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Research Article

Stability of the Jensen-Type Functional Equation in C^*-Algebras: A Fixed Point Approach

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Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and Lie C^*-algebras and also of derivations on C^*-algebras and Lie C^*-algebras for the Jensen-type functional equation $f((x+y)/2) + f((x-y)/2) = f(x)$.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. In 1982, Rassias [6] followed the innovative approach of the Rassias' theorem [4] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 7–27]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [28, 29]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

By the using fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [17, 30–33]).

This paper is organized as follows: in Sections 2 and 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and of derivations on C^*-algebras for the Jensen-type functional equation.

In Sections 4 and 5, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras and of derivations on Lie C^*-algebras for the Jensen-type functional equation.

2. Stability of Homomorphisms in C^*-Algebras

Throughout this section, assume that A is a C^*-algebra with norm $\|\cdot\|_A$ and that B is a C^*-algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$D_\mu f(x,y) := \mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) - f(\mu x) \tag{2.1}$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ such that*

$$\left\| D_\mu f(x,y) \right\|_B \leq \varphi(x,y), \tag{2.2}$$

$$\left\| f(xy) - f(x)f(y) \right\|_B \leq \varphi(x,y), \tag{2.3}$$

$$\left\| f(x^*) - f(x)^* \right\|_B \leq \varphi(x,x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in A$. *If there exists an* $L < 1$ *such that* $\varphi(x, y) \le 2L\varphi(x/2, y/2)$ *for all* $x, y \in A$, *then there exists a unique* C^**-algebra homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\|_B \le \frac{L}{1-L}\varphi(x, 0) \tag{2.5}$$

for all $x \in A$.

Proof. It follows from $\varphi(x, y) \le 2L\varphi(x/2, y/2)$ that

$$\lim_{j \to \infty} 2^{-j}\varphi(2^j x, 2^j y) = 0 \tag{2.6}$$

for all $x, y \in A$.

Consider the set

$$X := \{g : A \longrightarrow B\} \tag{2.7}$$

and introduce the *generalized metric* on X:

$$d(g, h) = \inf\left\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \le C\varphi(x, 0),\ \forall x \in A\right\}. \tag{2.8}$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{2.9}$$

for all $x \in A$.

By [28, Theorem 3.1],

$$d(Jg, Jh) \le Ld(g, h) \tag{2.10}$$

for all $g, h \in X$.

Letting $\mu = 1$ and $y = 0$ in (2.2), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_B \le \varphi(x, 0) \tag{2.11}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_B \le \frac{1}{2}\varphi(2x, 0) \le L\varphi(x, 0) \tag{2.12}$$

for all $x \in A$. Hence $d(f, Jf) \le L$.

By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.13}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}. \tag{2.14}$$

This implies that H is a unique mapping satisfying (2.13) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x, 0) \tag{2.15}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.16}$$

for all $x \in A$.

(3) $d(f, H) \le (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, H) \le \frac{L}{1-L}. \tag{2.17}$$

This implies that the inequality (2.5) holds.

It follows from (2.2), (2.6), and (2.16) that

$$\begin{aligned} \left\| H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) - H(x) \right\|_B &= \lim_{n\to\infty} \frac{1}{2^n} \left\| f\left(2^{n-1}(x+y)\right) + f\left(2^{n-1}(x-y)\right) - f(2^n x) \right\|_B \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0 \end{aligned} \tag{2.18}$$

for all $x, y \in A$. So

$$H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) = H(x) \tag{2.19}$$

for all $x, y \in A$. Letting $z = (x+y)/2$ and $w = (x-y)/2$ in (2.19), we get

$$H(z) + H(w) = H(z+w) \tag{2.20}$$

for all $z, w \in A$. So the mapping $H : A \to B$ is Cauchy additive, that is, $H(z+w) = H(z) + H(w)$ for all $z, w \in A$.

Letting $y = x$ in (2.2), we get

$$\mu f(x) = f(\mu x) \tag{2.21}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By a similar method to above, we get

$$\mu H(x) = H(\mu x) \tag{2.22}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.3) that

$$\begin{aligned} \left\| H(xy) - H(x)H(y) \right\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \left\| f(4^n xy) - f(2^n x) f(2^n y) \right\|_B \\ &\le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0 \end{aligned} \tag{2.23}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y) \tag{2.24}$$

for all $x, y \in A$.

