Chapter XIX
Abstract Equivalence Relations

by J. A. Makowsky and D. Mundici

For a regular logic $\mathscr{L}$, let $\sim\ =\ \equiv_{\mathscr{L}}$ be the equivalence relation obtained by saying that two structures are $\sim$-equivalent iff they satisfy the same sentences of $\mathscr{L}$. The isomorphism relation $\cong$ is automatically a refinement of $\sim$—that is, isomorphic structures are $\sim$-equivalent— $\sim$ itself is a refinement of elementary equivalence $\equiv$, and $\sim$ is preserved under both renaming and reduct. This last property simply means that upon renaming, or taking reducts of $\sim$-equivalent structures, we obtain $\sim$-equivalent structures. Furthermore, if $\mathscr{L}[\tau]$ is a set for every vocabulary τ, then the collection of equivalence classes given by $\sim$ on $\mathrm{Str}(\tau)$ has a cardinality. (Briefly, we say that $\sim$ is *bounded*). This paper is mainly concerned with abstract equivalence relations $\sim$ on $\bigcup_\tau \mathrm{Str}(\tau)$, having the above-mentioned properties as well as the *Robinson property* so that for every $\mathfrak{M}$, $\mathfrak{N}$, and τ with $\tau = \tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}}$,

$$\begin{aligned}&\text{if}\quad \mathfrak{M} \restriction \tau \sim \mathfrak{N} \restriction \tau \quad \text{then for some } \mathfrak{A},\\ &\mathfrak{M} \sim \mathfrak{A} \restriction \tau_{\mathfrak{M}} \quad \text{and} \quad \mathfrak{N} \sim \mathfrak{A} \restriction \tau_{\mathfrak{N}}.\end{aligned}$$

If $\sim\ =\ \equiv_{\mathscr{L}}$, then $\sim$ has the Robinson property iff $\mathscr{L}$ satisfies the Robinson consistency theorem. If, in addition, $\mathscr{L}[\tau]$ is a set for all τ, and if all sentences in $\mathscr{L}$ have a finite vocabulary, then the Robinson consistency theorem holds in $\mathscr{L}$ iff $\mathscr{L}$ is compact and has the interpolation property (see Corollary 1.4). Every bounded equivalence relation $\sim$ with the Robinson property satisfies the equation $\sim\ =\ \equiv_{\mathscr{L}}$ for *at most one* logic $\mathscr{L}$ (see Corollary 3.4). This result can be extended to equivalence relations corresponding to compact logics (see Theorem 3.11). Moreover, we have that $\sim\ =\ \equiv_{\mathscr{L}}$ for *exactly one* logic $\mathscr{L}$ iff $\sim$ is *separable* by quantifiers, in the sense that whenever $\mathfrak{M}$ and $\mathfrak{N}$ are not $\sim$-equivalent, there is a quantifier Q such that $\sim$ is a refinement of $\equiv_{\mathscr{L}(Q)}$ and $\mathfrak{M} \not\equiv_{\mathscr{L}(Q)} \mathfrak{N}$ (see (ii) of Theorem 3.10). Even if $\sim$ is not separable by quantifiers, there is still a strongest logic $\mathscr{L}$ such that $\sim$ refines $\equiv_{\mathscr{L}}$. This $\mathscr{L}$ is compact and can be written as $\mathscr{L} = \mathscr{L}\{Q \mid \sim \text{ is a refinement of } \equiv_{\mathscr{L}(Q)}\}$ (see Corollary 3.3 and (i) of Theorem 3.10).

The Robinson property of $\mathscr{L}$ can also be coupled with such properties as $[\omega]$-incompactness. Then $\equiv_{\mathscr{L}}$ will coincide with $\cong$ below the first uncountable measurable cardinal μ_0 (see Theorem 1.7), and the infinitary logic $\mathscr{L}_{\mu_0\omega}$ can be interpreted in $\mathscr{L}$ in some natural sense (refer to Theorem 1.12).

Some of the results in Section 1 can be extended to logics for enriched structures, such as topological, uniform, and proximity structures, as discussed in Section 2.

With any logic $\mathscr{L}$ we can associate an embedding relation $\to_{\mathscr{L}}$, where $\mathfrak{A} \to_{\mathscr{L}} \mathfrak{N}$ means that $\tau_{\mathfrak{N}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \equiv_{\mathscr{L}} \mathfrak{N}^+$ for some expansion $\mathfrak{N}^+$ of $\mathfrak{N} \restriction \tau_{\mathfrak{A}}$, with $\mathfrak{A}_A$ denoting, as usual, the diagram expansion of $\mathfrak{A}$. In Definition 4.1 we define embedding relations by abstracting these properties of the $\to_{\mathscr{L}}$ relation. Any such relation $\to$ generates an equivalence relation $\sim\, = \,\to^*$ by writing $\mathfrak{A} \sim \mathfrak{N}$ iff $\mathfrak{A}$ and $\mathfrak{N}$ are connected by a finite path of arrows. Conversely, every equivalence relation $\sim$ generates an embedding relation $\to\, = \,\sim^*$ by writing $\mathfrak{A} \to \mathfrak{N}$ iff $\tau_{\mathfrak{N}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \sim \mathfrak{N}^+$, for some expansion $\mathfrak{N}^+$ of $\mathfrak{N} \restriction \tau_{\mathfrak{A}}$. The mapping $*$ sends equivalence relations with the Robinson property into embedding relations with the *expanded amalgamation property* (AP^+) in a one-one way, the latter being a natural strengthening of the usual amalgamation property (AP). The mapping $*$ becomes a bijection with $** =$ identity, provided we restrict ourselves to embedding relations with AP^+ and such that $\to^{**} = \to$ (see Theorem 4.8). In particular, first-order logic $\mathscr{L}_{\omega\omega}$ is uniquely determined by the familiar elementary embeddability relation $\precsim$ (see Theorem 4.9).

Every countably generated logic $\mathscr{L} = \mathscr{L}(Q^i)_{i<\omega}$ determines, for each finite vocabulary τ, a sequence $\{\simeq_\tau^n\}_{n<\omega}$ of finite partitions over $\mathrm{Str}(\tau)$, by writing $\mathfrak{M} \simeq_\tau^n \mathfrak{N}$ iff $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same $\mathscr{L}[\tau]$-sentences of quantifier rank $\leq n$. We study an abstract notion of back-and forth systems (see Definition 5.1); the latter generalize the celebrated Fraïssé–Ehrenfeucht games for $\equiv$ (see Examples 5.2 and Theorem 5.3) and are in one-one correspondence with their associated logics, under the Robinson assumption (see Theorem 5.4). By use of Theorems 3.11 and 5.7 and the argument in Theorem 5.4 this correspondence can be extended to the realm of compact logics.

Any back-and-forth game G for $\mathscr{L}$-elementary equivalence determines not only a back-and-forth system in the above sense, but also a game $G(\mathfrak{A}, \mathfrak{B})$ for pairs of structures, or—equivalently—a decreasing sequence of sets of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. We regard the former as a global version of G (since each partition acts on the whole of $\mathrm{Str}(\tau)$), and the latter as a local version of G. Global and local versions have the same extreme generality (see Theorems 5.7 and 5.10) and are closely related, as is discussed in Section 5.

