enough to obtain the desired decay for the modified energy $F(t)$, which will be done below. The key point for showing our desired result is finding a Lyapunov functional L which is equivalent to $F(t)$. First, we introduce three functionals and establish several lemmas. So, let

$$\Phi_1(t) = \int_0^t G(\alpha; t-s)\, a(u(s), u(s))\, ds \tag{35}$$

with

$$\begin{aligned} G(\alpha; t) &= e^{-\alpha t} \int_t^{+\infty} e^{\alpha s} g(s)\, ds, \\ \Phi_2(t) &= \frac{1}{\rho+1} \int_\Omega \left|u'\right|^\rho u' u\, d\Omega + h \int_\Omega \nabla u' \nabla u\, d\Omega, \\ \Phi_3(t) &= -\frac{1}{\rho+1} \int_\Omega \left|u'\right|^\rho u' \int_0^t g(t-s)(u(t) - u(s))\, ds\, d\Omega \\ & \quad - h \int_\Omega \nabla u' \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds\, d\Omega. \end{aligned} \tag{36}$$

We define the modified energy by

$$L(t) = NF(t) + \sum_{i=1}^3 \gamma_i \Phi_i(t), \quad t \ge 0; \tag{37}$$

for some positive constants γ_i is to be specified later.

Lemma 4. *Assume that g satisfies (23) and (24). For $N > 0$ large enough, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that*

$$\alpha_1 F(t) \le L(t) \le \alpha_2 \left(F(t) + \Phi_1(t)\right), \quad \forall t \ge 0. \tag{38}$$

Proof. From Young inequality, we deduce

$$\begin{aligned} \left|\Phi_2(t)\right| \le {} & \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|u\|_{\rho+2}^{\rho+2} \\ & + \frac{h}{2}\left\|\nabla u'\right\|^2 + \frac{h}{2}\|\nabla u\|^2. \end{aligned} \tag{39}$$

Considering the embedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ and taking (22) into account, it holds that

$$\begin{aligned} \left|\Phi_2(t)\right| \le {} & \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} + \frac{C^{\rho+2}}{(\rho+1)(\rho+2)}\|\nabla u\|^{\rho+2} \\ & + \frac{h}{2}\left\|\nabla u'\right\|^2 + \frac{h}{2}\|\nabla u\|^2 \\ \le {} & \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} \\ & + \left(\frac{C^{\rho+2} C_s}{(\rho+1)(\rho+2)}\left(2C_s E(0)\right)^{\rho/2} + \frac{hC_s}{2}\right) \\ & \times a(u,u) + \frac{h}{2}\left\|\nabla u'\right\|^2, \end{aligned} \tag{40}$$

where C comes from the inequality $\|u\|_{\rho+2} \le C\|\nabla u\|$ for all $u \in H_0^1(\Omega)$. On the other hand, by Young inequality, Hölder inequality and (22) can be estimated as

$$\begin{aligned} & \left|\Phi_3(t)\right| \\ & \le \frac{h}{2}\left\|\nabla u'\right\|^2 + \frac{h}{2} \\ & \quad \times \int_\Omega \left(\int_0^t g(t-s)\left|\nabla u(t) - \nabla u(s)\right| ds\right)^2 d\Omega \\ & \quad + \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)} \\ & \quad \times \int_\Omega \left(\int_0^t g(t-s)\left|u(t) - u(s)\right| ds\right)^{\rho+2} d\Omega \\ & \le \frac{h}{2}\left\|\nabla u'\right\|^2 + \frac{hlC_s}{2} g \square \partial^2 u + \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} \\ & \quad + \frac{1}{(\rho+1)(\rho+2)}\left(\int_0^t g(s)\, ds\right)^{\rho+1} \\ & \quad \times \int_\Omega \int_0^t g(t-s)\left|u(t) - u(s)\right|^{\rho+2} ds\, d\Omega \\ & \le \frac{h}{2}\left\|\nabla u'\right\|^2 + \frac{1}{\rho+2}\left\|u'\right\|_{\rho+2}^{\rho+2} \\ & \quad + \left(\frac{hlC_s}{2} + \frac{l^{\rho+1} C^{\rho+2} C_s}{(\rho+1)(\rho+2)}\left(2C_s E(0)\right)^{\rho/2}\right) g \square \partial^2 u. \end{aligned} \tag{41}$$