It follows from (2.4) that

$$\begin{aligned} \left\| H(x^*) - H(x)^* \right\|_B &= \lim_{n\to\infty} \frac{1}{2^n} \left\| f(2^n x^*) - f(2^n x)^* \right\|_B \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n x) = 0 \end{aligned} \tag{2.25}$$

for all $x \in A$. So

$$H(x^*) = H(x)^* \tag{2.26}$$

for all $x \in A$.

Thus $H : A \to B$ is a C^*-algebra homomorphism satisfying (2.5), as desired. □

Corollary 2.2. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\left\| D_\mu f(x, y) \right\|_B \le \theta \left(\|x\|_A^r + \|y\|_A^r \right), \tag{2.27}$$

$$\left\| f(xy) - f(x)f(y) \right\|_B \le \theta \left(\|x\|_A^r + \|y\|_A^r \right), \tag{2.28}$$

$$\left\| f(x^*) - f(x)^* \right\|_B \le 2\theta \|x\|_A^r \tag{2.29}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in A$. *Then there exists a unique* C^**-algebra homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\|_B \leq \frac{2^r \theta}{2 - 2^r} \|x\|_A^r \tag{2.30}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{2.31}$$

for all $x, y \in A$. Then $L = 2^{r-1}$ and we get the desired result. □

Theorem 2.3. *Let* $f : A \to B$ *be a mapping for which there exists a function* $\varphi : A^2 \to [0, \infty)$ *satisfying* (2.2), (2.3), *and* (2.4). *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leq (1/2)L\varphi(2x, 2y)$ *for all* $x, y \in A$, *then there exists a unique* C^**-algebra homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\|_B \leq \frac{L}{2 - 2L} \varphi(x, 0) \tag{2.32}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{2.33}$$

for all $x \in A$.

It follows from (2.11) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \leq \varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{2}\varphi(x, 0) \tag{2.34}$$

for all $x \in A$. Hence $d(f, Jf) \leq L/2$.

By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.35}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}. \tag{2.36}$$

This implies that H is a unique mapping satisfying (2.35) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x,0) \tag{2.37}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x) \tag{2.38}$$

for all $x \in A$.

(3) $d(f,H) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{L}{2-2L}, \tag{2.39}$$

which implies that the inequality (2.32) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.27), (2.28) *and* (2.29)*. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{\theta}{2^r - 2}\|x\|_A^r \tag{2.40}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{2.41}$$

for all $x, y \in A$. Then L= 2^{1-r} and we get the desired result. □

Theorem 2.5. *Let $f : A \to B$ be an odd mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (2.2), (2.3), (2.4) *and* (2.6)*. If there exists an $L < 1$ such that $\varphi(x,3x) \le 2L\varphi(x/2,3x/2)$ for all $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{1}{2-2L}\varphi(x,3x) \tag{2.42}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \longrightarrow B\} \tag{2.43}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\left\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x,3x),\ \forall x \in A\right\}. \tag{2.44}$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{2.45}$$

for all $x \in A$.

By [28, Theorem 3.1],

$$d(Jg, Jh) \leq Ld(g,h) \tag{2.46}$$

for all $g, h \in X$.

Letting $\mu = 1$ and relpacing y by $3x$ in (2.2), we get

$$\|f(2x) - 2f(x)\|_B \leq \varphi(x,3x) \tag{2.47}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_B \leq \frac{1}{2}\varphi(x,3x) \tag{2.48}$$

for all $x \in A$. Hence $d(f, Jf) \leq 1/2$.

By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.49}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \left\{g \in X : d(f,g) < \infty\right\}. \tag{2.50}$$

This implies that H is a unique mapping satisfying (2.49) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x,3x) \tag{2.51}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.52}$$

for all $x \in A$.

(3) $d(f, H) \le (1/(1-L)) d(f, Jf)$, which implies the inequality

$$d(f, H) \le \frac{1}{2-2L}. \tag{2.53}$$

This implies that the inequality (2.42) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $0 < r < 1/2$ and θ be nonnegative real numbers, and let $f : A \to B$ be an odd mapping such that*

$$\begin{gathered} \left\| D_\mu f(x,y) \right\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \\ \left\| f(xy) - f(x)f(y) \right\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \\ \left\| f(x^*) - f(x)^* \right\|_B \le \theta \|x\|_A^{2r} \end{gathered} \tag{2.54}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\left\| f(x) - H(x) \right\|_B \le \frac{3^r \theta}{2 - 2^{2r}} \|x\|_A^{2r} \tag{2.55}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.56}$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. □

Theorem 2.7. *Let $f : A \to B$ be an odd mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying (2.2), (2.3) and (2.4). If there exists an $L < 1$ such that $\varphi(x, 3x) \le (1/2)L\varphi(2x, 6x)$ for all $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\left\| f(x) - H(x) \right\|_B \le \frac{L}{2-2L} \varphi(x, 3x) \tag{2.57}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{2.58}$$

for all $x \in A$.

