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Research Article

A Functional Equation Originating from Elliptic Curves

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We obtain the general solution and the stability of the functional equation $f(x+y+z,u+v+w)+f(x+y-z,u+v+w)+2f(x,u-w)+2f(y,v-w) = f(x+y,u+w)+f(x+y,v+w)+f(x+z,u+w)+f(x-z,u+v-w)+f(y+z,v+w)+f(y-z,u+v-w)$. The function $f(x,y)=x^3+ax+b-y^2$ having level curves as elliptic curves is a solution of the above functional equation.

1. Introduction

A graph of an equation of the form $y^2 = x^3 + ax + b$ is calledan *elliptic curve* [1], where a and b are constants. Since $f(x,y) = x^3 + ax + b - y^2$ has level curves as elliptic curves, functional equations having the mapping $f(x,y) = x^3 + ax + b - y^2$ as a solution are helpful to study cryptography and their applications.

Recently, Jun and Kim [2] solved the cubic functional equation

$$\begin{aligned} &g(x+y+z)+g(x+y-z)+2g(x)+2g(y)\\ &\quad = 2g(x+y)+g(x+z)+g(x-z)+g(y+z)+g(y-z). \end{aligned} \tag{1.1}$$

We consider the quadratic functional equation

$$g(x+y+z)+g(x-z)+g(y-z)=g(x+y-z)+g(x+z)+g(y+z), \tag{1.2}$$

and the 2-dimensional vector variable functional equation

$$\begin{aligned} & f(x+y+z,u+v+w)+f(x+y-z,u+v+w)+2f(x,u-w)+2f(y,v-w) \\ & \quad = f(x+y,u+w)+f(x+y,v+w)+f(x+z,u+w) \\ & \qquad + f(x-z,u+v-w)+f(y+z,v+w)+f(y-z,u+v-w). \end{aligned} \tag{1.3}$$

The function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x):=ax^3+bx+c$ is a particular solution of (1.1), and the function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x):=ax^2+b$ is a particular solution of (1.2). The function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y):=x^3+ax+b-y^2$ is a particular solution of (1.3). The functional (1.3) is a mixed type of (1.1) and (1.2).

Problems on solutions or stability of various functional equations have been extensively investigated by a number of mathematicians [3–12].

In this paper, let X, Y, and Z be real vector spaces. We find out the general solution and investigate the stability of (1.2) and (1.3).

2. Solutions

We find out the general solution of the functional (1.2) as follows.

Lemma 2.1. *A mapping $g : X \to Y$ satisfies* (1.2) *if and only if there exists a symmetric biadditive mapping $S : X\times X \to Y$ such that*

$$g(x)=S(x,x)+g(0) \tag{2.1}$$

for all $x\in X$.

Proof. We first assume that g is a solution of (1.2). Let $h : X \to Y$ be a mapping given by $h(x):=g(x)-g(0)$ for all $x\in X$. Then, $h(0)=0$ and

$$h(x+y+z)+h(x-z)+h(y-z)=h(x+y-z)+h(x+z)+h(y+z) \tag{2.2}$$

for all $x,y,z\in X$. Putting $x=y=0$ in (2.2), h is even. Replacing z by $x+y$ in (2.2),

$$h(2x+2y)+h(y)+h(x)=h(2x+y)+h(x+2y) \tag{2.3}$$

for all $x,y\in X$. Taking $z=x$ in (2.2), we get that

$$h(2x+y)+h(x-y)=h(y)+h(2x)+h(x+y) \tag{2.4}$$

for all $x,y\in X$. Interchanging x and y in (2.4), we see that

$$h(x+2y)+h(x-y)=h(x)+h(2y)+h(x+y) \tag{2.5}$$

for all $x,y\in X$. By (2.3), (2.4), and (2.5), we obtain that

$$\begin{aligned} h(2x+2y) &= h(2x+y)-h(y)+h(x+2y)-h(x) \\ &= h(2x)+h(x+y)-h(x-y)+h(2y)+h(x+y)-h(x-y) \\ &= h(2x)+h(2y)+2h(x+y)-2h(x-y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Letting $y = x$ in (2.6), we get that