As this chapter will show, the Robinson property is very strong. Indeed, one of the main open problems of abstract model theory—a problem originally posed by Feferman—asks whether compactness and interpolation together are strong enough to characterize first-order logic. A negative answer would exhibit a proper extension of $\mathscr{L}_{\omega\omega}$ still having many important features in common with $\mathscr{L}_{\omega\omega}$ (by the very results of this chapter) while a positive answer would characterize $\mathscr{L}_{\omega\omega}$ in terms of properties which are generally reputed to be desirable for a logic $\mathscr{L}$. As a matter of fact, compactness is related to the finiteness of sentences and proofs in $\mathscr{L}$ and makes available a number of methods for constructing models; interpolation (together with its most notable consequence, Δ-closure—or equivalently—truth-maximality) is related to the equilibrium between syntax and semantics in $\mathscr{L}$.

Whatever the ultimate answer to this problem, the techniques and results of this chapter can be applied to logics for enriched structures (see Section 2). Furthermore, even for ordinary structures, several theorems originally stated under the Robinson assumption, can now be proved under the (weaker) compactness, or

JEP assumption (see Theorems 1.1, 2.4, 3.11, Lemma 3.11.1, Corollary 3.12, Remark 4.10, and Theorem 5.7) by simply refining the methods developed for the study of the Robinson property. Sometimes there are even applications to first-order logic itself (see Corollary 3.5, Theorem 4.9, and Corollary 5.5).

Throughout this chapter logics are assumed to satisfy the occurrence axiom, stating that for each sentence φ in $\mathscr{L}$ there is a smallest $\tau = \tau_\varphi$ such that $\varphi \in \mathscr{L}[\tau]$. We will write $\mathscr{L}(Q^i)_{i \in I}$ instead of $\mathscr{L}_{\omega\omega}(Q^i)_{i \in I}$ and will always assume that Q^i is a quantifier with built in relativization, as in Proposition II.4.1.5. Given equivalence relations $\sim$ and $\sim'$, instead of saying that $\sim$ is a refinement of $\sim'$ we will usually say that $\sim$ is *finer* than $\sim'$ (or, that $\sim'$ is *coarser* than $\sim$). We will constantly work with many-sorted structures and logics, so that our expansions may very well involve new sorts. Vocabularies and universes of structures are always assumed to be sets, while $\mathscr{L}[\tau]$ may be a proper class. However, when we want to exclude this possibility for $\mathscr{L}$ we will simply say that $\mathscr{L}[\tau]$ is a set for each τ.

The first author would like to thank the Swiss National Science Foundation for their support. The second author wishes to express his gratitude to the Heidelberger Akademie der Wissenschaften for their financial support; and, in particular, he extends thanks to Professor Gert Müller for his constant support and stimulating suggestions.

1. Logics with the Robinson Property

In Chapter XVIII we saw that logics satisfying the amalgamation property are compact, provided they have the finite dependence property, or even if the dependence number is smaller than the first uncountable measurable cardinal. The amalgamation property is both a consequence of the Robinson property, and of the joint embedding property. In this section we will review the relationship between the latter two properties and compactness, since under these stronger hypotheses many of the proofs are simpler and generalize to the case of logics whose underlying structures need not be first-order structures. Recall that a logic $\mathscr{L}$ has the *joint embedding property* (abbreviated JEP) iff whenever $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ and $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are jointly embeddable in some structure $\mathfrak{M}$. That is, $\mathfrak{M} \models_{\mathscr{L}} \mathrm{Th}_{\mathscr{L}}(\mathfrak{A}_A) \cup \mathrm{Th}_{\mathscr{L}}(\mathfrak{B}_B)$, where $\mathfrak{A}_A$ (resp., $\mathfrak{B}_B$) is the diagram expansion of $\mathfrak{A}$ (resp., of $\mathfrak{B}$) in vocabulary $\tau_A = \tau \cup \{c_a\}_{a \in A}$ (resp., $\tau_B = \tau \cup \{c_b\}_{b \in B}$), and $\tau_A \cap \tau_B = \tau$.

1.1 Theorem. *Let $\mathscr{L}$ be a regular logic such that $\mathscr{L}[\tau]$ is a set for every τ. Assume that for every countable τ_0, $|\mathscr{L}[\tau_0]| \leq \lambda$ for some fixed λ. If $\mathscr{L}$ satisfies the JEP, then there are at most 2^λ many regular cardinals κ such that $\mathscr{L}$ is not $[\kappa]$-compact.*

Proof. Let S be a set of regular cardinals such that $\mathscr{L}$ is not $[\kappa]$-compact for each $\kappa \in S$. By Definition XVIII.1.2.1, κ is cofinally characterizable in $\mathscr{L}$; hence, there is a structure $\mathfrak{A}_\kappa = \langle \kappa, <, \ldots \rangle$ whose diagram expansion

$$\mathfrak{A}_A = \langle \kappa, <, \{c_\alpha\}_{\alpha<\kappa}, \ldots \rangle$$

has the following property:

(∗) whenever $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}_A$ the set $\{c_\alpha^{\mathfrak{B}}\}_{\alpha<\kappa}$ is unbounded in the order $<^{\mathfrak{B}}$.

Adding, if necessary, more sorts and elements, and using the regularity properties of $\mathscr{L}$, we can safely assume that the vocabulary τ_κ of $\mathfrak{A}_\kappa$ is countable (we can code n-ary relations into a single "universal" $(n + 1)$-ary relation). Indeed, we can safely assume that for some fixed countable vocabulary $\tau \supseteq \{<\}$, the τ_κ are all equal to τ, for any $\kappa \in S$. Let T_κ be the complete $\mathscr{L}$-theory of $\mathfrak{A}_\kappa$ in vocabulary τ. By hypothesis, there are at most 2^λ such theories. Therefore, if $|S| > 2^\lambda$, then for two regular cardinals $\mu < \nu$ in S, we must have that $\mathfrak{A}_\mu \equiv_{\mathscr{L}} \mathfrak{A}_\nu$. Let $\mathfrak{M} = \mathfrak{A}_\mu$, $\mathfrak{N} = \mathfrak{A}_\nu$. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are joint embeddable in a structure $\mathfrak{D}$ (*absurdum* hypothesis), say $\mathfrak{D} \models \mathrm{Th}(\mathfrak{M}_M) \cup \mathrm{Th}(\mathfrak{N}_N)$, and let $\tau_M = \tau \cup \{c_m\}_{m\in M}$ and $\tau_N = \tau \cup \{g_n\}_{n\in N}$ be the vocabularies of $\mathfrak{M}_M$ and $\mathfrak{N}_N$, respectively, with $\tau_M \cap \tau_N = \tau$. Since

$$\mathfrak{M}_M = \langle \mu, <, \{c_\alpha\}_{\alpha<\mu}, \ldots \rangle$$

the set $\{c_m\}_{m\in M}$ contains a subset $\{c_\alpha\}_{\alpha<\mu}$ whose interpretation in $\mathfrak{M}_M$ are the ordinals $\alpha < \mu$. Similarly, $\{g_n\}_{n\in N}$ contains a subset $\{g_\beta\}_{\beta<\nu}$ whose interpretation in $\mathfrak{N}_N$ are the ordinals $\beta < \nu$. Now consider the linear order $<^{\mathfrak{D}}$. Since $\mathfrak{D} \models \mathrm{Th}\,\mathfrak{M}_M$ then $\langle \mu, < \rangle \cong \langle \{c_\alpha^{\mathfrak{D}}\}_{\alpha<\mu}, <^{\mathfrak{D}} \rangle$, and the set $\{c_\alpha^{\mathfrak{D}}\}_{\alpha<\mu}$ is unbounded in $<^{\mathfrak{D}}$, by (∗). Similarly, from $\mathfrak{D} \models \mathrm{Th}\,\mathfrak{N}_N$, we obtain that $\langle \nu, < \rangle \cong \langle \{g_\beta^{\mathfrak{D}}\}_{\beta<\nu}, <^{\mathfrak{D}} \rangle$ and the set $\{g_\beta^{\mathfrak{D}}\}_{\beta<\nu}$ is unbounded in $<^{\mathfrak{D}}$. Therefore, we get that μ is cofinal in $\nu > \mu$, thus contradicting the assumed regularity of ν. Therefore, the JEP fails in $\mathscr{L}$ if $|S| > 2^\lambda$. □