Thus, from (40) and (41) we obtain

$$\begin{aligned} & \left|L(t) - NF(t) - \gamma_1 \Phi_1(t)\right| \\ & \le \frac{1}{\rho+2}\left(\gamma_2 + \gamma_3\right)\left\|u'\right\|_{\rho+2}^{\rho+2} + \frac{h}{2}\left(\gamma_2 + \gamma_3\right)\left\|\nabla u'\right\|^2 \\ & \quad + \left(\frac{C^{\rho+2} C_s}{(\rho+1)(\rho+2)}\left(2C_s E(0)\right)^{\rho/2} + \frac{hC_s}{2}\right)\gamma_2 a(u,u) \\ & \quad + \left(\frac{hlC_s}{2} + \frac{l^{\rho+1} C^{\rho+2} C_s}{(\rho+1)(\rho+2)}\left(2C_s E(0)\right)^{\rho/2}\right) \\ & \quad \times \gamma_3 g \square \partial^2 u \le c_0 F(t), \end{aligned} \tag{42}$$

where c_0 is a positive constant depending on γ_2, γ_3, h, ρ, l, C, and C_s. Choosing $N > 0$ large, we complete the proof of Lemma 4. □

Lemma 5. *For each $t_0 > 0$ and sufficiently large $N > 0$, there exists positive constant c_2 such that*

$$L'(t) \le -c_2 \left(F(t) + \Phi_1(t)\right), \quad \forall t \ge t_0. \tag{43}$$

Proof. By differentiating $\Phi_1(t)$ and using Young inequality, we get

$$\begin{aligned}
\Phi_1'(t) &= \overline{g}_\alpha a(u,u) - \alpha \Phi_1(t) \\
&\quad - \int_0^t g(t-s)\, a(u(s), u(s))\, ds, \\
&= \overline{g}_\alpha a(u,u) - \alpha\Phi_1(t) - g\square\partial^2 u \\
&\quad + \left(\int_0^t g(s)\,ds\right) a(u,u) \\
&\quad - 2\int_0^t g(t-s)\, a(u(t), u(s))\, ds \\
&\le -\alpha\Phi_1(t) - g\square\partial^2 u \\
&\quad + \left(\overline{g}_\alpha + \frac{1}{4\delta} - \int_0^t g(s)\,ds\right) a(u,u) + \delta l g\square\partial^2 u,
\end{aligned} \tag{44}$$

where $\overline{g}_\alpha = \int_0^\infty e^{\alpha s}|g(s)|ds$, $\alpha > 0$, and $\delta > 0$. Using (1)–(4), we have

$$\begin{aligned}
\Phi_2'(t) &= -a(u,u) + a(g * u, u) + ([u,v], u) \\
&\quad + \frac{1}{\rho+1}\left\|u'\right\|_{\rho+2}^{\rho+2} + h\left\|\nabla u'\right\|^2.
\end{aligned} \tag{45}$$

We use the following inequality:

$$\begin{aligned}
a(g * u, u) &= \int_0^t g(t-s)\, a(u(s) - u(t), u(t))\, ds \\
&\quad + \int_0^t g(s)\, ds a(u,u) \\
&\le \left(\eta + \int_0^t g(s)\,ds\right) a(u,u) + \frac{l}{4\eta} g\square\partial^2 u;
\end{aligned} \tag{46}$$

then we obtain

$$\begin{aligned}
\Phi_2'(t) &\le \frac{1}{\rho+1}\left\|u'\right\|_{\rho+2}^{\rho+2} + h\left\|\nabla u'\right\|^2 \\
&\quad - \left(1 - \eta - \int_0^t g(s)\,ds\right) a(u,u) - \|\Delta v\|^2 + \frac{l}{4\eta} g\square\partial^2 u,
\end{aligned} \tag{47}$$