$$h(2x) = 4h(x) \tag{2.7}$$

for all $x \in X$. By (2.6) and (2.7), we obtain that

$$h(x+y) + h(x-y) = 2h(x) + 2h(y) \tag{2.8}$$

for all $x, y \in X$. By [13], there exists a symmetric biadditive mapping $S : X \times X \to Y$ such that $h(x) = S(x, x)$ for all $x \in X$.

Conversely, we assume that there exists a symmetric biadditive mapping $S : X \times X \to Y$ such that $g(x) = S(x, x) + g(0)$ for all $x \in X$. Since S is biadditive,

$$\begin{aligned} & g(x+y+z) + g(x-z) + g(y-z) \\ &\quad = S(x+y+z, x+y+z) + S(x-z, x-z) + S(y-z, y-z) + 3g(0) \\ &\quad = 2\big[S(x,x) + S(x,y) + S(y,y)\big] + 3S(z,z) + 3g(0) \\ &\quad = S(x+y-z, x+y-z) + S(x+z, x+z) + S(y+z, y+z) + 3g(0) \\ &\quad = g(x+y-z) + g(x+z) + g(y+z) \end{aligned} \tag{2.9}$$

for all $x, y, z \in X$. □

Example 2.2. Let r/s be a rational number, where r and s are integers with $\gcd(r, s) = 1$. Define $H(r/s) := \max\{|r|, |s|\}$ and $g(r/s) := \log H(r/s)$. Let $\mathbb{Q}$ be the set of rational numbers and let $E(\mathbb{Q}) := \{\infty\} \cup \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \mid y^2 = x^3 + ax + b\}$ be an elliptic curve over $\mathbb{Q}$. The addition in $E(\mathbb{Q})$ is given in [1]. The height function $h : E(\mathbb{Q}) \to [0, \infty)$ is defined by

$$h(x, y) = g(x), \quad h(\infty) = 0 \tag{2.10}$$

for all $(x, y) \in E(\mathbb{Q})$. The canonical height function $\widehat{h} : E(\mathbb{Q}) \to [0, \infty)$ given by

$$\widehat{h}(P) = \frac{1}{2} \lim_{n\to\infty} \frac{1}{4^n} h(2^n P) \tag{2.11}$$

is a solution of (1.2).

We find out the general solution of the functional (1.3) as follows.

Theorem 2.3. *A mapping $f : X \times Y \to Z$ satisfies* (1.3) *if and only if there exist a symmetric multi-additive mapping $S_1 : X \times X \times X \to Z$, an additive mapping $A : X \to Z$, and a symmetric biadditive mapping $S_2 : Y \times Y \to Z$ such that*

$$f(x, y) = S_1(x, x, x) + A(x) + f(0, 0) + S_2(y, y) \tag{2.12}$$

for all $x \in X$ and all $y \in Y$.

Proof. We first assume that f satisfies (1.3). Define $f_1 : X \to Z$ and $f_2 : Y \to Z$ by $f_1(x) := f(x,0)$ for all $x \in X$ and $f_2(y) := f(0,y)$ for all $y \in Y$. Then, f_1 satisfies (1.1) and f_2 satisfies (1.2). By [2], there exist a symmetric multiadditive mapping $S_1 : X\times X\times X \to Z$ and an additive mapping $A : X \to Z$ such that

$$f_1(x) = S_1(x,x,x) + A(x) + f_1(0) \tag{2.13}$$

for all $x \in X$. By Lemma 2.1, there exists a symmetric biadditive mapping $S_2 : Y \times Y \to Z$ such that

