

PERSPECTIVES IN MATHEMATICAL LOGIC

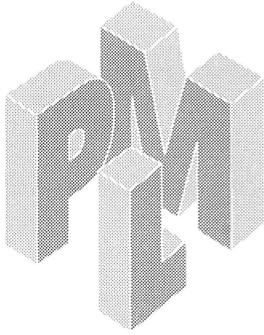
Saharon Shelah

Proper and Improper Forcing

Second Edition



Springer



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in
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Perspectives in Mathematical Logic

This series was founded in 1969 by the Omega Group consisting of R. O. Gandy, H. Hermes, A. Levy, G. H. Müller, G. E. Sacks and D. S. Scott. Initially sponsored by a grant from the Stiftung Volkswagenwerk, the series appeared under the auspices of the Heidelberger Akademie der Wissenschaften. Since 1986, *Perspectives in Mathematical Logic* is published under the auspices of the Association for Symbolic Logic.

Mathematical Logic is a subject which is both rich and varied. Its origins lie in philosophy and the foundations of mathematics. But during the last half century it has formed deep links with algebra, geometry, analysis and other branches of mathematics. More recently it has become a central theme in theoretical computer science, and its influence in linguistics is growing fast.

The books in the series differ in level. Some are introductory texts suitable for final year undergraduate or first year graduate courses, while others are specialized monographs. Some are expositions of well-established material, some are at the frontiers of research. Each offers an illuminating perspective for its intended audience.

Dedicated to My Beloved Son Omri

מקדש לבני האהוב עמרי

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