To be able to prove that no incompactness exists, we will need some set-theoretic hypotheses, as discussed in Section XVIII.1.3. However, if we assume the Robinson property, we can get even more. Recall that a logic $\mathscr{L}$ satisfies the *Robinson consistency theorem* (for short, $\mathscr{L}$ has the *Robinson property*) iff for arbitrary vocabularies τ, τ', τ'' and classes of sentences T, T', T'', if T is complete in τ and T' and T'' are consistent extensions of T in τ' and τ'', respectively, with $\tau = \tau' \cap \tau''$, then $T' \cup T''$ is consistent; (that is, $T' \cup T''$ has some model). Equivalently, we might assume also that T' and T'' in the above definition are complete. In fact, the Robinson property only depends on the complete theories of $\mathscr{L}$ and may thus be regarded as a property of the equivalence relation $\equiv_{\mathscr{L}}$. This notion will be pursued further in later sections, for in this section we will only be concerned with the effect of the Robinson property on logics.

1.2 Theorem. *Let $\mathscr{L}$ be a regular logic with dependence number $o(\mathscr{L}) \leq$ the first uncountable measurable cardinal μ_0—if it exists—or $o(\mathscr{L}) < \infty$ otherwise. If $\mathscr{L}$ has the Robinson property, then $\mathscr{L}$ has the finite dependence property.*

Proof. The proof follows immediately from Chapter XVIII, Corollary 3.3.5, Theorem 2.2.1, and Proposition 2.1.2. □

For logics of the form $\mathscr{L}(Q^i)_{i\in I}$ there is an easy, self-contained proof that the Robinson property implies compactness. This is given in the following result.

1.3 Theorem. *Let $\mathscr{L}$ be a regular logic. Assume that each sentence of $\mathscr{L}$ is of finite vocabulary—or even assume that $o(\mathscr{L})$ satisfies the hypotheses of Theorem 1.2. If $\mathscr{L}$ has the Robinson property, then $\mathscr{L}$ is compact.*

Proof. In the light of Theorem 1.2, it suffices to prove the theorem under the assumption that sentences in $\mathscr{L}$ are of finite vocabulary. Now assume that $\mathscr{L}$ has the Robinson property and is not compact (*absurdum* hypothesis). Let κ be the smallest cardinal such that $\mathscr{L}$ is not (κ, ω)-compact. There is a vocabulary τ and a set of sentences $T = \{\varphi_\alpha | \alpha < \kappa\} \subseteq \mathscr{L}[\tau]$ such that T has no model, while for each $\beta < \kappa$, the subtheory $T_\beta = \{\varphi_\alpha | \alpha < \beta\}$ does have a model $\mathfrak{A}_\beta$. In $\mathscr{L}$ we can replace function by relation symbols (by regularity); constant symbols are eliminable by using instead unary relations which represent singletons. This can be done in the usual manner for $\mathscr{L}_{\omega\omega}$ without using the substitution property. Thus, replace, for example, $\psi(c, d)$ by $\exists! cRc \wedge \exists! dSd \wedge \forall c, d(Rc \wedge Sd \to \psi(c, d))$. For arbitrary ψ', we proceed similarly, recalling that $|\tau_{\psi'}| < \omega$, where $\tau_{\psi'}$ is the smallest vocabulary of ψ', as given by the occurrence axiom. For the sake of notational simplicity we will also assume that τ is single-sorted (the proof for the many-sorted case only requires some additional notation). Without loss of generality the $\mathfrak{A}_\beta$'s have pairwise disjoint universes. Recalling that τ may be assumed to contain only relation symbols, define the disjoint union $\mathfrak{A} = \langle A, \ldots \rangle$ of the $\mathfrak{A}_\beta$'s by

$$A = \bigcup_{\beta<\kappa} A_\beta, \qquad R^{\mathfrak{A}} = \bigcup_{\beta<\kappa} R^{\mathfrak{A}_\beta} \quad \text{for each} \quad R \in \tau.$$

Define the function $f\colon A \to \kappa$ by $f(a) = \beta$ iff $a \in A_\beta$, for each $a \in A$, $\beta < \kappa$, and let $\mathfrak{M}$ be the two-sorted structure given by

$$\mathfrak{M} = [\mathfrak{A}, \langle \kappa, <, c_\beta \rangle_{\beta<\kappa}, f],$$

where, as usual, symbols are identified with their natural interpretation; and, in particular, $c_\beta^{\mathfrak{M}} = \beta$ for every $\beta < \kappa$.

Claim. *Whenever $\mathfrak{N} \equiv_{\mathscr{L}} \mathfrak{M}$, the set $\{c_\beta^{\mathfrak{N}}\}_{\beta<\kappa}$ is unbounded in $<^{\mathfrak{N}}$.*

Proof of Claim. Otherwise (*absurdum* hypothesis) let $\mathfrak{N}$ be a counterexample so that, for some fixed element n in the second sort of $\mathfrak{N}$, we have

$$\langle \mathfrak{N}, n \rangle \models_{\mathscr{L}} c_\beta < n \quad \text{for each} \quad \beta < \kappa.$$

For every $\beta < \kappa$, let ψ_β be the sentence of $\mathscr{L}$ given by

$$\psi_\beta \underset{\text{def}}{=} \forall z(c_\beta < z \to \varphi_\beta^{\{x | f(x) = z\}}).$$

This is the only place in this proof where we use the assumption that $\mathscr{L}$ is closed under relativization. Indeed, we only need that $\mathscr{L}$ be closed under relativization to atomic sentences. Observe that for each $\beta < \kappa$, we have $\mathfrak{M} \models_{\mathscr{L}} \psi_\beta$. Hence, $\mathfrak{N} \models_{\mathscr{L}} \psi_\beta$. Then for each $\beta < \kappa$,

$$\langle \mathfrak{N}, n \rangle \models_{\mathscr{L}} \varphi_\beta^{\{x | f(x) = n\}},$$

which implies that $\mathfrak{N} | \{x \in N | \langle \mathfrak{N}, x \rangle \models_{\mathscr{L}} f(x) = n\} \models_{\mathscr{L}} T$. This contradicts the assumed inconsistency of T and our claim is thus established. Now expand $\mathfrak{M}$ to $\mathfrak{M}' \in \mathrm{Str}(\tau')$, $\tau' = \tau_{\mathfrak{M}} \cup \tau_0$ with $\tau_0 = \{P_\beta\}_{\beta < \kappa}$ a set of new unary relations, to be interpreted in $\mathfrak{M}'$ as initial segments,

$$P_\beta^{\mathfrak{M}'} = \{\alpha | \alpha < \beta\} \quad \text{for each} \quad \beta < \kappa.$$