$$f_2(y) = S_2(y,y) + f_2(0) \tag{2.14}$$

for all $y \in Y$. Note that $f_1(0) = f_2(0) = f(0,0)$ and that $f(0,0) \neq 0$ in general. Let $k : X \times Y \to Z$ be a mapping given by $k(x,y) := f(x,y) - f(0,0)$ for all $x \in X$ and all $y \in Y$. Then, $k(0,0) = 0$ and

$$\begin{aligned} k(x+y+z,u+v+w) &+ k(x+y-z,u+v+w) + 2k(x,u-w) + 2k(y,v-w) \\ &= k(x+y,u+w) + k(x+y,v+w) + k(x+z,u+w) \\ &\quad + k(x-z,u+v-w) + k(y+z,v+w) + k(y-z,u+v-w) \end{aligned} \tag{2.15}$$

for all $x,y,z \in X$ and all $u,v,w \in Y$.

Since $k(0,u) = S_2(u,u)$ for all $u \in Y$, $k(0,-u) = k(0,u)$ and $k(0,2u) = 4k(0,u)$ for all $u \in Y$. Letting $y = z = u = v = 0$ in (2.15), we get

$$k(x,w) = k(x,-w) \tag{2.16}$$

for all $x \in X$ and all $w \in Y$. Putting $y = z = 0$ in (2.15), we get

$$\begin{aligned} 2k(x,u+v+w) &+ 2k(x,u-w) + 2k(0,v-w) \\ &= 2k(x,u+w) + k(x,v+w) + k(0,v+w) + k(x,u+v-w) + k(0,u+v-w) \end{aligned} \tag{2.17}$$

for all $x \in X$ and all $u,v,w \in Y$. Setting $v = w = -u$ in (2.17) and using (2.16),

$$k(x,u) + k(x,2u) = 2k(x,0) + 5k(0,u) \tag{2.18}$$

for all $x \in X$ and all $u \in Y$. Taking $v = -w = -u$ in (2.17) and using (2.16),

$$k(x,u) + k(x,0) + 7k(0,u) = 2k(x,2u) \tag{2.19}$$

for all $x \in X$ and all $u \in Y$. By (2.18) and (2.19), $k(x,u) = k(x,0) + k(0,u)$ for all $x \in X$ and all $u \in Y$. Hence, we obtain that

$$\begin{aligned} f(x,y) &= k(x,y) + f(0,0) \\ &= k(x,0) + k(0,y) + f(0,0) \\ &= f_1(x) - f_1(0) + f_2(y) - f_2(0) + f(0,0) \\ &= S_1(x,x,x) + A(x) + f(0,0) + S_2(y,y) \end{aligned} \tag{2.20}$$

for all $x \in X$ and all $y \in Y$.

Conversely, we assume that there exist a symmetric multiadditive mapping $S_1 : X \times X \times X \to Z$, an additive mapping $A : X \to Z$, and a symmetric biadditive mapping $S_2 : Y \times Y \to Z$ such that

$$f(x,y) = S_1(x,x,x) + A(x) + f(0,0) + S_2(y,y) \tag{2.21}$$

for all $x \in X$ and all $y \in Y$. Since S_1 is symmetric multiadditive and S_2 is symmetric biadditive,

