Thirteenth International Conference on
Geometry, Integrability and Quantization
June 3–8, 2011, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2012, pp 233–244

COHOMOGENEITY TWO RIEMANNIAN MANIFOLDS OF NON-POSITIVE CURVATURE

REZA MIRZAIE

Department of Mathematics, Imam Khomeini International University, Qazvin, Iran

Abstract. We consider a Riemannian manifold M ($\dim M \geq 3$), which is flat or has negative sectional curvature. We suppose that there is a closed and connected subgroup G of $\mathrm{Iso}(M)$ such that $\dim(M/G) = 2$. Then we study some topological properties of M and the orbits of the action of G on M.

1. Introduction

Let M^n be a connected and complete Riemannian manifold of dimension n, and G be a closed and connected subgroup of the Lie group of all isometries of M. If $x \in M$ then we denote by $G(x) = \{gx\ ;\ g \in G\}$ the orbit containing x.

If $\max\{\dim G(x)\ ;\ x \in M\} = n - k$, then M is called a ***C_k-G*-manifold** (G-manifold of cohomogeneity k) and we will denote it by $\mathrm{Coh}(G, M) = k$. If M is a C_k-G-manifold then the orbit space $M/G = \{G(x)\ ;\ x \in M\}$ is a topological space of dimension k. When k is small, we expect close relations between topological properties of M and the orbits of the action of G on M. If M is a C_0-G-manifold then the action of G on M is transitive, so M is a homogeneous G-manifold and it is diffeomorphic to G/G_x (where $x \in M$ and $G_x = \{g \in G\ ;\ gx = x\}$). Thus, topological properties of homogeneous G-manifolds are closely related to Lie group theory. If M is a homogeneous G-manifold of non-positive curvature, it is diffeomorphic to $\mathbb{R}^{n_1} \times \mathbb{T}^{n_2}$, $n_1 + n_2 = n$ ([20]). The study of C_1-G-manifolds goes back to 1957 and a paper due to Mostert [14]. Mostert characterized the orbit space of C_1- G-manifolds, when G is compact. Later, other mathematicians generalized the Mostert's theorem to G-manifolds with non-compact G. There are many interesting results on topological properties of the orbits of C_1-G-manifolds under conditions on the sectional curvature of M. If M is a C_1-G-manifold of negative curvature then it is proved (see [17]) that either M is simply connected or the fundamental group of M is isomorphic to $\mathbb{Z}^p$ for some positive integer p. In the later case, if $p > 1$ then each orbit is diffeomorphic to $\mathbb{R}^{n-1-p} \times \mathbb{T}^p$, $n = \dim M$, and M is diffeomorphic to $\mathbb{R}^{n-p} \times \mathbb{T}^p$. If $p = 1$, then there is an orbit diffeomorphic to $\mathbb{S}^1$ and the other orbits are covered by $\mathbb{S}^{n-2} \times \mathbb{R}$. Topological properties of flat C_1-G-manifolds have been studied in [13].

In the present article, I summarize some of my results [9-13] about C_2-G-manifolds which are flat or have negative curvatures.

2. Flat C_2-G-manifolds

In the following, M^n is a Riemannian manifold of dimension n, G is a closed and connected subgroup of Iso(M), π: $M \to M/G$ denotes the projection on to the orbit space. If $G, H \subset$ Iso(M) and for each $x \in M$, $G(x) = H(x)$, then we say that G and H are orbit equivalent on M and we denote it by $G \simeq H$.

Fact 2.1 (See [2,18]). Let M be a Riemannian manifold and $\hat{M}$ be the Riemannian universal covering of M by the covering map k: $\hat{M} \to M$, and let G be a closed and connected subgroup of Iso(M). Then there is a connected covering $\hat{G}$ for G such that $\hat{G}$ acts isometrically on $\hat{M}$ and the following assertions are true

1) Coh(G, M) =Coh$(\hat{G}, \hat{M})$
2) If $D = \hat{G}(x)$ is a $\hat{G}$-orbit in $\hat{M}$ then $k(D)$ is a G-orbit in M, and each G-orbit in M is equal to $k(D)$ for some $\hat{G}$-orbit D in $\hat{M}$
3) If Δ is the deck transformation group of the covering k: $\hat{M} \to M$, then for each $\delta \in \Delta$ and each $g \in \hat{G}$, $\delta o g = g o \delta$. Thus δ maps $\hat{G}$-orbits in $\hat{M}$ on to $\hat{G}$-orbits.

If M is a C_k-G-manifold, then there are two types of points in M called principal and singular points (for definition and details about singular and principal points, we refer to [2, 8]). If x is a principal (singular) point then $\pi(x)$ is an interior (boundary) point of M/G, the orbit $G(x)$ is called a principal (singular) orbit and $\dim G(x) = n - m$ ($\dim G(x) \leq n - m$). The union of all principal orbits is an open and dense subset of M.

