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Research Article

Some Properties of $l^p(A, X)$ Spaces

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We provide a representation of elements of the space $l^p(A, X)$ for a locally convex space X and $1 \le p < \infty$ and determine its continuous dual for normed space X and $1 < p < \infty$. In particular, we study the extension and characterization of isometries on $l^p(\mathrm{N}, \mathrm{X})$ space, when X is a normed space with an unconditional basis and with a symmetric norm. In addition, we give a simple proof of the main result of G. Ding (2002).

1. Introduction

Let X be a Hausdorff locally convex space, let R be a family of seminorms on X determining its topology and, let A be a set. We say that x belongs to $l^p(A, X)$ if and only if

$$\sum_{a \in A} [(r \circ x)(a)]^p < \infty \tag{1.1}$$

for each r in R, where $1 \le p < +\infty$. Obviously, $l^p(A, X)$ is a Hausdorff locally convex space with the seminorms $(\sum_{a \in A} [(r \circ x)(a)]^p)^{1/p}$, for each r in R. When $p = 1$, Yilmaz in [1] investigated some structural properties of the function space $l^1(A, X)$ for a Hausdorff locally convex space X and obtained the continuous duals of $l^1(A, X)$ and $c_0(A, X)$ for a normed space X. It should be mentioned that [2] is a powerful tool in the detailed investigation of mentioned function spaces.

Let X be a real F space with the F-norm $\|x\|$ and with an unconditional basis $\{e_n\}$. The norm $\|x\|$ is called symmetric if, for any permutation $\{p_n\}$ and for an arbitrary sequence $\{\varepsilon_n\}$ of numbers equal either to 1 or to -1, the following equality holds (see [3]):

$$\|t_1 e_1 + \cdots + t_n e_n + \cdots\| = \left\|\varepsilon_1 t_1 e_{p_1} + \cdots + \varepsilon_n t_n e_{p_n} + \cdots\right\|. \tag{1.2}$$

As follows from the definition of symmetric norms, the operator V defined by the formula

$$V(t_1e_1 + \cdots + t_ne_n + \cdots) = \varepsilon_1 t_1 e_{p_1} + \cdots + \varepsilon_n t_n e_{p_n} + \cdots \tag{1.3}$$

is an isometry of the X onto itself.

Let E and F be normed spaces. A mapping $V : E \to F$ is called an isometry if $\|Vx - Vy\| = \|x - y\|$ for all $x, y \in E$ (see, e.g., [4]). The classical Mazur-Ulam theorem in [5] describes the relation between isometry and linearity and states that every onto isometry V between two normed spaces with $V(0) = 0$ is linear. So far, this has been generalized in several directions (see, e.g., [6]). One of them is the study of the isometric extension problem.

Mankiewicz in [7] showed that an isometry which maps a connected subset of a normed space X onto an open subset of another normed space Y can be extended to an affine isometry from X to Y. In 1987, Tingley [8] posed the problem of extending an isometry between unit spheres as follows.

> Let E and F be two real Banach spaces. Suppose that V_0 is a surjective isometry between the two unit spheres $S_1(E)$ and $S_1(F)$. Is V_0 necessarily a restriction of a linear or affine transformation to $S_1(E)$?

It is very difficult to answer this question, even in two dimensional cases. In the same paper, Tingley proved that if E and F are finite-dimensional Banach spaces and $V_0 : S_1(E) \to S_1(F)$ is a surjective isometry, then $V_0(x) = -V_0(-x)$ for all $x \in S_1(E)$. In [9], Ding gave an affirmative answer to Tingley problem, when E and F are Hilbert spaces. In the case E and F are metric vector spaces, the corresponding extension problem was investigated in [10] and [11]. See [12] for some related results.

In this paper we obtain some structural properties of $l^p(A, X)$ for $1 < p < \infty$. We mainly provide a representation of the elements of $l^p(A, X)$ space and obtain continuous duals of $l^p(A, X)$ for a normed space X, where $1 < p < \infty$. We also study the extension and characterization of isometries on $l^p(\mathbf{N}, X)$ space, when X is a normed space with an unconditional basis and with a symmetric norm. Finally, we give a simple proof of an isometric extension theorem of [9].

2. Some Results of $l^p(A, X)$ Spaces

In this section we obtain some structural properties of the function space $l^p(A, X)(1 \le p < \infty)$. For this purpose, we need a lemma that will be used in the proofs of our main results. We begin with the following well-known result (see [3]).

Lemma 2.1. *Let X be a real infinite-dimensional F-space with a basis $\{e_n\}$ and with a symmetric norm $\|x\|$. Then either X is a Hilbert space or each isometry is of type*(1.3).