$$\begin{aligned}
f(x&+y+z,u+v+w) + f(x+y-z,u+v+w) + 2f(x,u-w) + 2f(y,v-w) \\
&= S_1(x+y+z,x+y+z,x+y+z) + S_1(x+y-z,x+y-z,x+y-z) \\
&\quad + 2S_1(x,x,x) + 2S_1(y,y,y) + A(x+y+z) + A(x+y-z) \\
&\quad + 2A(x) + 2A(y) + 2S_2(u+v+w,u+v+w) \\
&\quad + 2S_2(u-w,u-w) + 2S_2(v-w,v-w) + 6f(0,0) \\
&= 4S_1(x,x,x) + 4S_1(y,y,y) + 6S_1(x,x,y) + 6S_1(x,y,y) + 6S_1(x,z,z) \\
&\quad + 6S_1(y,z,z) + 4A(x) + 4A(y) + 4S_2(u,u) + 4S_2(u,v) + 4S_2(v,v) \\
&\quad + 6S_2(w,w) + 6f(0,0) \qquad\qquad (2.22) \\
&= 2S_1(x+y,x+y,x+y) + S_1(x+z,x+z,x+z) + S_1(x-z,x-z,x-z) \\
&\quad + S_1(y+z,y+z,y+z) + S_1(y-z,y-z,y-z) \\
&\quad + 2A(x+y) + A(x+z) + A(x-z) + A(y+z) + A(y-z) \\
&\quad + 2S_2(u+w,u+w) + 2S_2(v+w,v+w) + 2S_2(u+v-w,u+v-w) \\
&\quad + 6f(0,0) \\
&= f(x+y,u+w) + f(x+y,v+w) + f(x+z,u+w) + f(x-z,u+v-w) \\
&\quad + f(y+z,v+w) + f(y-z,u+v-w)
\end{aligned}$$

for all $x,y,z \in X$ and all $u,v,w \in Y$. □

Example 2.4. Let X and Y be the vector spaces $M_3(\mathbb{C})$ and $M_2(\mathbb{C})$, respectively. Consider a function $f : X \times Y \to \mathbb{C}$ given by $f(x,y) := \det(x) + \det(y)$ for all $x \in X$ and all $y \in Y$. One can easily verify that f satisfies (1.3). By Theorem 2.3, there exist a symmetric multiadditive mapping $S_1 : X \times X \times X \to \mathbb{C}$, an additive mapping $A : X \to \mathbb{C}$, and a symmetric biadditive mapping $S_2 : Y \times Y \to \mathbb{C}$ such that

$$f(x,y) = S_1(x,x,x) + A(x) + f(0,0) + S_2(y,y) \tag{2.23}$$

for all $x \in X$ and all $y \in Y$. In fact,

$$\begin{aligned}
S_1(x,y,z) &= \frac{1}{24}\left[\det(x+y+z) + \det(x-y-z) - \det(x+y-z) - \det(x-y+z)\right], \\
A(x) &= f(0,0) = 0, \qquad\qquad (2.24) \\
S_2(u,v) &= \frac{1}{4}\left[\det(u+v) - \det(u-v)\right]
\end{aligned}$$

for all $x,y,z \in X$ and all $u,v \in Y$

3. Stability

From now on, let Y be a Banach space, and let $\varphi : X^3 \to [0,\infty)$ be a function satisfying

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{3.1}$$

for all $x, y, z \in X$.

Lemma 3.1. *Let $g : X \to Y$ be a mapping such that*

$$\|g(x+y+z) + g(x-z) + g(y-z) - g(x+y-z) - g(x+z) - g(y+z)\| \le \varphi(x,y,z) \tag{3.2}$$

for all $x, y, z \in X$. Then, there exists a unique quadratic mapping $G : X \to Y$ satisfying (1.2) such that

$$\|g(x) - G(x)\| \le \widetilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}, x\right) + 2\widetilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + 2\widetilde{\varphi}\left(0, 0, \frac{x}{2}\right) + \frac{1}{3}[\varphi(0,0,0) + \|g(0)\|] \tag{3.3}$$

for all $x \in X$. The mapping G is given by $G(x) := \lim_{j\to\infty} \frac{1}{4^j} g(2^j x)$ for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.2), we have

$$\|g(-z) - g(z)\| \le \varphi(0,0,z) \tag{3.4}$$

for all $z \in X$.