Theorem 1 ([13]). a) *If G is a closed and connected subgroup of* Iso$(\mathbb{R}^n)$ *such that $\mathbb{R}^n$ is a C_1-G-manifold, then either each principal orbit is isometric to $\mathbb{R}^{n-1}$ and there is not singular orbit, or each principal orbit is diffeomorphic to $\mathbb{S}^{n-m-1} \times \mathbb{R}^m$, for some $m \geq 0$ and there is a unique singular orbit isometric to $\mathbb{R}^m$.*

b) *If G is a closed and connected subgroup of* Iso(M) *and M is a flat C_1-G-manifold then there is a non-negative integer l such that $\pi_1(M) = \mathbb{Z}^l$.*

Theorem 2 ([20]). *If M is a homogeneous Riemannian manifold of non-positive curvature, then it is diffeomorphic to $\mathbb{T}^m \times \mathbb{R}^r$ for some non-negative integers m, t, where $\mathbb{T}^m$ denotes the m-torus.*

Let G be a connected subgroup of $\mathrm{Iso}(\mathbb{R}^n)$ and d, e be positive integers such that $d + e = n$. If G is not compact and it is a subgroup of $\mathrm{SO}(d) \times \mathbb{R}^e$, then we say that G is *d-helicoidical* on $\mathbb{R}^n$. Let

$$K = \{A \in \mathrm{SO}(d)\ ;\ (A, b) \in G, \text{for some}\, b \in \mathbb{R}^e\}$$

$$T = \{b \in \mathbb{R}^e\ ;\ (A, b) \in G, \text{for some}\, A \in \mathrm{SO}(d)\}$$

If $x = (x_1, x_2) \in (\mathbb{R}^d - \{o\}) \times \mathbb{R}^e$, $\mathbb{T}(x_2) = \mathbb{R}^e$ and $K(x_1) = \mathbb{S}^{d-1}(|x_1|)$, then $G(x)$ is called a **d-helix** in $\mathbb{R}^n$.

Let G be a closed and connected subgroup of $\mathrm{Iso}(\mathbb{R}^n)$. We say that G has compact (or helicoidical) factor, if there is an integer $0 < m < n$ and $G_1 \subset \mathrm{Iso}(\mathbb{R}^{n-m})$, $G_2 \subset \mathrm{Iso}(\mathbb{R}^m)$, such that

1) G_2 is compact (or helicoidical on $\mathbb{R}^m$)
2) $G \simeq G_2 \times G_1$
3) For some(so each) $x \in \mathbb{R}^{n-m}$, $G_1(x) = \mathbb{R}^{n-m}$.

Theorem 3. *Let M^n, $n \geq 3$, be a complete connected non-simply connected and flat Riemannian manifold, which is a C_2-G-manifold under the action of a closed and connected Lie group G of isometries. Then one of the following is true*

a) $\pi_1(M) = Z$ *and each principal orbit is isometric to* $\mathbb{S}^{n-2}(c)$, *for some* $c > 0$ (*c depends on orbits*).
b) *There is a positive integer l, such that* $\pi_1(M) = \mathbb{Z}^l$ *and one of the following is true*:
   - b1) *There is a positive integer* m, $2 < m < n$, *such that each principal orbit is covered by* $N^{m-2}(c) \times \mathbb{R}^{n-m}$, *where* $N^{m-2}(c)$ *is a homogeneous hypersurface of* $\mathbb{S}^{m-1}(c)$ ($c > 0$ *depends on orbits*). *There is a unique orbit diffeomorphic to* $\mathbb{T}^l \times \mathbb{R}^{n-m-l}$.
   - b2) *Each principal orbit is covered by* $\mathbb{S}^r \times \mathbb{R}^{n-r-2}$, *for some positive integer* r.
   - b3) *Each principal orbit is covered by* $H \times \mathbb{R}^{n-m}$, *such that* H *is a helix in* $\mathbb{R}^m$. *There is an orbit diffeomorphic to* $\mathbb{T}^l \times \mathbb{R}^t$, *for some non-negative integer* t.
c) *Each orbit is diffeomorphic to* $\mathbb{R}^t \times \mathbb{T}^l$, *for some nonnegative integer* t ($t = n - l - 2$, *if the orbit is principal*).

Sketch of the proof: $\hat{M} = \mathbb{R}^n$ is the universal covering of M. Consider $\hat{G}$ as in Fact 2.1, and let $k\colon \mathbb{R}^n \to M$ be the covering map, and Δ be the deck transformation group of the covering $k\colon \mathbb{R}^n \to M$ (i.e, $\pi_1(M)$ is isomorphic to Δ). We can show that one of the following is true

1) $\hat{G}$ is compact or it has compact factor on $\mathbb{R}^n$
2) $\hat{G}$ is helicoidical or it has helicoidical factor on $\mathbb{R}^n$
3) All $\hat{G}$-orbits are euclidean.

We consider 1), 2) and 3) separately.