where $\eta > 0$. Similarly we deduce

$$\begin{aligned}
\Phi_3'(t) &= \int_0^t g(t-s)\, a(u(t) - u(s), u(t))\, ds \\
&\quad - \int_0^t g(t-s)\,(u(t) - u(s), [u,v])\, ds \\
&\quad - \int_0^t g(t-s)\, a\left(u(t) - u(s), \int_0^t g(t-\tau)\, u(\tau)\, d\tau\right) ds \\
&\quad - h\int_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s), \nabla u'(t)\right) ds \\
&\quad - h\int_0^t g(s)\,ds\left\|\nabla u'\right\|^2 - \frac{1}{\rho+1} \\
&\quad \times \int_0^t g'(t-s)\left(u(t) - u(s), \left|u'\right|^\rho u'\right) ds \\
&\quad - \frac{1}{\rho+1}\int_0^t g(s)\,ds\left\|u'\right\|_{\rho+2}^{\rho+2} \\
&= \left(1 - \int_0^t g(s)\,ds\right)\int_0^t g(t-s) \\
&\quad \times a(u(t) - u(s), u(t))\, ds \\
&\quad - \int_0^t g(t-s)\,(u(t) - u(s), [u,v])\, ds \\
&\quad + \int_0^t g(t-s) \\
&\quad \times a\left(u(t) - u(s), \int_0^t g(t-\tau)\,(u(t) - u(\tau))\, d\tau\right) ds \\
&\quad - h\int_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s), \nabla u'(t)\right) ds \\
&\quad - h\int_0^t g(s)\,ds\left\|\nabla u'\right\|^2 - \frac{1}{\rho+1} \\
&\quad \times \int_0^t g'(t-s)\left(u(t) - u(s), \left|u'\right|^\rho u'\right) ds \\
&\quad - \frac{1}{\rho+1}\int_0^t g(s)\,ds\left\|u'\right\|_{\rho+2}^{\rho+2} \\
&:= I_1 + I_2 + \cdots + I_5 - h\int_0^t g(s)\,ds\left\|\nabla u'\right\|^2 \\
&\quad - \frac{1}{\rho+1}\int_0^t g(s)\,ds\left\|u'\right\|_{\rho+2}^{\rho+2}.
\end{aligned} \tag{48}$$

Now, we estimate the terms in the right hand side of (48). The Young and Hölder inequalities and (22) give that

$$\begin{aligned}
\left|I_1 + I_3\right| &\le \left(1 - \int_0^t g(s)\,ds\right)\left(\eta a(u,u) + \frac{l}{4\eta} g\square\partial^2 u\right) \\
&\quad + l g\square\partial^2 u, \\
\left|I_4\right| &\le h\eta\left\|\nabla u'\right\|^2 \\
&\quad + \frac{h}{4\eta}\int_\Omega\left(\int_0^t g'(t-s)\,|\nabla u(t) - \nabla u(s)|\,ds\right)^2 d\Omega
\end{aligned}$$

$$
\begin{aligned}
&\le h\eta\left\|\nabla u'\right\|^2-\frac{g(0)\,C_s h}{4\eta}g'\Box\partial^2 u,\\
\left|I_5\right| &\le \frac{\eta}{\rho+1}\left\|u'\right\|_{2(\rho+1)}^{2(\rho+1)}+\frac{1}{4\eta(\rho+1)}\\
&\quad\times\int_\Omega\left(\int_0^t g'(t-s)\left|u(t)-u(s)\right|ds\right)^2 d\Omega\\
&\le \frac{\eta C^{2(\rho+1)}}{\rho+1}\left\|\nabla u'\right\|^{2(\rho+1)}-\frac{g(0)}{4\eta(\rho+1)}\\
&\quad\times\int_\Omega\int_0^t g'(t-s)\left|u(t)-u(s)\right|^2 ds\,d\Omega\\
&\le \frac{a_0\eta}{\rho+1}\left\|\nabla u'\right\|^2-\frac{g(0)\,C_p}{4\eta(\rho+1)}g'\Box\partial^2 u,
\end{aligned}
\tag{49}
$$

where $a_0 = C^{2(\rho+1)}(2h^{-1}E(0))^{\rho} > 0$. From Lemmas 1 and 2 and (22), we obtain

$$
\begin{aligned}
\left|I_2\right| &= \left|\left(\int_0^t g(t-s)(u(t)-u(s))\,ds,[u,v]\right)\right|\\
&\le \eta\|[u,v]\|^2+\frac{1}{4\eta}\left\|\int_0^t g(t-s)(u(t)-u(s))\,ds\right\|^2\\
&\le \eta\left(c\|u\|_{H^2(\Omega)}\|v\|_{W^{2,\infty}(\Omega)}\right)^2+\frac{lC_p}{4\eta}g\Box\partial^2 u\\
&\le \eta C_0 a(u,u)+\frac{lC_p}{4\eta}g\Box\partial^2 u.
\end{aligned}
\tag{50}
$$