It follows from (2.47) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_B \leq \varphi\left(\frac{x}{2}, \frac{3x}{2}\right) \leq \frac{L}{2}\varphi(x, 3x) \tag{2.59}$$

for all $x \in A$. Hence $d(f, Jf) \leq L/2$.

By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.60}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}. \tag{2.61}$$

This implies that H is a unique mapping satisfying (2.60) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, 3x) \tag{2.62}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x) \tag{2.63}$$

for all $x \in A$.

(3) $d(f, H) \leq (1/(1 - L))d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{L}{2 - 2L}, \tag{2.64}$$

which implies that the inequality (2.57) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be an odd mapping satisfying* (2.54). *Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\left\| f(x) - H(x) \right\|_B \le \frac{\theta}{2^{2r} - 2} \|x\|_A^{2r} \tag{2.65}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.7 by taking

$$\varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.66}$$

for all $x, y \in A$. Then $L= 2^{1-2r}$ and we get the desired result. □

3. Stability of Derivations on C^*-Algebras

Throughout this section, assume that A is a C^*-algebra with norm $\| \cdot \|_A$.

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on C^*-algebras for the functional equation $D_\mu f(x, y) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that*

$$\left\| D_\mu f(x, y) \right\|_A \le \varphi(x, y), \tag{3.1}$$

$$\left\| f(xy) - f(x)y - xf(y) \right\|_A \le \varphi(x, y) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, y) \le 2L\varphi(x/2, y/2)$ for all $x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\left\| f(x) - \delta(x) \right\|_A \le \frac{L}{1 - L} \varphi(x, 0) \tag{3.3}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{3.4}$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned}\left\|\delta(xy)-\delta(x)y-x\delta(y)\right\|_A &= \lim_{n\to\infty}\frac{1}{4^n}\left\|f(4^nxy)-f(2^nx)\cdot 2^ny-2^nxf(2^ny)\right\|_A \\ &\le \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny)\le\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,2^ny)=0\end{aligned}\tag{3.5}$$

for all $x, y \in A$. So

$$\delta(xy)=\delta(x)y+x\delta(y)\tag{3.6}$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 3.2. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\left\|D_\mu f(x,y)\right\|_A\le\theta\left(\|x\|_A^r+\|y\|_A^r\right),\tag{3.7}$$

$$\left\|f(xy)-f(x)y-xf(y)\right\|_A\le\theta\left(\|x\|_A^r+\|y\|_A^r\right)\tag{3.8}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\left\|f(x)-\delta(x)\right\|_A\le\frac{2^r\theta}{2-2^r}\|x\|_A^r\tag{3.9}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y):=\theta\left(\|x\|_A^r+\|y\|_A^r\right)\tag{3.10}$$

for all $x, y \in A$. Then $L= 2^{r-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (3.1) *and* (3.2). *If there exists an $L < 1$ such that $\varphi(x,y) \le (1/2)L\varphi(2x,2x)$ for all $x, y \in A$, then there exists a unique derivation $\delta : A \to A$ such that*

$$\left\|f(x)-\delta(x)\right\|_A\le\frac{L}{2-2L}\varphi(x,0)\tag{3.11}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

Corollary 3.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.7) *and* (3.8). *Then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{\theta}{2^r - 2}\|x\|_A^r \tag{3.12}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{3.13}$$

for all $x, y \in A$. Then $L= 2^{1-r}$ and we get the desired result. □

Remark 3.5. For inequalities controlled by the product of powers of norms, one can obtain similar results to Theorems 2.5 and 2.7 and Corollaries 2.6 and 2.8.

4. Stability of Homomorphisms in Lie C^*-Algebras

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := (xy - yx)/2$ on $\mathcal{C}$, is called a *Lie C^*-algebra* (see [13–15]).

Definition 4.1. Let A and B be Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a Lie C^*-algebra homomorphism if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in A$.

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$ and that B is a Lie C^*-algebra with norm $\|\cdot\|_B$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras for the functional equation $D_\mu f(x, y) = 0$.