Replacing z by $x + y$ in (3.2), we get

$$\|g(2x+2y) + g(-y) + g(-x) - g(0) - g(2x+y) - g(x+2y)\| \le \varphi(x,y,x+y) \tag{3.5}$$

for all $x, y \in X$. Taking $z = x$ in (3.2), we see that

$$\|g(2x+y) + g(0) + g(y-x) - g(y) - g(2x) - g(x+y)\| \le \varphi(x,y,x) \tag{3.6}$$

for all $x, y \in X$. Interchanging x and y in (3.6), we see that

$$\|g(x+2y) + g(0) + g(x-y) - g(x) - g(2y) - g(x+y)\| \le \varphi(y,x,x) \tag{3.7}$$

for all $x, y \in X$. By (3.5), (3.6), and (3.7),

$$\begin{aligned}
&\|g(2x+2y) - 2g(x+y) - g(2x) - g(2y)\| \\
&\quad \le \varphi(x,y,x+y) + \varphi(x,y,x) + \varphi(y,x,x) \\
&\qquad + \|g(-x) + g(x)\| + \|g(-y) - g(y)\| + \|g(y-x) - g(x-y)\| + \|g(0)\| \\
&\quad \le \varphi(x,y,x+y) + \varphi(x,y,x) + \varphi(y,x,x) + \varphi(0,0,x) + \varphi(0,0,y) + \varphi(0,0,x-y) + \|g(0)\|
\end{aligned} \tag{3.8}$$

for all $x, y \in X$. Putting $y = x$ in the above inequality, we get

$$\|g(4x) - 4g(2x)\| \le \varphi(x,x,2x) + 2\varphi(x,x,x) + 2\varphi(0,0,x) + \varphi(0,0,0) + \|g(0)\| \tag{3.9}$$

for all $x \in X$. So,

$$\left\| g(x) - \frac{1}{4}g(2x) \right\| \leq \frac{1}{4}\varphi^*(x), \tag{3.10}$$

where

$$\varphi^*(x) := \varphi\left(\frac{x}{2}, \frac{x}{2}, x\right) + 2\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + 2\varphi\left(0, 0, \frac{x}{2}\right) + \varphi(0,0,0) + \|g(0)\| \tag{3.11}$$

for all $x \in X$.

Thus, we obtain

$$\left\| \frac{1}{4^j}g(2^j x) - \frac{1}{4^{j+1}}g(2^{j+1}x) \right\| \leq \frac{1}{4^{j+1}}\varphi^*(2^j x) \tag{3.12}$$

for all $x \in X$ and all j. For given integers l, m $(0 \leq l < m)$, we get

$$\left\| \frac{1}{4^l}g(2^l x) - \frac{1}{4^m}g(2^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\varphi^*(2^j x) \tag{3.13}$$

for all $x \in X$. By (3.13), the sequence $\{(1/4^j)g(2^j x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(1/4^j)g(2^j x)\}$ converges for all $x \in X$. Define $G : X \to Y$ by

$$G(x) := \lim_{j\to\infty} \frac{1}{4^j}g(2^j x) \tag{3.14}$$

for all $x \in X$. By (3.2), we have

$$\begin{aligned} &\Big\| \frac{1}{4^j}\big[g(2^j(x+y+z)) + g(2^j(x-z)) + g(2^j(y-z)) - g(2^j(x+y-z)) \\ &\qquad - g(2^j(x+z)) - g(2^j(y+z))\big] \Big\| \leq \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) \end{aligned} \tag{3.15}$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ and using (3.1), we see that G satisfies (1.2). Setting $l = 0$ and taking $m \to \infty$ in (3.13), one can obtain the inequality (3.3). If $H : X \to Y$ is another quadratic mapping satisfying (1.2) and (3.3), we obtain

$$\begin{aligned} \|G(x) - H(x)\| &= \frac{1}{4^n}\|G(2^n x) - H(2^n x)\| \\ &\leq \frac{1}{4^n}\|G(2^n x) - g(2^n x)\| + \frac{1}{4^n}\|g(2^n x) - H(2^n x)\| \\ &\leq \frac{2}{4^n}\big[\tilde{\varphi}(2^{n-1}x, 2^{n-1}x, 2^n x) + 2\tilde{\varphi}(2^{n-1}x, 2^{n-1}x, 2^{n-1}x) + 2\tilde{\varphi}(0, 0, 2^{n-1}x) \\ &\qquad + \frac{1}{3}\big[\varphi(0,0,0) + \|g(0)\|\big]\big] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{3.16}$$