Summarizing these estimates with (48), we deduce that

$$
\begin{aligned}
\Phi_3'(t) &\le \left(h\eta+\frac{a_0\eta}{\rho+1}-h\int_0^t g(s)\,ds\right)\left\|\nabla u'\right\|^2\\
&\quad+\left(l+\frac{l}{4\eta}+\frac{lC_p}{4\eta}\right)g\Box\partial^2 u\\
&\quad+\eta(1+C_0)\,a(u,u)-\frac{g(0)}{4\eta}\left(C_s h+\frac{C_p}{\rho+1}\right)g'\Box\partial^2 u\\
&\quad-\frac{1}{\rho+1}\int_0^t g(s)\,ds\left\|u'\right\|_{\rho+2}^{\rho+2}.
\end{aligned}
\tag{51}
$$

Since g is continuous and positive, for any $t \ge t_0 > 0$ we have

$$
\int_0^t g(s)\,ds \ge \int_0^{t_0} g(s)\,ds := g_0 > 0. \tag{52}
$$

Thus, making use of (52) and combining (33), (37), (44), (47), and (51), we obtain

$$
\begin{aligned}
L'(t) &\le -\left(\left(hg_0-h\eta-\frac{\eta a_0}{\rho+1}\right)\gamma_3-h\gamma_2\right)\left\|\nabla u'\right\|^2\\
&\quad+\left(\frac{N}{2}-\frac{g(0)}{4\eta}\left(C_s h+\frac{C_p}{\rho+1}\right)\gamma_3\right)g'\Box\partial^2 u\\
&\quad-\left[\frac{N}{2}g(t)+\left(1-\eta-\int_0^t g(s)\,ds\right)\gamma_2\right.\\
&\quad\left.-\left(\overline{g}_\alpha+\frac{1}{4\delta}-g_0\right)\gamma_1-\eta(1+C_0)\gamma_3\right]a(u,u)\\
&\quad-\alpha\gamma_1\Phi_1(t)-\gamma_2\|\Delta v\|^2\\
&\quad-\left[(1-\delta l)\gamma_1-\frac{l\gamma_2}{4\eta}-\left(l+\frac{l}{4\eta}+\frac{lC_p}{4\eta}\right)\gamma_3\right]g\Box\partial^2 u\\
&\quad-\frac{1}{\rho+1}\left(g_0\gamma_3-\gamma_2\right)\left\|u'\right\|_{\rho+2}^{\rho+2},\quad \forall t\ge t_0.
\end{aligned}
\tag{53}
$$

We first take $\gamma_2 > 0$ and $\delta > 0$ so small that

$$
g_0\gamma_3-\gamma_2>0,\qquad 1-\delta l>0, \tag{54}
$$

respectively. And then, we choose $\eta > 0$ and $\gamma_3 > 0$ so small that

$$
\begin{gathered}
\left(g_0-\eta-\frac{\eta a_0}{(\rho+1)h}\right)\gamma_3-\gamma_2>0,\\
1-\eta-\int_0^t g(s)\,ds>0,
\end{gathered}
\tag{55}
$$

respectively. We then pick γ_1 large enough so that

$$
(1-\delta l)\gamma_1-\frac{l\gamma_2}{4\eta}-\left(l+\frac{l}{4\eta}+\frac{lC_p}{4\eta}\right)\gamma_3>0. \tag{56}
$$

Finally, taking $N > 0$ large enough and by (53), we conclude that

$$
L'(t)\le -c_2\left(F(t)+\Phi_1(t)\right),\quad \forall t\ge t_0, \tag{57}
$$

for some $c_2 > 0$. □

Our main result reads as follows.

Theorem 6. *Suppose that g satisfies* (23) *and* (24). *Then, for each $t_0 > 0$, there exist two positive constants C_1 and β such that*

$$
E(t)\le C_1 e^{-\beta t},\quad \forall t\ge t_0. \tag{58}
$$

Proof. From (38) and (43), we have

$$
L'(t)\le -\frac{c_2}{\alpha_2}L(t),\quad \forall t\ge t_0. \tag{59}
$$

Integrating this over (t_0, t), we obtain

$$
L(t)\le L(t_0)\,e^{-\beta(t-t_0)},\quad \forall t\ge t_0, \tag{60}
$$

with $\beta = c_2/\alpha_2$. Consequently, (34), (38), and (60) yield the result in Theorem 6. □

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