Theorem 4.2. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (2.2) *such that*

$$\|f([x, y]) - [f(x), f(y)]\|_B \leq \varphi(x, y) \tag{4.1}$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, y) \leq 2L\varphi(x/2, y/2)$ for all $x, y \in A$, then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying (2.5).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (2.5). The mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{4.2}$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned}\left\|H([x,y]) - [H(x),H(y)]\right\|_B &= \lim_{n\to\infty}\frac{1}{4^n}\left\|f(4^n[x,y]) - [f(2^nx),f(2^ny)]\right\|_B\\ &\le \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny) \le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,2^ny) = 0\end{aligned} \tag{4.3}$$

for all $x,y \in A$. So

$$H([x,y]) = [H(x),H(y)] \tag{4.4}$$

for all $x,y \in A$.

Thus $H : A \to B$ is a Lie C^*-algebra homomorphism satisfying (2.5), as desired. □

Corollary 4.3. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.27) *such that*

$$\left\|f([x,y]) - [f(x),f(y)]\right\|_B \le \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{4.5}$$

for all $x,y \in A$. Then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying (2.30).

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x,y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{4.6}$$

for all $x,y \in A$. Then $L = 2^{r-1}$ and we get the desired result. □

Theorem 4.4. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (2.2) *and* (4.1). *If there exists an $L < 1$ such that $\varphi(x,y) \le (1/2)L\varphi(2x,2y)$ for all $x,y \in A$, then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.32).

Proof. The proof is similar to the proofs of Theorems 2.3 and 4.2. □

Corollary 4.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.27) *and* (4.5). *Then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.40).

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x,y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{4.7}$$

for all $x,y \in A$. Then $L = 2^{1-r}$ and we get the desired result. □

Remark 4.6. For inequalities controlled by the product of powers of norms, one can obtain similar results to Theorems 2.5 and 2.7 and Corollaries 2.6 and 2.8.

5. Stability of Lie Derivations on C^*-Algebras

Definition 5.1. Let A be a Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a Lie derivation if $\delta([x,y]) = [\delta(x),y] + [x,\delta(y)]$ for all $x,y \in A$.

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$.

We prove the generalized Hyers-Ulam stability of derivations on Lie C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 5.2. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (3.1) *such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \le \varphi(x,y) \tag{5.1}$$

for all $x,y \in A$. If there exists an $L < 1$ such that $\varphi(x,y) \le 2L\varphi(x/2,y/2)$ for all $x,y \in A$. Then there exists a unique Lie derivation $\delta : A \to A$ satisfying (3.3).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{5.2}$$

for all $x \in A$.

It follows from (5.1) that

$$\begin{aligned}
&\|\delta([x,y]) - [\delta(x),y] - [x,\delta(y)]\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n[x,y]) - [f(2^n x),2^n y] - [2^n x, f(2^n y)]\|_A \\
&\quad \le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x,2^n y) \le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x,2^n y) = 0
\end{aligned} \tag{5.3}$$

for all $x,y \in A$. So

$$\delta([x,y]) = [\delta(x),y] + [x,\delta(y)] \tag{5.4}$$

for all $x,y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 5.3. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.7) *such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \le \theta(\|x\|_A^r + \|y\|_A^r) \tag{5.5}$$

for all $x,y \in A$. Then there exists a unique Lie derivation $\delta : A \to A$ satisfying (3.9).

Proof. The proof follows from Theorem 5.2 by taking

$$\varphi(x,y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{5.6}$$

for all $x, y \in A$. Then $L = 2^{r-1}$ and we get the desired result. □

Theorem 5.4. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (3.1) *and* (5.1). *If there exists an $L < 1$ such that $\varphi(x,y) \le (1/2)L\varphi(2x,2y)$ for all $x, y \in A$, then there exists a unique Lie derivation $\delta : A \to A$ satisfying* (3.11).

Proof. The proof is similar to the proofs of Theorems 2.3 and 5.2. □

Corollary 5.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.7) *and* (5.5). *Then there exists a unique Lie derivation $\delta : A \to A$ satisfying* (3.12).

Proof. The proof follows from Theorem 5.4 by taking

$$\varphi(x,y) := \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{5.7}$$

for all $x, y \in A$. Then $L = 2^{1-r}$ and we get the desired result. □

Remark 5.6. For inequalities controlled by the product of powers of norms, one can obtain similar results to Theorems 2.5 and 2.7 and Corollaries 2.6 and 2.8.

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