for all $x \in X$. Hence, G is a unique quadratic mapping, as desired. □

Example 3.2. If the height function h in Example 2.2 satisfies

$$\left\| h(P+Q+R)+h(P-R)+h(Q-R)-h(P+Q-R)-h(P+R)-h(Q+R) \right\| \le c \tag{3.17}$$

for all $P, Q, R \in E(\mathbb{Q})$, then there exists a unique quadratic function $\widetilde{h} : E(\mathbb{Q}) \to [0,\infty)$ satisfying (1.2) such that

$$\left\| h(P)-\widetilde{h}(P) \right\| \le 2c \tag{3.18}$$

for all $P \in E(\mathbb{Q})$.

Let $\varphi : X^3 \times Y^3 \to [0,\infty)$ be a function satisfying

$$\widetilde{\varphi}(x,y,z,u,v,w) := \sum_{j=0}^{\infty} \left[\frac{1}{2^{j+1}} \varphi_1\left(2^j x, 2^j y, 2^j z\right) + \frac{1}{4^{j+1}} \varphi_2\left(2^j u, 2^j v, 2^j w\right) \right] < \infty, \tag{3.19}$$

where

$$\varphi_1(x,y,z) := \varphi(x,y,z,0,0,0), \qquad \varphi_2(u,v,w) := \varphi(0,0,0,u,v,w) \tag{3.20}$$

for all $x, y, z \in X$ and all $u, v, w \in Y$. Also, let $\widetilde{\varphi}_1 : X^3 \to [0,\infty)$, and $\widetilde{\varphi}_2 : Y^3 \to [0,\infty)$ be two functions given by

$$\widetilde{\varphi}_1(x,y,z) := \widetilde{\varphi}(x,y,z,0,0,0), \qquad \widetilde{\varphi}_2(u,v,w) := \widetilde{\varphi}(0,0,0,u,v,w) \tag{3.21}$$

for all $x, y, z \in X$ and all $u, v, w \in Y$.

Theorem 3.3. *Let $f : X \times Y \to Z$ be a mapping such that*

$$\begin{aligned} &\| f(x+y+z,u+v+w)+f(x+y-z,u+v+w)+2f(x,u-w)+2f(y,v-w) \\ &\quad - f(x+y,u+w)-f(x+y,v+w)-f(x+z,u+w)-f(x-z,u+v-w) \\ &\quad - f(y+z,v+w)-f(y-z,u+v-w) \| \\ &\le \varphi(x,y,z,u,v,w) \end{aligned} \tag{3.22}$$

for all $x, y, z \in X$ and all $u, v, w \in Y$. Then, there exists a mapping $F : X \times Y \to Z$ satisfying (1.3) *such that*

$$\begin{aligned} \| f(x,y)-F(x,y) \| \le{}& \varphi(x,0,0,y,0,0)+\widetilde{\varphi}_1\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)+\frac{1}{3}\widetilde{\varphi}_1\left(x,\frac{x}{2},\frac{x}{2}\right) \\ &+\widetilde{\varphi}_2\left(\frac{y}{2},\frac{y}{2},y\right)+2\widetilde{\varphi}_2\left(\frac{y}{2},\frac{y}{2},\frac{y}{2}\right)+2\widetilde{\varphi}_2\left(0,0,\frac{y}{2}\right) \\ &+\frac{1}{3}\left[\varphi_2(0,0,0)+\|f(0,0)\|\right] \end{aligned} \tag{3.23}$$

for all $x \in X$ and all $y \in Y$. The mapping F is given by

$$F(x,y) := \lim_{j\to\infty} \left[\frac{4^{j+1}-1}{3\cdot 8^j} f\left(2^j x,0\right) - \frac{4^j-1}{6\cdot 8^j} f\left(2^{j+1}x,0\right) + \frac{1}{4^j} f\left(0,2^j y\right) \right] - f(0,0) \tag{3.24}$$

for all $x \in X$ and all $y \in Y$.