1) If $\hat{G}$ is compact, we get part a) of the theorem (see [10]). If $\hat{G}$ is not compact but has compact factor, there are subgroups $\hat{G}_1$ of Iso($\mathbb{R}^{n-m}$) and $\hat{G}_2$ of Iso($\mathbb{R}^m$), for some positive integer $m < n$, such that $\hat{G}_2$ is compact, $\hat{G}_1$ acts transitively on $\mathbb{R}^{n-m}$ and $\hat{G} \simeq \hat{G}_2 \times \hat{G}_1$. Since $\hat{G}_2$ is compact it has a fixed point in $\mathbb{R}^m$, which without lose of generality we assume that the origin of $\mathbb{R}^m$ is a fixed point of $\hat{G}_2$ (i.e, $\hat{G}_2 \subset \mathrm{SO}(m)$). So, $\mathbb{R}^m$ is a C_2-$\hat{G}_2$-manifold. If $m = 2$ then $\hat{G}_2$ is trivial and all $\hat{G}$ orbits are euclidean (isometric to $\mathbb{R}^{n-2}$) which we will consider in the case 3). If $m > 2$, put $F = \{x \in \mathbb{R}^m \ ; \ \hat{G}_2(x) = x\}$. F is a totally geodesic submanifold of $\mathbb{R}^n$, so it is isomorphic to $\mathbb{R}^k$, for some $k < m$. Since $\dim F < 2$ (see [11], Lemma 2.6), then $F = \{o\}$ or F is isometric to $\mathbb{R}$. Suppose $F = \{o\}$ and put $W = \{o\} \times \mathbb{R}^{n-m} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}, \quad D = k(W)$. Since W is a $\hat{G}$-orbit, D must be a G-orbit. Therefore, D is a flat homogeneous Riemannian manifold which is diffeomorphic to $\mathbb{R}^{n-m-l} \times \mathbb{T}^l$ for some integer l, so $\pi_1(D) = \mathbb{Z}^l$. W is the unique $\hat{G}$-orbit with dimension $n - m$. Then $\Delta(W) = W$ and $\Delta = \pi_1(D) = \mathbb{Z}^l$. Therefore, $\pi_1(M) = \mathbb{Z}^l$. If $o \neq x_2 \in \mathbb{R}^m$ then $\hat{G}_2(x_2) \subset \mathbb{S}^{m-1}(|x_2|)$ and $\mathbb{S}^{m-1}(|x_2|)$ is a C_1-$\hat{G}_2$-manifold. Thus $\hat{G}_2(x_2)$ is a homogeneous hypersurface of $\mathbb{S}^{m-1}(|x_2|)$, which we denote it by $N^{m-2}(|x_2|)$. Therefore, each principal orbit in M is covered by $N^{m-2}(c) \times \mathbb{R}^{n-m}$ for some $c > 0$ related to orbits. These yield to part b1) of the theorem. Now, suppose that F is isometric to $\mathbb{R}$ and put $A = F \times \mathbb{R}^{n-m} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ and $B = k(A)$. Since A is a C_1-$\hat{G}$-manifold, B is a C_1-G-manifold. Since B is flat, by Theorem 1, there is a non-negative integer l such that $\pi_1(B) = \mathbb{Z}^l$. Consider a point $x = (x_2, x_1) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. If $x_2 \in F$ then $\hat{G}(x) = \{x_2\} \times \mathbb{R}^{n-m} \cong \mathbb{R}^{n-m}$. If $x_2 \in \mathbb{R}^m - F$, then $\hat{G}(x) = \hat{G}_2(x_2) \times \mathbb{R}^{n-m}$, with $\dim \hat{G}_2(x_2) \geq 1$, so by dimensional reasons for each $x_2 \in F$, there is $x_2' \in F$ such that $\delta(\{x_2\} \times \mathbb{R}^{n-m}) = \{x_2'\} \times \mathbb{R}^{n-m}$. Thus $\Delta(A) = A$ and $\pi_1(M) = \Delta = \pi_1(B) = \mathbb{Z}^l$. Let $x = (x_2, x_1) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ be a principal orbit. Each $g \in \hat{G}_2$ is a rotation around the line F, so $\hat{G}_2(x_2)$ is a sphere included in a hyperplane of $\mathbb{R}^m$ which is perpendicular to F. Thus, $\hat{G}(x_2)$ is isometric to $\mathbb{S}^{m-2}(c)$ for some positive number c, and $\hat{G}(x)$ must be isometric to $\mathbb{S}^{m-2}(c) \times \mathbb{R}^{n-m}$. If we put $m - 2 = r$, then we get part b2) of the theorem.

2) Let $m(\leq n)$ be a positive integer and $\hat{G} \simeq \hat{G}_2 \times \hat{G}_1$ such that $\hat{G}_2$ be helicoidical on $\mathbb{R}^m$ and $\hat{G}_1$ be transitive on $\mathbb{R}^{n-m}$. If $m = 2$, then G_2 is trivial and $\hat{G}$ orbits are euclidean, which is the case 3). If $m > 2$, then $\hat{G}_2$ is orbit equivalent (on $\mathbb{R}^m$) to a subgroup of $\mathrm{SO}(d) \times \mathbb{R}^{m-d}$ for some positive integer d. Put

$$K = \{A \in \mathrm{SO}(d)\ ;\ (A, b) \in \hat{G}_2 \text{ for some } b \in \mathbb{R}^{m-d}\}$$
$$T = \{b \in \mathbb{R}^{m-d}\ ;\ (A, b) \in \hat{G}_2 \ \text{ for some } A \in \mathrm{SO}(d)\}.$$

Then, either all $\hat{G}_2$ orbits (so $\hat{G}$ orbits) are euclidean (which is the case 3)), or one of the following is true (see [8])

I) $d > 1$, each principal $\hat{G}_2$-orbit in R^m is diffeomorphic to $\mathbb{S}^{d-1} \times \mathbb{R}^{m-d-1}$ and the other $\hat{G}_2$-orbits of $\mathbb{R}^m$ are isometric to $\mathbb{R}^{m-d-1}$. The union of all orbits which are isometric to $\mathbb{R}^{m-d-1}$ is a submanifold W of $\mathbb{R}^m$, such that W is isometric to $\mathbb{R}^{m-d}$, $\hat{G}_2(W) = W$ and $\mathrm{Coh}(\hat{G}_2, W) = 1$.