Let $T' = \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) \cup \{\forall x(P_\beta x \leftrightarrow x < c_\beta) | \beta < \kappa\}$, and observe that $\mathfrak{M}' \models_{\mathscr{L}} T'$. On the other hand, let $\tau'' = \tau_0 \cup \{c\}$, with c a new constant, and let

$$T'' = \{\neg P_\beta c | \beta < \kappa\}.$$

Consider the structure $\mathfrak{M}''$ of vocabulary τ'' given by

$$\mathfrak{M}'' = \langle \kappa \cup \{c\}, P_\beta^{\mathfrak{M}'} \rangle_{\beta < \kappa};$$

that is, $\mathfrak{M}''$ is obtained by adding one element at the end of κ and by interpreting each P_β exactly as in $\mathfrak{M}'$. Then we have that $\mathfrak{M}'' \models_{\mathscr{L}} T''$. For every finite $\tau^* \subseteq \tau_0$, we have that $\mathfrak{M}' \upharpoonright \tau^* \cong \mathfrak{M}'' \upharpoonright \tau^*$ (it is easy to get an isomorphism). Hence,

$$\mathfrak{M}' \upharpoonright \tau^* \equiv_{\mathscr{L}} \mathfrak{M}'' \upharpoonright \tau^*$$

by the isomorphism property of logics. Whence $\mathfrak{M}' \upharpoonright \tau_0 \equiv_{\mathscr{L}} \mathfrak{M}'' \upharpoonright \tau_0$, recalling that each sentence of $\mathscr{L}$ is of finite vocabulary. Now $\tau_{\mathfrak{M}'} \cap \tau_{\mathfrak{M}''} = \tau' \cap \tau'' = \tau_0$. Hence, by the assumed Robinson property of $\mathscr{L}$, there is $\mathfrak{D}$ of vocabulary $\tau' \cup \tau''$ such that $\mathfrak{D} \upharpoonright \tau' \equiv_{\mathscr{L}} \mathfrak{M}'$ and $\mathfrak{D} \upharpoonright \tau'' \equiv_{\mathscr{L}} \mathfrak{M}''$. In particular, $\mathfrak{D} \models_{\mathscr{L}} T' \cup T''$, and $c^{\mathfrak{D}}$ is a strict upper bound for the $\{c_\beta^{\mathfrak{D}}\}_{\beta < \kappa}$. But $\mathfrak{D} \upharpoonright \tau_{\mathfrak{M}} \equiv_{\mathscr{L}} \mathfrak{M}$ then stands as a counterexample to our claim. Therefore, $\mathscr{L}$ is compact. $\square$

1.4 Corollary. *Let $\mathscr{L}$ be a logic satisfying the hypotheses of Theorem* 1.3. *Assume further that $\mathscr{L}[\tau]$ is a set for every τ. Then $\mathscr{L}$ has the Robinson property iff $\mathscr{L}$ is compact and satisfies Craig's interpolation theorem.*

Proof. This proof requires use of Theorem 1.3 and Proposition II.7.1.5. The assumption that $\mathscr{L}[\tau]$ is a set for every τ is needed in the proof that compactness plus Robinson property imply interpolation. In order to apply compactness, we must guarantee that complete theories are *sets* of sentences. $\square$

1.5 Remark. Although it is stated only for regular $\mathscr{L}$, Theorem 1.3 still holds if the relativization axiom is replaced by the weaker requirement that $\mathscr{L}$ allow relativizations to atomic sentences. Also, the substitution axiom can be relaxed for the requirement that in $\mathscr{L}$ we are allowed to replace a function f by a relation R representing the graph of f. This will be important in the sequel (see Section 3.11.2, and Theorem 5.4).

1.6 Corollary. *Assume that $\mathscr{L}$ is a logic with the Robinson property, and that $\mathscr{L}[\tau]$ is a set for every τ, and $|\tau_\varphi| < \omega$ for every sentence φ. Assume further that $\mathscr{L}$ is closed under the atom, Boole, and particularization property of Definition* II.1.2.1. *Then the following are equivalent:*

(i) *$\mathscr{L}$ is closed under relativization to atomic sentences and allows elimination of function symbols;*
(ii) *$\mathscr{L}$ is regular.*

Proof. That (ii) implies (i) is evident in the light of Definition II.1.2.3. To prove that (i) implies (ii), we first note that, by Remark 1.5, Theorem 1.3 can be applied to $\mathscr{L}$. Since $\mathscr{L}[\tau]$ is always a set, then $\mathscr{L}$ satisfies Craig's interpolation theorem, by Corollary 1.4; and, in particular, $\mathscr{L}$ is Δ-closed (Definition II.7.2.1), whence regularity follows immediately. □

We now look at logics $\mathscr{L}$ which satisfy the Robinson property but are not $[\omega]$-compact. In contrast to the above results, no restriction is here imposed on the size of $\mathscr{L}[\tau]$ or on that of $o(\mathscr{L})$. On the other hand, we require that relativization in $\mathscr{L}$ incorporate τ-closure; that is, $\mathfrak{B} \models_{\mathscr{L}} \varphi^{\{x|\alpha(x)\}}$ implies that the set $B' = \{b \in B \mid \langle \mathfrak{B}, b\rangle \models_{\mathscr{L}} \alpha(x)\}$ contains all the constants of τ_φ; and, for each $f \in \tau_\varphi$, if $b_1, \ldots, b_n \in B'$, then $f(b_1, \ldots, b_n) \in B'$ (see Barwise [1974a, p. 235], and Flum [1975b, p. 294]). All the infinitary logics mentioned in the literature have this property; for logics in which all sentences have a finite vocabulary, the present form of relativization is exactly the same as the usual relativization as defined in Definition II.1.2.2, since τ-closure is expressible by a first-order sentence whenever τ is finite.

1.7 Theorem. *If $\mathscr{L}$ is a regular logic with the Robinson property and $\mathscr{L}$ is not $[\omega]$-compact, then for every $\mathfrak{A}$, $\mathfrak{B}$ with $|\mathfrak{A}|$ of cardinality $< \mu_0$, we have that $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ implies $\mathfrak{A} \cong \mathfrak{B}$. If no uncountable measurable cardinal exists, then $\equiv_{\mathscr{L}} = \cong$.*

Proof. To prove this theorem, we establish three formal claims.

Claim 1. *$\langle \omega, <, c_n\rangle_{n<\omega}$ is characterized up to isomorphism by its own complete theory in $\mathscr{L}$.*

Proof of Claim 1. The proof is reminiscent of the proof of the first part of Theorem 1.3. Let the pair $T = \{\varphi_i \mid i < \omega\}$, Δ be a counterexample to $[\omega]$-compactness in $\mathscr{L}$. For every $m < \omega$, let $T_m = \{\varphi_i \mid i < m\} \cup \Delta$, and let $\mathfrak{A}_m \models T_m$. For the moment, assume the vocabulary τ of $T \cup \Delta$ is single-sorted and has only relations. Assume further that the universes of the $\mathfrak{A}_m$'s are pairwise disjoint. Define the disjoint union $\mathfrak{A} = \langle A, \ldots\rangle$ of the $\mathfrak{A}_m$'s by

$$A = \bigcup_{m<\omega} A_m, \qquad R^{\mathfrak{A}} = \bigcup_{m<\omega} R^{\mathfrak{A}_m} \quad \text{for each} \quad R \in \tau.$$