Proof. Letting $y = z = v = w = 0$ in (3.22), we have

$$\|f(x,u) - f(x,0) - f(0,u) + f(0,0)\| \leq \varphi(x,0,0,u,0,0) \tag{3.25}$$

for all $x \in X$ and all $u \in Y$.

Define $f_1, f_2 : X \to Y$ by $f_1(x) := f(x,0)$ and $f_2(u) := f(0,u)$ for all $x \in X$ and all $u \in Y$. Putting $u = v = w = 0$ in (3.22), we have

$$\begin{aligned}&\|f_1(x+y+z) + f_1(x+y-z) + 2f_1(x) + 2f_1(y)\\ &\quad - 2f_1(x+y) - f_1(x+z) - f_1(x-z) - f_1(y+z) - f_1(y-z)\|\\ &\leq \varphi_1(x,y,z)\end{aligned} \tag{3.26}$$

for all $x, y, z \in X$. Setting $x = y = z = 0$ in (3.22), we have

$$\|f_2(u+v+w) + f_2(u-w) + f_2(v-w) - f_2(u+v-w) - f_2(u+w) - f_2(v+w)\| \leq \varphi_2(u,v,w) \tag{3.27}$$

for all $u, v, w \in Y$.

By [2], there exists a cubic mapping $F_1 : X \to Z$ satisfying (1.1) such that

$$\|f_1(x) - F_1(x)\| \leq \widetilde{\varphi}_1\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \frac{1}{3}\widetilde{\varphi}_1\left(x, \frac{x}{2}, \frac{x}{2}\right) \tag{3.28}$$

for all $x \in X$. By Lemma 3.1, there exists a quadratic mapping $F_2 : Y \to Z$ satisfying (1.2) such that

$$\|f_2(y) - F_2(y)\| \leq \widetilde{\varphi}_2\left(\frac{y}{2}, \frac{y}{2}, y\right) + 2\widetilde{\varphi}_2\left(\frac{y}{2}, \frac{y}{2}, \frac{y}{2}\right) + 2\widetilde{\varphi}_2\left(0, 0, \frac{y}{2}\right) + \frac{1}{3}\left[\varphi_2(0,0,0) + \|f(0,0)\|\right] \tag{3.29}$$

for all $y \in Y$.

If we define

$$F(x,y) := F_1(x) + F_2(y) - f(0,0) \tag{3.30}$$

for all $x \in X$ and all $y \in Y$, we conclude that

$$\begin{aligned}\|f(x,y) - F(x,y)\| &\leq \varphi(x,0,0,y,0,0) + \widetilde{\varphi}_1\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \frac{1}{3}\widetilde{\varphi}_1\left(x, \frac{x}{2}, \frac{x}{2}\right)\\ &\quad + \widetilde{\varphi}_2\left(\frac{y}{2}, \frac{y}{2}, y\right) + 2\widetilde{\varphi}_2\left(\frac{y}{2}, \frac{y}{2}, \frac{y}{2}\right) + 2\widetilde{\varphi}_2\left(0, 0, \frac{y}{2}\right)\\ &\quad + \frac{1}{3}\left[\varphi_2(0,0,0) + \|f(0,0)\|\right]\end{aligned} \tag{3.31}$$

for all $x \in X$ and all $y \in Y$. □

Let $\varphi' : X^3 \to [0, \infty)$ be a function satisfying

$$\sum_{j=0}^{\infty} 4^{j+1}\varphi'(2^{-j}x, 2^{-j}y, 2^{-j}z) < \infty \tag{3.32}$$

for all $x, y, z \in X$. For the function φ', we can obtain similar results to Lemma 3.1 and Theorem 3.3.

As a corollary, one can obtain a result when the control mapping is a summation of terms $\|\cdot\|^p$, in which p is a suitable constant by referring to [14].

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