II) $d > 2$ and each principal $\hat{G}_2$-orbit of $\mathbb{R}^m$ is isometric to $N^{d-2}(c) \times \mathbb{R}^{m-d}$. Where $N^{d-2}(c)$ is a homogeneous hypersurface of $\mathbb{S}^{d-1}(c)$ $(c > 0)$. There is a unique $\hat{G}_2$-orbit V in $\mathbb{R}^m$, which is isometric to $\mathbb{R}^{m-d}$.

III) $d > 1$ and each principal $\hat{G}_2$-orbit in $\mathbb{R}^m$ is isometric to a d-helix in $\mathbb{R}^m$. There is a unique $\hat{G}_2$-orbit V isometric to $\mathbb{R}^{m-d}$.

We consider I), II), III) separately.

I) Put $D = W \times \mathbb{R}^{n-m}$ and $B = k(D)$. Since $\mathrm{Coh}(\hat{G}_2, W) = 1$, then $\mathrm{Coh}(\hat{G}, W \times \mathbb{R}^{n-m}) = 1$. Thus B is a flat cohomogeneity one G-manifold, so by Theorem 1, there is a non-negative integer l such that $\pi_1(D) = \mathbb{Z}^l$. Now let $(x_2, x_1) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. If $x_2 \in W$, then $\hat{G}(x) = \hat{G}_2(x_2) \times \hat{G}_1(x_1)$ is isometric to $\mathbb{R}^{m-d-1} \times \mathbb{R}^{n-m} = \mathbb{R}^{n-d-1}$, and if $x_2 \in \mathbb{R}^m - W$, then $\hat{G}(x) = \hat{G}_2(x_2) \times \hat{G}_1(x_1)$ is diffeomorphic to $\mathbb{S}^{d-1} \times \mathbb{R}^{m-d-1} \times \mathbb{R}^{n-m} = \mathbb{S}^{d-1} \times \mathbb{R}^{n-d-1}$. Since each $\delta \in \Delta$ maps $\hat{G}$-orbits of $\mathbb{R}^m \times \mathbb{R}^{n-m}$ on to $\hat{G}$-orbits , and the $\hat{G}$-orbits in $D = W \times \mathbb{R}^{n-m}$ are not isometric to $\hat{G}$-orbits in $(\mathbb{R}^m - W) \times \mathbb{R}^{n-m}$, then $\Delta(D) = D$. Thus

$$\pi_1(M) = \Delta = \pi_1(B) = \mathbb{Z}^l.$$

Since principal $\hat{G}$-orbits of $\mathbb{R}^n$ are diffeomorphic to $\mathbb{S}^{d-1} \times \mathbb{R}^{n-d-1}$ then we get part b2) of the theorem.

II) Let $P = V \times \mathbb{R}^{n-m}$ and $C = k(P)$. P is the unique $\hat{G}$-orbit of $\mathbb{R}^n$ which is isometric to $\mathbb{R}^{m-d} \times \mathbb{R}^{n-m} \simeq \mathbb{R}^{n-d}$. Thus C is a flat G-orbit in M, and it must be diffeomorphic to $\mathbb{T}^l \times \mathbb{R}^{n-d-l}$, for some non-negative integer l. Since each $\delta \in \Delta$ maps $\hat{G}$-orbits on to $\hat{G}$-orbits, we get from uniqueness of P that $\Delta(P) = P$. Thus

$$\pi_1(M) = \Delta = \pi_1(C) = \mathbb{Z}^l.$$

Therefore, we get part b1) of the theorem.

III) From uniqueness of V we can prove in the same way as II) that $\pi_1(M) = \mathbb{Z}^l$, for some positive integer l, and there is a unique G-orbit in M diffeomorphic to $\mathbb{T}^l \times \mathbb{R}^r$ for some integer r. Thus we get part b3) of the theorem.

3) Consider a G-orbit B in M. There is a $\hat{G}$-orbit D in $\mathbb{R}^n$ such that $B = k(D)$. Since D is flat then B is flat and homogeneous, and it must be diffeomorphic to $\mathbb{R}^t \times \mathbb{T}^l$ for some integers t, l. This is part c) of the theorem.

3. C_2-G-manifolds of Negative Curvature

If M is a Riemannian manifold and $\delta \in Iso(M)$, the squared displacement function d^2_δ: $M \to M$ is defined by

$$d^2_\delta(x) = d(x, \delta x).$$

Fact 3.1 (see [5]). If M is a simply connected Riemannian manifold of negative curvature and $\delta \in \mathrm{Iso}(M)$, then one of the followings is true

1) d^2_δ has no minimum point.
2) Minimum point set of d^2_δ is equal to the fixed point set of δ.
3) minimum point set of d^2_δ is the image of a geodesic γ translated by δ (i.e., there is a positive number t_0 such that for all t, $\delta(\gamma(t)) = \gamma(t + t_0)$).

The isometries 1), 2), and 3) are called **parabolic**, **elliptic** and **axial**, respectively.

We recall (see [5]) that infinity $M(\infty)$ of a simply connected Riemannian manifold M of nonpositive curvature is the classes of asymptotic geodesics. For each geodesic γ we denote by $[\gamma]$ the asymptotic class of geodesics containing γ. If $x \in M$, then there is a unique (up to parametrization) geodesic γ_x in the class $[\gamma]$ containing x, and there is a unique hypersurface S_x containing x and perpendicular to all elements of $[\gamma]$. S_x is called a horosphere.

Fact 3.2 (see [3, 5]).

a) Let M be a simply connected Riemannian manifold of negative curvature.