Let $f: A \to \omega$ be defined by $f(a) = m$ iff $a \in A_m$. Let finally the two-sorted structure $\mathfrak{M}$ be given by $\mathfrak{M} = [\mathfrak{A}, \langle \omega, <, c_m \rangle_{m<\omega}, f]$. By arguing as in the proof of Claim 1 in Theorem 1.3, we see that whenever $\mathfrak{N} \equiv_{\mathscr{L}} \mathfrak{M}$, the $\{c_m^{\mathfrak{N}}\}_{m<\omega}$ are unbounded in $<^{\mathfrak{N}}$. We can now prove that whenever $\mathfrak{D} \equiv_{\mathscr{L}} \langle \omega, <, c_n \rangle_{n<\omega}$, we also have $\mathfrak{D} \cong \langle \omega, <, c_n \rangle_{n<\omega}$. As a matter of fact, if this were not the case and $\mathfrak{D}$ were a counterexample, then we expand $\mathfrak{D}$ to $\mathfrak{D}^+ = \langle \mathfrak{D}, c, g \rangle$, where g maps the set W of predecessors of c one–one onto $W \cup \{c\}$. We expand $\langle \omega, <, c_n \rangle_{n<\omega}$ to the structure $\mathfrak{M}$ defined above. Using the Robinson property of $\mathscr{L}$, we exhibit $\mathfrak{B}$ such that $\mathfrak{B} \upharpoonright \tau_{\mathfrak{M}} \equiv_{\mathscr{L}} \mathfrak{M}$ and $\mathfrak{B} \upharpoonright \tau_{\mathfrak{D}^+} \equiv_{\mathscr{L}} \mathfrak{D}^+$. Then the $\{c_n^{\mathfrak{B}}\}_{n<\omega}$ are unbounded in $<^{\mathfrak{B}}$, by the discussion above. But they are also bounded by $c^{\mathfrak{B}}$, because $\mathfrak{B} \models_{\mathscr{L}} c > c_n$ for all $n < \omega$—a contradiction which establishes our claim in case τ is single-sorted and only contains relation symbols. The many-sorted case (for τ only containing relations) can be established, with the help of additional notation. We now consider in detail the case in which some sentence ψ of T is such that the set τ_ψ of symbols occurring in ψ, as given by the occurrence axiom, also contains constants (but no functions). If there are only finitely many such constants, then we can get rid of them by using unary relations and renamings, without using the substitution property of $\mathscr{L}$ (see the proof of Theorem 1.3). Otherwise, if τ_ψ has infinitely many constants, display them as $\{b_\alpha\}_{\alpha<\kappa}$, for some $\kappa \geq \omega$. Recalling that we assumed that relativization incorporates τ_ψ-closure, whenever U is a new relation, we have

(1) $\quad (\psi \vee \neg \psi)^{\{x|Ux\}}$ is equivalent to $Ub_0 \wedge Ub_1 \wedge \cdots$.

Similarly, letting θ be the sentence in $\mathscr{L}$ given by

(2) $\quad \theta \underset{\text{def}}{=} \forall y \neg((\psi \vee \neg\psi)^{\{x|x \neq y\}})$,

we must have

(3) $\quad \theta$ is equivalent to $\forall y(y = b_0 \vee y = b_1 \vee \cdots)$.

Add a new relation V and let theory Γ be given by

(4) $\quad \Gamma \underset{\text{def}}{=} \{Vb_\beta | \beta < \omega\} \cup \{\neg Vb_\gamma | \gamma \geq \omega\}$.

By (2) and (3), for every structure $\mathfrak{S}$, we have

(5) $\quad \mathfrak{S} \models \Gamma \cup \{\theta\}$ implies $V^{\mathfrak{S}} = \{b_\beta^{\mathfrak{S}}\}_{\beta<\omega}$.

Let T' be defined by $T' \underset{\text{def}}{=} \mathrm{Th}_{\mathscr{L}}\langle \omega, <, c_n \rangle_{n<\omega} \cup \Gamma \cup \{\theta, \eta\}$, where η says that f is a one–one mapping from the new sort of the $\{c_n\}_{n<\omega}$ onto $V = \{b_\beta\}_{\beta<\omega}$. Then, by (5), each model of T' will be an expansion of $\langle \omega, <, c_n \rangle_{n<\omega}$, the latter being defined on a new sort. We now complete the proof of Claim 1. Assume $\mathrm{Th}_{\mathscr{L}}\langle \omega, <, c_n \rangle_{n<\omega}$ has a non-standard model $\mathfrak{D}$ (*absurdum* hypothesis). Expand $\mathfrak{D}$ to $\mathfrak{D}' = \langle \mathfrak{D}, c, g \rangle$, where c is a strict upper bound for the $\{c_n\}_{n<\omega}$, and g maps the set K of predecessors of c one–one onto $K \cup \{c\}$. Then $\mathrm{Th}_{\mathscr{L}}\, \mathfrak{D}' \cup T'$ has no models, thus contradicting the Robinson property of $\mathscr{L}$. This completes the proof of Claim 1 (the case in which τ_ψ has function symbols can be reduced to the cases considered above).

Claim 2. *Let $\kappa \geq \omega$ be an arbitrary cardinal. Assume $\langle \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$ is characterized (up to isomorphism by its own complete theory in $\mathscr{L}$), and also each $\mathfrak{A}'$ with $|A'| < \kappa$ is characterized. Then every $\mathfrak{A}$ with $|A| = \kappa$ is characterized.*

Proof of Claim 2. To establish this claim, we consider two cases, the first being the

Special Case. Here $\mathfrak{A} = \langle \kappa, <, c_\alpha, R^{\mathfrak{A}}, \ldots \rangle_{\alpha<\kappa}$ is a single-sorted expansion of $\langle \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$. Then let $\tau = \tau_{\mathfrak{A}}$ and assume $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}$, but $\mathfrak{B} \not\cong \mathfrak{A}$ *(absurdum* hypothesis). By assumption (and by the reduct and isomorphism axioms given in Definition II.1.1.1) we can safely write $\mathfrak{B} = \langle \kappa, <, c_\alpha, R^{\mathfrak{B}}, \ldots \rangle_{\alpha<\kappa}$. Since $\mathfrak{B} \not\cong \mathfrak{A}$, then without loss of generality we must have $R^{\mathfrak{A}} \neq R^{\mathfrak{B}}$. For the sake of definiteness assume that R is a unary relation (the other cases being treated similarly). We then have that for some $\beta < \kappa$, $R^{\mathfrak{A}}\beta$ holds and $R^{\mathfrak{B}}\beta$ does not (or vice versa). Now by the assumed characterizability of $\langle \beta, <, c_\alpha \rangle_{\alpha<\beta}$ we have that $c_\beta^{\mathfrak{A}} = c_\beta^{\mathfrak{B}} = \beta$, so that $\mathfrak{A} \models_{\mathscr{L}} Rc_\beta$ and $\mathfrak{B} \models_{\mathscr{L}} \neg Rc_\beta$, thus contradicting $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$. Consider now the