Now we are in position to state and prove the main results in this section.

Theorem 2.2. *Let X be a Hausdorff locally convex space, let R be a family of seminorms on X determining its topology, and let A be a set. Then each $x \in l^p(A,X)$ $(1 \le p < \infty)$ is represented by*

$$x = \Sigma_{a\in A}(I_a \circ x)(a), \tag{2.1}$$

where $I_a : X \to l^p(A,X)$ is defined by

$$I_a(t)(b) = \begin{cases} t, & b = a, \\ 0, & b \neq a, \end{cases} \quad b \in A. \tag{2.2}$$

Proof. We denote by $\mathcal{F}$ the family of all finite subsets of the index set A. We write $x = \Sigma_{a\in A}(I_a \circ x)(a)$ if the net $(\sum_{a\in F}(I_a \circ x)(a) : F \in \mathcal{F})$ converges to x. Define

$$S_F(x) = \sum_{a\in F}(I_a \circ x)(a) \tag{2.3}$$

for a finite subset F of A. We must prove that the net $(S_F(x) : \mathcal{F})$ converges to x in $l^p(A,X)$. By the definition of $S_F(x)$, we have

$$S_F(x)(a) = \begin{cases} x(a), & a \in F, \\ 0, & a \in A \setminus F. \end{cases} \tag{2.4}$$

For $U \in \mathcal{N}_0(l^p(A,X))$ (where $\mathcal{N}_0(l^p(A,X))$ denotes a base of neighborhoods of the origin of $l^p(A,X)$), there exist $\varepsilon > 0$ and $r_1, r_2, \dots, r_n \in R$ such that

$$U \supseteq \bigcap_{i=1}^{n}\left\{z : \sum_{a\in A}[(r_i \circ z)(a)]^p < \varepsilon\right\}. \tag{2.5}$$

Since $\sum_{a\in A}[(r \circ x)(a)]^p < \infty$ for each $r \in R$, then for $i(1 \le i \le n)$, we can find $F_i \in \mathcal{F}$ such that

$$\sum_{a\in A\setminus F_i}[(r_i \circ x)(a)]^p < \varepsilon. \tag{2.6}$$

Hence, setting $F_0 := \bigcup_{i=1}^{n} F_i$, we have

$$\sum_{a\in A}[(r_i \circ [x - S_F(x)])(a)]^p = \sum_{a\in A\setminus F}[(r_i \circ x)(a)]^p < \varepsilon \tag{2.7}$$

for each $F \supseteq F_0$. This implies $x - S_F(x) \in U$. That is $x = \Sigma_{a\in A}(I_a \circ x)(a)$. □

Remark 2.3. If X is a normed space and $\|\|_p$ denotes the norm of $l^p(A,X)$, it holds that $\|I_a(t)\|_p = \|t\|$ and $\|I_a\| = 1$.

Theorem 2.4. *Let X be a normed space and let A be a set. Then for each $f \in l^p(A, X)'$, there exists $\psi \in l^q(A, X')$ such that*

$$f(x) = \sum_{a\in A} \psi(a)[x(a)], \tag{2.8}$$

and $l^p(A, X)' = l^q(A, X')$, where $1/p + 1/q = 1$ and $1 < p < \infty$.

Proof. By Theorem 2.2, $x \in l^p(A, X)$ is represented by

$$x = \sum_{a\in A} I_a[x(a)]. \tag{2.9}$$

If $f \in l^p(A, X)'$, then

$$f(x) = \sum_{a\in A} f \circ I_a[x(a)]. \tag{2.10}$$

Define $\psi : A \to X'$ by $\psi(a) = f \circ I_a$. Next, we prove that $\psi \in l^q(A, X')$.

Let F be an arbitrary finite subset of A. Since Bishop and Phelps showed that the norm-attainers are dense in $B(X, Y)$ for every Banach space X when $Y = \mathbb{F}$ (the symbol $\mathbb{F}$ denotes a field that can be either $\mathbb{R}$ and $\mathbb{C}$), there exists $\xi(a)$ in the closed unit ball of X such that

$$\|\psi(a)\| = |\psi(a)[\xi(a)]| \tag{2.11}$$

for each $a \in F$. Let us write $\psi(a)[\xi(a)]$ in the polar form, that is,

$$\psi(a)[\xi(a)] = e^{i\theta_a} |\psi(a)[\xi(a)]|, \tag{2.12}$$

and define the function x from A to X by

$$x(a) = \begin{cases} \|\psi(a)\|^{q-1} e^{-i\theta_a} \xi(a), & \text{if } a \in F \text{ and } \psi(a)[\xi(a)] \neq 0, \\ 0, & \text{if } a \notin F \text{ or } \psi(a)[\xi(a)] = 0. \end{cases} \tag{2.13}$$