1) If g is an axial isometry of M, then the geodesic γ with the property $g(\gamma) = \gamma$ is unique.
2) If g is a parabolic isometry of M, then there is a unique class of asymptotic geodesics $[\gamma]$ such that $g[\gamma] = [\gamma]$.

b) Let G be a connected and solvable Lie subgroup of isometries of a simply connected and negatively curved Riemannian manifold M. Then one of the followings is true

1) $\mathrm{Fix}(G, M) \neq \emptyset$.
2) There is a unique G-invariant geodesic.
3) There is a unique class of asymptotic geodesics $[\gamma]$ such that $G[\gamma] = [\gamma]$.

Corollary 1 ([12, 18]). *If M is a simply connected Riemannian manifold of negative curvature and G is a closed and connected subgroup of* $\mathrm{Iso}(M)$ *such that* $\mathrm{Fix}(G, M) = \emptyset$, *then there is at most one totally geodesic G-orbit in M.*

Corollary 2. *If M is a negatively curved, non-simply connected, Riemannian manifold and $\widetilde{M}$ is the universal covering of M, then for each deck transformation δ there is a geodesic γ in $\widetilde{M}$ such that $\delta\gamma = \gamma$.*

Proof: Let $x_0 \in M$ and $[\alpha] \in \pi_1(M, x_0)$. Suppose that $[\alpha]$ is the corresponding element of δ in the canonical isomorphism between Δ and $\pi_1(M, x_0)$ (see [15] p. 186). Let β: $[0, 1] \to M$ be a geodesic segment such that $\beta(0) = \beta(1) = x_0$ and $[\beta] = [\alpha]$. Let $\kappa(\widetilde{x}) = x_0$ and $\widetilde{\beta}$ be the unique lift of β to $\widetilde{M}$ such that $\widetilde{\beta}(0) = \widetilde{x}$. It follows from the elementary properties of covering spaces that $\delta(\widetilde{x}) = \widetilde{\beta}(1)$. Now, if γ is the extension of geodesic segment $\widetilde{\beta}$ to a geodesic in $\widetilde{M}$ then $\delta(\gamma) = \gamma$. ■

Lemma 1 ([11]). *Let M be a Riemannian manifold of negative curvature, $n = \mathrm{dim} M \geq 3$, and $\widetilde{M}$ be its universal covering. If there is a geodesic γ on $\widetilde{M}$ and an element δ in the center of the deck transformation group Δ, such that $\delta\gamma = \gamma$, then M is diffeomorphic to one of the following spaces*

$$\mathbb{S}^1 \times \mathbb{R}^{n-1}, \qquad B^2 \times \mathbb{R}^{n-2}$$

where B^2 is the mobius band.

Theorem 4 ([11]). *Let M^{n+2} be a complete negatively curved and non-simply connected Riemannian manifold which is of cohomogeneity two under the action of a closed and connected Lie subgroup of isometries. If* $\mathrm{Fix}(G, M) \neq \emptyset$, *then*

a) *M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{n+1}$ or $B^2 \times \mathbb{R}^n$(B^2 is the mobius band)*
b) $\mathrm{Fix}(G, M)$ *is diffeomorphic to $\mathbb{S}^1$*
c) *Each principal orbit is diffeomorphic to $\mathbb{S}^n$.*

Remark 1. *By* Theorem 3.7 a) *in* [17], *if M is a non-simply connected and complete Riemannian manifold of negative curvature, which is of cohomogeneity one under the action of a connected and closed subgroup of isometries, and if there is not any singular orbit, then there are positive integers p, s such that M is diffeomorphic to $\mathbb{R}^p \times \mathbb{R}^{s+1}$ and each orbit is diffeomorphic to $\mathbb{R}^p \times \mathbb{R}^s$, $p + s =$* $\dim M - 1$.

Theorem 5 ([12]). *Let M^n, $n \geq 3$, be a complete negatively curved Riemannian manifold and G be a closed, connected and non-semisimple subgroup of isometries of M^n. If M is a cohomogeneity two G-manifold such that the singular orbits (if there are any) are fixed points of G. Then one of the following is true*

1) *M is simply connected (diffeomorphic to $\mathbb{R}^n$).*

2) *M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$($B^2$ is the mobious band). Each principal orbit is diffeomorphic to $\mathbb{S}^{n-2}$. Union of singular orbits* $\mathrm{Fix}(G, M)$ *is diffeomorphic to $\mathbb{S}^1$.*
3) *M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ or $B^2 \times \mathbb{R}$. All orbits are diffeomorphic to $\mathbb{S}^1$.*
4) *$\pi_1(M) = \mathbb{Z}^p$ for some positive integer p, and all orbits are diffeomorphic to $\mathbb{R}^{n-2-p} \times \mathbb{T}^p$.*