General Case. Here we assume that $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, $|A| = \kappa$. For the moment, let $\mathfrak{A}$ be single-sorted; let $|B| = \lambda$. Then we must have that $\lambda \leq \kappa$; for, otherwise, by expanding $\mathfrak{A}$ to $\mathfrak{A}^+ = \langle \mathfrak{A}, \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$ and $\mathfrak{B}$ to $\mathfrak{B}' = \langle \mathfrak{B}, b_\beta \rangle_{\beta<\lambda}$, using the Robinson property, we exhibit $\mathfrak{M}$ with $\mathfrak{M} \upharpoonright \tau_{\mathfrak{A}^+} \equiv_{\mathscr{L}} \mathfrak{A}^+$ and $\mathfrak{M} \upharpoonright \tau_{\mathfrak{B}'} \equiv_{\mathscr{L}} \mathfrak{B}'$. Hence, by hypothesis $\mathfrak{M} \upharpoonright \{<\} \cong \langle \kappa, < \rangle$ and $|M| \geq \lambda > \kappa$, since $b_\beta^{\mathfrak{M}} \neq b_\alpha^{\mathfrak{M}}$ for $\beta \neq \alpha$. This is a contradiction. Having seen that $|B| = \lambda \leq \kappa$, we now expand $\mathfrak{B}$ to $\mathfrak{B}^+ = \langle \mathfrak{B}, \lambda, <', d_\beta \rangle_{\beta<\lambda}$, where $<'$ is a new binary relation symbol having the natural interpretation in $\mathfrak{B}^+$. By the Robinson property, we let $\mathfrak{N}$ be such that $\mathfrak{N} \upharpoonright \tau_{\mathfrak{A}^+} \equiv_{\mathscr{L}} \mathfrak{A}^+$ and $\mathfrak{N} \upharpoonright \tau_{\mathfrak{B}^+} \equiv_{\mathscr{L}} \mathfrak{B}^+$. Now $\mathfrak{A}^+$ is taken care of by the special case just considered, and so is $\mathfrak{B}^+$—unless $\lambda < \kappa$, in which case $\mathfrak{B}^+$ is characterized up to isomorphism by hypothesis. In definitive, we have that $\mathfrak{N} \upharpoonright \tau_{\mathfrak{A}^+} \cong \mathfrak{A}^+$ and $\mathfrak{N} \upharpoonright \tau_{\mathfrak{B}^+} \cong \mathfrak{B}^+$. By taking reducts, we finally obtain $\mathfrak{A} \cong \mathfrak{N} \upharpoonright \tau_{\mathfrak{A}} = \mathfrak{N} \upharpoonright \tau_{\mathfrak{B}} \cong \mathfrak{B}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are many-sorted, one proceeds similarly, by first excluding the possibility of $\mathfrak{B}$ having sorts of cardinality $> \kappa$, and by adding one copy of $\langle |S|, <', \ldots \rangle$ over each sort S in $\mathfrak{B}$. This completes the proof of Claim 2.

Claim 3. *All structures of cardinality $<\mu_0$ are characterized.*

Proof of Claim 3. Let κ be the least cardinal such that there are two $\mathscr{L}$-equivalent non-isomorphic structures $\mathfrak{A}'$ and $\mathfrak{A}''$, with $\kappa = |A'| \leq |A''|$. Clearly $\kappa \geq \omega$, and by Claim 2 it follows that $\mathfrak{A} = \langle \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$ is not characterized. By Claim 1, we see that κ is uncountable. We will now prove that κ is measurable. Let $\mathfrak{B} = \langle B, <, c_\alpha \rangle_{\alpha<\kappa}$ with $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}$ and $\mathfrak{B} \not\cong \mathfrak{A}$. Using standard arguments of model theory, along with the characterizability of each ordinal $\beta < \kappa$, we conclude that there must be some $b \in B$ such that $\mathfrak{B}^+ \models_{\mathscr{L}} b > c_\alpha$ for all $\alpha < \kappa$, where $\mathfrak{B}^+ = \langle \mathfrak{B}, b \rangle$. Expand $\mathfrak{A}$ to $\mathfrak{A}^+$, adding symbols for all unary functions and relations on κ, as follows:

$$\mathfrak{A}^+ = \langle \kappa, <, c_\alpha, U_s, f_j \rangle_{\alpha<\kappa, s \in P(\kappa), j \in {}^\kappa\kappa}$$

with

$$U_s^{\mathfrak{A}^+} = s \quad \text{and} \quad f_j^{\mathfrak{A}^+} = j,$$

where $P(\kappa)$ is the power set of κ. Using the Robinson property, let $\mathfrak{M}$ be such that $\mathfrak{M} \upharpoonright \tau_{\mathfrak{B}^+} \equiv_{\mathscr{L}} \mathfrak{B}^+$ and $\mathfrak{M} \upharpoonright \tau_{\mathfrak{A}^+} \equiv_{\mathscr{L}} \mathfrak{A}^+$. Define $D \subseteq P(\kappa)$ by

$$s \in D \quad \text{iff} \quad s \subseteq \kappa \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} U_s(b);$$

that is, $s \in D$ iff the unary relation U_s whose interpretation is s in $\mathfrak{A}^+$ has $b^{\mathfrak{M}}$ among its elements when interpreted in $\mathfrak{M}$. Clearly D is a nonprincipal ultrafilter on κ. We now show that D is κ-complete. If not (*absurdum* hypothesis), D is μ-descendingly incomplete for some $\mu < \kappa$; that is, there is a descending chain $D' = \{s'_\alpha\}_{\alpha<\mu}$ with $s'_\alpha \in D$ and $\bigcap_{\alpha<\mu} s'_\alpha \notin D$. Hence, without loss of generality, $\bigcap_{\alpha<\mu} s'_\alpha = \varnothing$. Without loss of generality, we may also assume that for every limit ordinal $\varepsilon < \mu$, $\bigcap_{\alpha<\varepsilon} s'_\alpha = s'_\varepsilon$. Define $h\colon \kappa \to \mu$ by $h(\beta) = \alpha$ iff $\beta \in s'_\alpha \backslash s'_{\alpha+1}$, for $\beta < \kappa$, $\alpha < \mu$, so that intuitively h tells us how long an element $\beta \in \kappa$ stays in the descending, and eventually vanishing, chain D'. h is well defined, by our assumption that for every β the first η such that $\beta \notin s'_\eta$ is a successor ordinal. Let $U_\alpha = U_{s'_\alpha}$ $(\alpha < \mu)$. Then, for every $\alpha < \mu$, we have

$$\begin{aligned}
&\mathfrak{A}^+, \mathfrak{M} \models_{\mathscr{L}} \forall x(h(x) \leq c_\alpha \to \neg U_{\alpha+1}(x)),\\
&\qquad \mathfrak{M} \models_{\mathscr{L}} h(b) > c_\alpha \quad \text{since} \quad \mathfrak{M} \models_{\mathscr{L}} U_{\alpha+1}(b),\\
&\qquad \mathfrak{M} \models_{\mathscr{L}} \forall x(x < c_\mu \to h(b) > x) \quad \text{since } \mu < \kappa \text{ is characterizable},\\
&\mathfrak{M}, \mathfrak{A}^+ \models_{\mathscr{L}} \exists y(\forall x(x < c_\mu \to h(y) > x)),
\end{aligned}$$

so that $\bigcap_{\alpha<\mu} s'_\alpha \neq \varnothing$—a contradiction. Therefore, D is κ-complete, and κ is measurable, indeed uncountable and measurable. This completes the proof of Claim 3 and of the theorem as well. □

1.8 Corollary. *If $\mathscr{L}$ is a regular logic with the Robinson property, and there are $<\mu_0$ many sentences in the pure identity language of $\mathscr{L}$, then $\mathscr{L}$ is $[\omega]$-compact.* □

1.9 Examples. (a) The logic $\mathscr{L}_{\infty\infty} = \mathscr{L}$ has the Robinson property and is not $[\omega]$-compact. Here $\mathscr{L}[\tau]$ is a proper class and $\equiv_{\mathscr{L}} = \cong$.

(b) If κ is an extendible cardinal, then $\mathscr{L}^{\mathrm{II}}_{\kappa\kappa}$, infinitary logic with conjunctions and quantifications of elements and relations of length $<\kappa$, has the Robinson property and is not $[\omega]$-compact. Here $\mathscr{L}^{\mathrm{II}}_{\kappa\kappa}$-equivalence coincides with isomorphism below κ, and $\kappa \geq \mu_0$. Indeed $\kappa \geq$ the first supercompact cardinal (see Magidor [1971], and Examples XVIII.3.3.7).