Obviously, $x \in l^p(A, X)$. Therefore, for this x, we have

$$\begin{aligned}
|f(x)| &= \left|\sum_{a\in A}\psi(a)[x(a)]\right| \\
&= \left|\sum_{a\in F}\|\psi(a)\|^{q-1}e^{-i\theta_a}e^{i\theta_a}|\psi(a)[\xi(a)]|\right| \\
&= \sum_{a\in F}\|\psi(a)\|^{q} \\
&\leq \|f\| \ \ \|x\| \\
&\leq \|f\|\left(\sum_{a\in F}\left(\|\psi(a)\|^{q-1}\right)^p\right)^{1/p} \\
&= \|f\|\left(\sum_{a\in F}\|\psi(a)\|^{q}\right)^{1/p}.
\end{aligned} \tag{2.14}$$

Thus

$$\left(\sum_{a\in F}\|\psi(a)\|^{q}\right)^{1/q} \leq \|f\| < \infty. \tag{2.15}$$

Since F is an arbitrary finite subset of A, we have

$$\|\psi\| = \left(\sum_{a\in A}\|\psi(a)\|^{q}\right)^{1/q} \leq \|f\| < \infty, \tag{2.16}$$

and so $\psi \in l^q(A, X')$. Moreover, by Hölder inequality, we have

$$|f(x)| \leq \sum_{a\in A}\|\psi(a)\|\|x(a)\| \leq \left(\sum_{a\in A}\|\psi(a)\|^{q}\right)^{1/q}\left(\sum_{a\in A}\|x(a)\|^{p}\right)^{1/p} = \|\psi\|\|x\|, \tag{2.17}$$

from which we get

$$\|f\| \leq \|\psi\|. \tag{2.18}$$

Combining (2.15) and (2.18) yields $\|f\| = \|\psi\|$. Thus we define a linear isometry $T : l^p(A, X)' \to l^q(A, X')$ with $Tf = \psi$. To prove that T is surjective. Indeed, for $\psi \in l^q(A, X')$, there exists f defined on $l^p(A, X)$ such that

$$f(x) = \sum_{a\in A}\psi(a)[x(a)], \tag{2.19}$$

that is, $Tf = \varphi$. By Mazur-Ulam theorem (see [5]), T is a linear isometry from $l^p(A,X)'$ onto $l^q(A,X')$, thus

$$l^p(A,X)' = l^q(A,X'). \tag{2.20}$$

The proof of this Theorem is finished. □

Theorem 2.5. *Let X be a normed space with an unconditional basis and with a symmetric norm. Then $l^p(\mathbf{N},X)$ is also a normed space with an unconditional basis and with a symmetric norm. Moreover, either $l^p(\mathbf{N},X)$ is a Hilbert space or each isometry is of type* (1.3).

Proof. Suppose that $\{e_k\}$ is an unconditional basis for X with $\|e_k\| = 1$. Let

$$e_{\text{ik}} = \underbrace{(o,\ldots,e_k,o,\ldots)}_{i\text{th place}}. \tag{2.21}$$

By Theorem 2.2, if $x(i) = \Sigma_{k=1}^{\infty} a_{\text{ik}} e_k$ then $x \in l^p(\mathbf{N},X)$ is represented by

$$x = \sum_{\substack{i\in N\\ k\in N}} a_{\text{ik}} e_{\text{ik}}, \tag{2.22}$$

that is $\{e_{\text{ik}}\}_{i\in\mathbf{N},k\in\mathbf{N}}$ is a basis for $l^p(\mathbf{N},X)$. Note that $x = \sum_{i\in N,\,k\in N} a_{\text{ik}} e_{\text{ik}}$ is an unconditionally convergent series in $l^p(\mathbf{N},X)$ and that $\{e_k\}$ is an unconditional basis for X. Thus $\{e_{\text{ik}}\}_{i\in\mathbf{N},k\in\mathbf{N}}$ is an unconditional basis for $l^p(\mathbf{N},X)$. by the definition of norm on $l^p(\mathbf{N},X)$ and symmetry of norm on X it follows that

$$\left\|\sum a_{\text{ik}} e_{\text{ik}}\right\| = \left(\sum \|a_{\text{ik}} e_{\text{ik}}\|^p\right)^{1/p} = \left(\sum |a_{\text{ik}}|^p\right)^{1/p}. \tag{2.23}$$