Sketch of the proof: Following Fact 2.1, let $\widetilde{M}$ be the universal Riemannian covering manifold of M with the deck transformation group Δ and let $\widetilde{G}$ be the corresponding connected covering of G which acts isometrically and by cohomogeneity two on $\widetilde{M}$. If $\mathrm{Fix}(\widetilde{G}, \widetilde{M}) \neq \emptyset$ then $\mathrm{Fix}(G, M) \neq \emptyset$, so by Theorem 4, we get the parts 1) or 2) of the theorem. Now, suppose that $\mathrm{Fix}(\widetilde{G}, \widetilde{M}) = \emptyset$. By assumptions of the theorem, if there is a singualr orbit, it must be a fixed point. So all $\widetilde{G}$-orbits in $\widetilde{M}$ must be $(n-2)$-dimensional. Since G is non-semisimple, $\widetilde{G}$ is non-semisimple. Let H be a solvable normal subgroup of $\widetilde{G}$ and put $N = \mathrm{Fix}(H, \widetilde{M})$. We consider the following two cases separately

$$\mathbf{a})\ N = \emptyset, \qquad \mathbf{b})\ N \neq \emptyset.$$

a) By Fact 3.2 b), one of the following is true:

a-i) There is a unique geodesic γ such that $H(\gamma) = \gamma$.

a-ii) There is a unique class of asymptotic geodesics $[\gamma]$ such that $H[\gamma] = [\gamma]$.

a-i) From normality of H in $\widetilde{G}$ and uniqueness of γ, we get that $\widetilde{G}(\gamma) = \gamma$. Since $\mathrm{Fix}(\widetilde{G}, \widetilde{M}) = \emptyset$ then γ is a $\widetilde{G}$-orbit in $\widetilde{M}$. But all orbits are $(n-2)$-dimensional and the orbit γ is of dimension one. Thus all orbits are of dimension one and $n - 2 = 1$. Each $\delta \in \Delta$ maps $\widetilde{G}$-orbits onto $\widetilde{G}$-orbits. So $\delta(\gamma)$ is a $\widetilde{G}$-orbit. Since by Corollary 1, γ is the unique geodesic orbit, then $\delta(\gamma) = \gamma$. Thus $\Delta\gamma = \gamma$ and $\pi_1(M) = \mathbb{Z}$ (see [4], Theorem 3.4, §261). Now, by Lemma 1, M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ or $B^2 \times \mathbb{R}$. Since all G-orbits of M are regular (and diffeomorphic to each other) and the G-orbit $\frac{\gamma}{\Delta}$ is diffeomorphic to $\gamma/\mathbb{Z} = \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$, all G-orbits are diffeomorphic to $\mathbb{S}^1$. This is part 3) of the theorem.

a-ii) By Corollary 2, each $\delta \in \Delta$ is axial. Consider a $\delta \in \Delta$ and Let λ be the unique geodesic in $\widetilde{M}$ such that $\delta(\lambda) = \lambda$. Since the elements of Δ and $\widetilde{G}$ are commutative, for each $g \in \widetilde{G}$ we have

$$\delta(g\lambda) = g(\delta\lambda) = g\lambda.$$

Since λ with the property $\delta(\lambda) = \lambda$ is unique, we get that $g\lambda = \lambda$. So λ is a $\widetilde{G}$-orbit, and we get the part 3) of the theorem in a similar way in a-i).

b) N is a nontrivial totally geodesic submanifold of $\widetilde{M}$. If $g \in \widetilde{G}$, $h \in H$ and $x \in N$, then

$$g^{-1}hg(x) = x \Rightarrow hg(x) = g(x) \Rightarrow g(x) \in N.$$

Thus $\widetilde{G}(N) = N$. All orbits are of dimension $n-2$. So if $x \in N$, then

$$n-2 = \dim\widetilde{G}(x) \leq \dim N < \dim\widetilde{M} = n \Rightarrow \dim N = n-2 \ \ or \ \ n-1.$$

Now, consider two cases $\dim N = n$ and $\dim N = n+1$ separately.

b-j) $\dim N = n-2$.

In this case, N is a $\widetilde{G}$-orbit. If $n-2=1$, in a similar way in (a-*i*) we get part (3) of the theorem. Suppose $n-2 \geq 2$ and put $N_1 = \kappa(N)$. By Corollary 1, N is the unique totally geodesic $\widetilde{G}$-orbit in $\widetilde{M}$. Thus, for each $\delta \in \Delta$, $\delta(N) = N$, so $N_1 = N/\Delta$. But N_1 is a totally geodesic G-orbit in M, so it must be simply connected (since by Kobayashi's theorem in [6] homogeneous manifolds of negative curvature are simply connected). Therefore, Δ is trivial and M is simply connected. This is the part 1) of the theorem.

b-jj) $\dim N = n-1$ Since all orbits are of dimension $n-2$, N is a negatively curved cohomogeneity one $\widetilde{G}$-manifold. Consider following two cases:

b-jj-1) There is a $\delta \in \Delta$ and $x \in \widetilde{M}$ such that $\delta\widetilde{G}(x) \neq \widetilde{G}(x)$.

b-jj-2) For each $\delta \in \Delta$ and $x \in \widetilde{M}$, $\delta\widetilde{G}(x) = \widetilde{G}(x)$.

b-jj-1) From the fact that δ maps orbits on to orbits, we get that $\delta\widetilde{G}(x) = \widetilde{G}(y)$, $y \in \widetilde{M}$ (i.e., $\widetilde{G}(x) \cap \widetilde{G}(y) = \emptyset$). By Proposition 4.2 in [1], the minimum point set of the following function is at most the image of a geodesic

$$f_\delta : \widetilde{M} \to \mathbb{R}, \qquad f_\delta(x) = d^2(x, \delta(x)).$$