The above examples, together with Theorem 1.7, simply tell us that if $\mathscr{L}$ fares well with the interpolation or definability properties, but does not do so with compactness, then its expressive power is extremely strong below some measurable cardinal. The prototype of this sort of result is Scott's theorem which yields for each countable structure $\mathfrak{A}$ a sentence $\varphi_{\mathfrak{A}}$ of $\mathscr{L}_{\omega_1\omega}$ whose countable models are exactly those which are isomorphic to $\mathfrak{A}$ (see Theorem VIII.4.1.1). A partial converse is given by the following result.

1.10 Theorem. *Let $\mathscr{L}$ be a logic such that for every countable structure $\mathfrak{A}$ there is a sentence $\varphi_{\mathfrak{A}}$ of vocabulary $\tau_{\mathfrak{A}}$ having the property that for any countable $\mathfrak{B} \in \mathrm{Str}(\tau_{\mathfrak{A}})$, $\mathfrak{B} \models_{\mathscr{L}} \varphi_{\mathfrak{A}}$ implies $\mathfrak{B} \cong \mathfrak{A}$. Then $\Delta\mathscr{L}$ is an extension of $\mathscr{L}_{\omega_1\omega}$.*

Proof. See Section XVII.3.2. □

Since $\mathscr{L}_{\omega_1\omega}$ is Δ-closed, Theorem 1.10 implies that $\mathscr{L}_{\omega_1\omega}$ can be characterized as the smallest Δ-closed logic satisfying Scott's theorem. Using Theorem 1.7, we can now prove an analogue of Theorem 1.10.

1.11 Definition. Let $\mathscr{L}$, $\mathscr{L}'$ be logics. We say that $\mathscr{L}'$ is *weakly interpretable in* $\mathscr{L}$ iff for every sentence $\varphi \in \mathscr{L}'[\tau]$ there is a vocabulary $\sigma \supseteq \tau$ and a set of sentences $\Sigma \subseteq \mathscr{L}[\sigma]$ such that $\mathrm{Mod}(\varphi) = (\mathrm{Mod}(\Sigma)) \upharpoonright \tau$.

1.12 Theorem. *If $\mathscr{L}$ is a regular logic with the Robinson property which is not $[\omega]$-compact, then*:

(i) *$\mathscr{L}_{\mu_0\omega}$ is weakly interpretable in $\mathscr{L}$; and,*
(ii) *$\equiv_{\mathscr{L}}$ is finer than $\equiv_{\mathscr{L}_{\mu_0\omega}}$; that is, $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ implies $\mathfrak{A} \equiv_{\mathscr{L}_{\mu_0\omega}} \mathfrak{B}$.*

Proof. For (i). If $\varphi \in \mathscr{L}_{\mu_0\omega}[\tau]$, then we can assume that $\varphi \in H(\kappa)$, for some $\kappa < \mu_0$. We now follow Feferman [1974a, b, 1975] and find an expansion $\mathfrak{M}_\varphi$ of $\langle H(\kappa), \in\rangle$ and a set $\Sigma_\varphi \subseteq \mathscr{L}[\sigma]$, for some $\sigma \supseteq \tau$, such that for all $\mathfrak{A} \in \mathrm{Str}(\tau)$ we have that

$$\mathfrak{A} \models \varphi \quad \text{iff the pair} \quad \langle \mathfrak{A}, \mathfrak{M}_\varphi\rangle \models \Sigma_\varphi.$$

The existence of $\mathfrak{M}_\varphi$ and Σ_φ with the required properties is guaranteed by Theorem 1.7.

Deny (ii). Then there is $\mathfrak{A} \in \mathrm{Str}(\tau)$ and $\varphi \in \mathscr{L}_{\mu_0\omega}[\tau]$ such that if we let $T = \mathrm{Th}_{\mathscr{L}}(\mathfrak{A})$, then both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ have a model. Let $\sigma, \sigma' \supseteq \tau, \Sigma_\varphi \subseteq \mathscr{L}[\sigma]$ and $\Sigma_{\neg\varphi} \subseteq \mathscr{L}[\sigma']$ be as in the proof of (i), and $\sigma \cap \sigma' = \tau$. By (i) each of $T \cup \Sigma_\varphi$ and $T \cup \Sigma_{\neg\varphi}$ has a model, and by the Robinson property of $\mathscr{L}$ we can write $\mathfrak{B} \models_{\mathscr{L}} T \cup \Sigma_\varphi \cup \Sigma_{\neg\varphi}$, for some $\mathfrak{B}$. Hence, $\mathfrak{B} \models T \cup \{\varphi, \neg\varphi\}$—a contradiction. □

To some extent, Theorem 1.12 clarifies how a non-$[\omega]$-compact logic with the Robinson property resembles an infinitary logic built on a measurable cardinal. Indeed, the only known examples of such logics involve an extendible cardinal (see Example 1.9(b)). Shelah has constructed a logic $\mathscr{L}$ with the amalgamation property, (a property which is weaker than the Robinson property) and which still does not contain $\mathscr{L}_{\omega_1\omega_1}$. This result was given in a private communication, and it seems an interesting problem to explore it with the view of making possible improvements of Theorem 1.12.

1.13 Notes and Remarks. A more detailed proof of Theorem 1.1 can be extracted from Mundici [1982b, pp. 64–66], where it is shown that compactness = JEP (for logics where $\mathscr{L}[\tau]$ is a set) under such set-theoretical hypotheses as $V = L$ or $\neg 0^{\#}$. Theorem 1.2 was originally proved by Makowsky–Shelah [1983]. Theorem 1.3 and Corollary 1.4 are independently due to Mundici [1982b], and Makowsky–Shelah [1983]. The proof presented here is given by Lindström in a private communication. Theorem 1.7 is due to Mundici [1982f] (see also [1982a] for results on the many-sorted case). The proof given here uses a number of ingredients from

Rabin [1959], Keisler [1963b], Lindström [1968] and Makowsky–Shelah [1979b]. In this latter reference, a variant of Corollary 1.8 was proven using a weaker notion of Robinson property together with the Feferman–Vaught property and different assumptions about $|\mathscr{L}[\tau]|$. For Example 1.9(b) see Magidor [1971] and Makowsky–Shelah [1983]. Theorem 1.10 is due to Makowsky [1973] and Barwise [1974a]. Actually, the theorem still holds under the weaker hypothesis that we can characterize by a sentence of $\mathscr{L}$ every structure of the form $\langle \omega, <, P\rangle$ with P an arbitrary subset of ω (see Makowsky–Shelah–Stavi [1976]). Theorem 1.12 is an unpublished result of Makowsky.

2. *Abstract Model Theory for Enriched Structures*

This short section is devoted to extending the results of Section 1 to logics for enriched structures, such as topological, uniform, proximity structures (see Chapter XV). The reader who is only interested in the usual (first-order) structures may safely proceed to Section 3 at first reading.