For any permutation of positive integers $\{p_{\text{ik}}\}$, we have

$$\left\|\sum \varepsilon_{\text{ik}} a_{\text{ik}} e_{p_{\text{ik}}}\right\| = \left(\sum |a_{\text{ik}}|^p\right)^{1/p}, \tag{2.24}$$

thus $l^p(\mathbf{N},X)$ has symmetric norm. By Lemma 2.1, either $l^p(\mathbf{N},X)$ is a Hilbert space or each isometry is of type (1.3). □

3. A Simple Proof of an Isometric Extension Result in Hilbert Space

Lemma 3.1. *Let E and F be normed spaces and let V_0 be an isometric operator mapping $S_1(E)$ into $S_1(F)$. If for any $\lambda \in \mathbf{R}$ and any $x,y \in S_1(E)$,*

$$\|V_0 x - |\lambda| V_0 y\| \le \|x - |\lambda| y\|, \tag{3.1}$$

then V_0 can be isometrically extended to the whole space. Furthermore, when V_0 is surjective, V_0 can be linearly and isometrically extended to the whole space.

Proof. Set

$$Vx = \begin{cases} \|x\| V_0\left(\dfrac{x}{\|x\|}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{3.2}$$

It is easy to see that $\|Vx - Vy\| \leq \|x - y\|$ for all $x, y \in E$. In particular, when $\|x\| = \|y\|$ either x or y is zero element, we have

$$\left\|Vx - Vy\right\| = \left\|x - y\right\|. \tag{3.3}$$

Thus, it suffices to prove (3.3) whenever $\|x\| > \|y\| > 0$.

Suppose, on the contrary, there exist $x_0, y_0 \in E$ such that $\|x_0\| > \|y_0\| > 0$ and $\|Vx_0 - Vy_0\| < \|x_0 - y_0\|$. Define a function on $\mathbf{R}$ by

$$\varphi(\lambda) = \left\|x_0 + \lambda(y_0 - x_0)\right\|. \tag{3.4}$$

The facts that $\varphi(\lambda)$ is a continuous function, $\varphi(1) = \|y_0\| < \|x_0\|$ and $\lim_{\lambda \to +\infty} \varphi(\lambda) = +\infty$ assure that there exists $\lambda_0 \in (1, +\infty)$ such that $\varphi(\lambda_0) = \|x_0\|$ (by the intermediate value theorem). Let $z_0 = x_0 + \lambda_0(y_0 - x_0)$. We see that x_0, y_0, and z_0 lie on a straight line and $\|z_0\| = \|x_0\|$. Hence

$$\begin{aligned} \|z_0 - x_0\| &= \left\|z_0 - y_0\right\| + \left\|y_0 - x_0\right\| \\ &> \left\|Vz_0 - Vy_0\right\| + \left\|Vx_0 - Vy_0\right\| \geq \|Vz_0 - Vx_0\| = \|z_0 - x_0\|, \end{aligned} \tag{3.5}$$

a contradiction. Thus V_0 can be isometrically extended to the whole space, and V is an extension of V_0.

If V_0 is surjective, then the conclusion follows easily from the Mazur-Ulam Theorem. □

Theorem 3.2. *Suppose that E and F are Hilbert spaces and V_0 is a surjective isometric operator mapping $S_1(E)$ onto $S_1(F)$. Then V_0 can be linearly and isometrically extended to the whole space.*

Proof. Since V_0 is an isometry, we have for all x, y in $S_1(E)$ that

$$\langle V_0(x) - V_0(y), V_0(x) - V_0(y) \rangle = \langle x - y, x - y \rangle, \tag{3.6}$$

that is,

$$2 - \langle V_0(x), V_0(y) \rangle - \langle V_0(y), V_0(x) \rangle = 2 - \langle x, y \rangle - \langle y, x \rangle, \tag{3.7}$$

and thus we have

$$\langle V_0(x), V_0(y) \rangle + \langle V_0(y), V_0(x) \rangle = \langle x, y \rangle + \langle y, x \rangle. \tag{3.8}$$

The last equality gives that

$$\begin{aligned}
&\langle V_0(x), V_0(x)\rangle - \lambda\langle V_0(x), V_0(y)\rangle - \lambda\langle V_0(y), V_0(x)\rangle + \lambda^2\langle V_0(y), V_0(y)\rangle \\
&\quad = 1 + \lambda^2 - \lambda\langle V_0(x), V_0(y)\rangle - \lambda\langle V_0(y), V_0(x)\rangle \\
&\quad = 1 + \lambda^2 - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle \\
&\quad = \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle.
\end{aligned} \tag{3.9}$$

Thus

$$\|V_0(x) - \lambda V_0(y)\| = \|x - \lambda y\| \tag{3.10}$$

holds for all λ in **R**. Now we can apply Lemma 3.1 to obtain the desired result. □

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