So we can find a geodesic γ such that the image of γ is not the minimum point set of f_δ and $\gamma(0) \in G(x)$, $\gamma(1) \in G(y)$. Put $g(t) = f_\delta(\gamma(t))$. Since the elements of Δ and $\widetilde{G}$ are commutative, f_δ is constant along orbits (because $f_\delta(gx) = d^2(gx, \delta gx) = d^2(gx, g\delta x) = d^2(x, \delta x) = f_\delta(x)$). Since $\delta(\gamma(0)) \in G(\gamma(1))$, then $f_\delta(\delta\gamma(0)) = f_\delta(\gamma(1))$. Thus

$$g(0) = f_\delta(\gamma(0)) = d^2(\gamma(0), \delta(\gamma(0))) = d^2(\delta(\gamma(0)), \delta^2(\gamma(0)))$$
$$= f_\delta(\delta\gamma(0)) = f_\delta(\gamma(1)) = g(1).$$

Since g is strictly convex (see [1]), it has a unique minimum point $t_0 \in (0,1)$. Therefore, $\widetilde{G}(\gamma(t_0))$ is the minimum point set of f_δ, which must be a geodesic. Then $\widetilde{G}(\gamma(t_0))$ is a (geodesic) one dimensional $\widetilde{G}$-orbit. Then in a similar way in a-i) we get part 3) of the theorem.

b-jj-2) Put $N_1 = \kappa(N)$. Since for each $\delta \in \Delta$, $\delta(N) = N$ then $\pi_1(M) = \pi_1(N_1)$. N_1 is a cohomogeneity one G-manifold of negative curvature, without singular orbits. So, by Remark 1, each G-orbit in N_1 is diffeomorphic to $\mathbb{T}^p \times \mathbb{R}^s$, $p + s = \dim N - 1 = n - 2$, and N_1 is diffeomorphic to $\mathbb{T}^p \times \mathbb{R}^{s+1}$. These yield to part 4) of the theorem.

4. C_2-G-manifolds of Constant Negative Curvature

Theorem 6. *Let $M^n(c)$, $n \geq 3$, be a complete Riemannian manifold of constant sectional curvature $c < 0$ and let G be a connected and closed Lie subgroup of isometries which acts by cohomogeneity two on M. Then one of the following is true*

a) *M is simply connected, i.e, $M = H^n(c)$*
b) *Each orbit is diffeomorphic to $\mathbb{R}^m \times \mathbb{T}^{n-2-m}$, for some nonnegative integer m, and M is a union of totally geodesic cohomogeneiy one Riemannian G-submanifolds*
c) *$\pi_1(M) = \mathbb{Z}$ and either there is an orbit diffeomorphic to $\mathbb{S}^1$ or $\mathrm{Fix}(G, M) = \mathbb{S}^1$*
d) *$\pi_1(M) = \mathbb{Z}^k$ for some positive integer k, and M is a union of the following two types of orbits*
   d1) *The orbits which are diffeomophic to $\mathbb{R}^{m-k} \times \mathbb{T}^k$ for some positive integer m. Union of this type of orbits is a totally geodesic submanifold of M*
   d2) *The orbits covered by $\mathbb{S}^{n-2-m} \times \mathbb{R}^m$.*

Sketch of the proof: $H^n(c)$ is the universal Riemannian covering manifold of M. Let Δ be the deck transformation group and $\widetilde{G}$ be the corresponding connected covering of G, which acts isometrically and by cohomogeneity two on $H^n(c)$ (as mentioned in Fact 2.1). By the main theorem of [18], we have three cases below

i) $\widetilde{G}$ has a fixed point.
ii) $\widetilde{G}$ has a unique nontrivial totally geodesic orbit.
iii) All orbits are included in horospheres centered at the same point at the infinity.

We study each case separately.

i) Let $F = \{x \in H^n(c)\ ;\ \widetilde{G}(x) = x\}$. If $\dim F \geq 2$, then the cohomogeneity of the action of $\widetilde{G}$ on $H^n(c)$ is ≥ 3 (see [11]), which is a contradiction. If $\dim(F) = 1$, then F is the image of a geodesic λ. Since each δ in Δ commutes with elements of $\widetilde{G}$ we get $\Delta(\lambda) = \lambda$. So $\pi_1(M) = \mathbb{Z}$. The set $B = F/\Delta$ (which is diffeomorphic to $\mathbb{S}^1$) is equal to $\mathrm{Fix}(G, M)$. This is part c) of the theorem. If $\dim(F) = 0$, then F is a one point set, so M is simply connected and we get part a) of the theorem.

ii) We get from uniqueness of P that $\Delta(P) = P$. If $\dim P = 1$, then P is a geodesic and we get part c) of the theorem in the same way as i). If $\dim P > 1$, then $k(P)$ is homogeneous and of negative curvature. Then it is simply connected and the covering map $k\colon P \to k(P)$ must be trivial. Therefore, the covering map $H^n(c) \to M$ is trivial and M is simply connected (part a) of the theorem).

iii) Let Q_t be a one-parameter family of horospheres, such that $\widetilde{G}(Q_t) = Q(t)$ (see [18]). Since the action of $\widetilde{G}$ on $H^n(c)$ is of cohomogeneity two, we can show that for each t the action of $\widetilde{G}$ on Q_t is of cohomogeneity one. So one of the following cases is true ([13])

1) *Each orbit in Q_t, $t \in \mathbb{R}$, is isometric to $\mathbb{R}^{n-2}$*
2) *There is $m < n-2$ such that one orbit of Q_t, $t \in \mathbb{R}$, is isometric to $\mathbb{R}^m$, and the other orbits are diffeomorphic to $\mathbb{S}^{n-2-m} \times \mathbb{R}^m$.*

1) Consider an orbit D in M. We have $D = k(V)$, where V is a $\widetilde{G}$-orbit in $H^n(c)$. Since V is isometric to $\mathbb{R}^{n-2}$ and D is flat (and homogeneous). So it is diffeomorphic to $\mathbb{R}^m \times \mathbb{T}^{n-2-m}$. We can show that for each t, there is a $\widetilde{G}$-orbit V_t in Q_t, such that $T = \bigcup_t V_t$ is a totally geodesic cohomogeneity one $\widetilde{G}$-submanifold of $H^n(c)$. Therefore, $k(T)$ is a totally geodesic cohomogeneity one G-submanifold of M. Since $H^n(c)$ is a union of such submanifols T, we get part b) of the theorem.

2) Let V_t be the orbit in Q_t which is isometric to R^m. Then the set $\widetilde{N} = \bigcup_t V_t$ is a totally geodesic $\widetilde{G}$-submanifold of $H^n(c)$. So $N = k(\widetilde{N})$ is a totally geodesic G-submanifold of M. Since $\dim\widetilde{N} = \dim V_t + 1$, then $\widetilde{N}$ is a cohomogeneity one $\widetilde{G}$-submanifold. $H^n(c) = \widetilde{N} \bigcup (H^n(c) - \widetilde{N})$ is a union of two types of orbits. Orbits in $\widetilde{N}$ which are isometric to $\mathbb{R}^{n-2}$, and the orbits in $(H^n(c) - \widetilde{N})$ which are diffeomorphic to $\mathbb{R}^m \times \mathbb{S}^{n-2-m}$. Since each δ in Δ maps orbits to orbits, by dimensional reasons we have

$$\Delta(\widetilde{N}) = \widetilde{N}, \qquad \Delta(H^n(c) - \widetilde{N}) = H^n(c) - \widetilde{N}.$$

Therefore, we can show that one of the parts (a) or (c) of the theorem is true, or we have

$$M = \frac{H^n(c)}{\Delta} = \frac{\widetilde{N}}{\Delta} \bigcup \frac{H^n(c) - \widetilde{N}}{\Delta}.$$

The orbits of $\widetilde{N}/\Delta(=N)$ are diffeomorphic to $\mathbb{R}^r \times \mathbb{T}^k$ and the orbits in $\frac{H^n(c)-\widetilde{N}}{\Delta}$ are covered by $\mathbb{R}^m \times \mathbb{S}^{n-2-m}$. Thus we get part d) of the theorem.

References

[1] Bishop R. and O'Neill B., *Manifolds of Negative Curvature*, Trans. Amer. Math. Soc. **145** (1969) 1–49.

[2] Bredon. G., *Introdution to Compact Transformation Groups*, Acad. Press, New York 1972.

[3] Byers W., *Isometry Group of Riemannian Manifolds of Negative Curvature*, Proc. Amer. Math. Soc. **45** (1976) 281–285.

[4] do Carmo M., *Riemannian Geometry*, Birkhhauser, Boston 1992.

[5] Eberlin P. and O'Neil B., *Visibility Manifolds*, Pacific J. Math. **46**(1973) 45–109.

[6] Kobayashi S., *Homogeneous Riemannian Manifolds of Negative Curvature*, Tohoku Math. J. **14** (1962) 413–415.

[7] Michor P., *Isometric Actions of Lie Groups and Invariants*, Lectures at the University of Vienna, 1996/1997 `http://www.mat.univie.ac.at/~michor/tgbook.ps`

[8] Mirzaie R., *On Orbits of Isometric Actions on Flat Riemannian Manifolds*, to appear in Kyushu Math. J.

[9] Mirzaie R., *On Riemannian Manifolds of Constant Negative Curvature*, J. Korean Math. Soc. **48** (2011) 23–31.

[10] Mirzaie R., *Cohomogeneity Two Actions on Flat Riemannian Manifolds*, Acta Mathematica Sinica **23** (2007) 1587–1592.

[11] Mirzaie R., *On Negatively Curved Riemannian Manifolds of Low Cohomogeneity*, Hokkaido Math. J. **38** (2009) 797–803.

[12] Mirzaie R., *On Homogeneous Submanifolds of Negatively Curved Riemannian Manifolds*, submitted.

[13] Mirzaie R. and Kashani S., *On Cohomogeneity One Flat Riemannian Manifolds*, Glasgow Math. J. **44** (2002) 185–190.

[14] Mostert P., *On a Compact Lie Group Action on Manifolds*, Ann. Math. **65** (1957) 447–455.

[15] O'Neil B., *Semi-Riemannian Geomerty with Applications to Relativity*, Academic Press, New York 1983.

[16] Palais R. and Terng Ch., *A Genereal Theory of Canonical Forms*, Am. Math. Soc. **300** (1987) 771–789.

[17] Podesta F. and Spiro A., *Some Topological Properties of Cohomogeneity One Manifolds with Negative Curvature*, Ann. Global. Anal. Geom. **14** (1966) 69–79.

[18] di Scala A. and Olmos C., *The Geometry of Homogeneous Submanifolds of Hyperbolic Space*, Math. Z. **237** (2001) 199–209.

[19] Searle C., *Cohomogeneity and Positive Curvature in Low Dimensions*, Math. Z. **214** (1993) 491–498.

[20] Wolf J., *Spaces of Constant Curvature*, University of California, Berkely 1977.