For an arbitrary nonempty set B, the *superstructure* V_ω^B of B is given by $V_0^B = B$, $V_{n+1}^B = V_n^B \cup PV_n^B$, $V_\omega^B = \bigcup_n V_n^B$, where P is the power-set operation. An *enriched structure* of vocabulary τ is a pair $\mathfrak{M}' = \langle \mathfrak{M}, \mu\rangle$ where $\mathfrak{M} \in \mathrm{Str}(\tau)$ is an *ordinary* structure (as defined in Chapter II), and $\mu \in V_\omega^M$. The many-sorted case is an immediate generalization of this notion. Examples of enriched structures are topological, weak, uniform, monotone, proximity, ordinary structures, as well as the structures studied in Chang [1973] in the framework of modal model theory. The *forgetful functor* $\|\cdot\|$ transforms $\mathfrak{M}'$ into $\mathfrak{M}$; the operations of reduct, renaming, diagram expansion, disjoint union (for structures of disjoint vocabularies) are the same as in the ordinary case. A *strict expansion* of $\mathfrak{M}' = \langle \mathfrak{M}, \mu\rangle$ is any structure $\mathfrak{M}'' = \langle \mathfrak{M}^+, \mu\rangle$, where $\mathfrak{M}^+$ is an expansion of $\mathfrak{M}$. The *ordinary semantic domain* is the function $\mathscr{O}$ assigning to every vocabulary τ the category $\mathscr{O}(\tau) = \langle \mathrm{Str}(\tau), \mathrm{Emb}(\tau)\rangle$ whose arrows are the isomorphic embeddings equipped with composition. More generally we consider

2.1 Definition. A *semantic domain* is a function $\mathscr{C}$ assigning to every vocabulary τ a category $\mathscr{C}(\tau) = \langle \mathrm{Ob}(\tau), \mathrm{Ar}(\tau)\rangle$ whose objects are enriched structures of vocabulary τ and whose arrows, called the *isomorphic embeddings* of $\mathscr{C}$, are functions equipped with composition, satisfying the following seven conditions:

(a) $\|\cdot\|$ preserves identities and commutative diagrams;
(b) $\cup_\tau \mathrm{Ob}(\tau)$ is closed under reduct, renaming, strict expansion, formation of disjoint pairs, and *disjoint union*; that is, for every set $\{\mathfrak{B}_\alpha\}_{\alpha<\kappa} \subseteq \mathrm{Ob}(\tau)$, τ without constants, there are $\mathfrak{B} \in \mathrm{Ob}(\tau)$ and arrows $g_\alpha: \mathfrak{B}_\alpha \to \mathfrak{B}$ $(\alpha < \kappa)$ having pairwise disjoint ranges whose union is B (here we essentially require that all the operations on structures used in the proof of Theorem 1.3 are also available for our enriched structures);

(c) for each ordinary structure $\mathfrak{A}$ there is a structure $\mathfrak{B}$ in $\mathscr{C}$ with the same vocabulary and such that $\|\mathfrak{B}\| = \mathfrak{A}$ (this amounts to requiring that $\mathscr{C}$ extends the ordinary semantic domain);
(d) $g: \mathfrak{M} \to \mathfrak{N}$ iff $g: \mathfrak{M}^\rho \to \mathfrak{N}^\rho$ for any renaming ρ;
(e) $g: \mathfrak{M} \to \mathfrak{N}$ iff $g: \mathfrak{M} \restriction \tau \to \mathfrak{N} \restriction \tau$ for all finite $\tau \subseteq \tau_{\mathfrak{M}} = \tau_{\mathfrak{N}}$;
(f) $f: \mathfrak{M} \to \mathfrak{N}, g: \mathfrak{A} \to \mathfrak{B}$ and $\tau_{\mathfrak{M}} \cap \tau_{\mathfrak{A}} = \varnothing$ imply $f \cup g: [\mathfrak{M}, \mathfrak{A}] \to [\mathfrak{N}, \mathfrak{B}]$;
(g) $g: \mathfrak{M} \to \mathfrak{N}$ implies $g: \mathfrak{M}_M \to \mathfrak{N}_{g(M)}$, $\mathfrak{M}_M =$ diagram expansion of $\mathfrak{M}$.

We also say that $\mathscr{C}$ *has substructures* iff $\mathscr{C}$ satisfies the following two additional conditions:

(h) whenever $B' \subseteq B$ is the range of an isomorphic embedding into $\|\mathfrak{B}\|$ (with respect to $\mathscr{O}$), then B' is also the range of some isomorphic embedding into the whole of $\mathfrak{B}$ (with respect to $\mathscr{C}$);
(i) whenever $\mathfrak{M} \xrightarrow{f} \mathfrak{B} \xleftarrow{g} \mathfrak{N}$ and $\mathrm{range}(f) \subseteq \mathrm{range}(g)$, then there exists $h: \mathfrak{M} \to \mathfrak{N}$ such that $f = g \circ h$.

2.2 Examples. The following are semantic domains with substructures: the category of topological structures with homeomorphic embeddings (see Chapter XV); the monotone structures with monotone embeddings (see Makowsky–Tulipani [1977]); the uniform structures with uniformly continuous embeddings (see Flum–Ziegler [1980]), the proximity structures with proximity-preserving embeddings.

The notion of a logic $\mathscr{L}$ over a semantic domain $\mathscr{C}$ is exactly the same as for the ordinary case (see Chapter II), except for the definition of relativization, which requires a little more care:

2.3 Definition. A logic $\mathscr{L}$ over $\mathscr{C}$ has *relativization* iff $\mathscr{C}$ has substructures and for every boolean combination α of atomic sentences with $\tau_\alpha \supseteq \{x\}$, and every sentence $\varphi \in \mathscr{L}[\tau_\varphi]$ there is $\psi \in \mathscr{L}[\tau']$ (with $\tau' = \tau_\varphi \cup (\tau_\alpha \setminus \{x\})$), denoted $\psi = \varphi^{\{x|\alpha\}}$, such that for all $\mathfrak{B} \in \mathrm{Str}(\tau')$, $\mathfrak{B} \models_{\mathscr{L}} \psi$ iff $\{b \in B \mid \|\langle \mathfrak{B}, b\rangle\| \models \alpha\}$ is the range of an isomorphic embedding $g: \mathfrak{N} \to \mathfrak{B} \restriction \tau_\varphi$, for some $\mathfrak{N} \in \mathrm{Str}(\tau_\varphi)$ with $\mathfrak{N} \models_{\mathscr{L}} \varphi$.

The assumption that $\mathscr{C}$ has substructures ensures that $\mathfrak{N}$ in the above definition is unique up to the isomorphism relation in $\mathscr{C}$. Furthermore we have incorporated τ-closure in relativization. In other words, $\mathfrak{B} \models_{\mathscr{L}} \varphi^{\{x|\alpha\}}$ implies that the substructure $\mathfrak{B} \mid \alpha^{\mathfrak{B}}$ contains all the constants of τ_φ and is closed under all the functions of τ_φ. We can recover the ordinary definition given in Definition II.1.2.2 simply by noting that for ordinary structures the following holds, for any isomorphic embedding g:

$$g: \mathfrak{N} \to \mathfrak{B} \restriction \tau_\varphi \quad \text{iff} \quad \mathfrak{N} \cong (\mathfrak{B} \restriction \tau_\varphi) \mid \mathrm{range}(g).$$

In Chapter XV the reader encountered a logic $\mathscr{L}^t$ which stands to topological structures as $\mathscr{L}_{\omega\omega}$ stands to ordinary structures. In Chapter III it is shown that large portions of ordinary abstract model theory can be extended to the realm of enriched structures. As for extensions of the results of Section 1, we have:

2.4 Theorem. *For $\mathscr{C}$ an arbitrary semantic domain with substructures, and $\mathscr{L}$ a logic over $\mathscr{C}$ obeying the hypotheses of Theorem* 1.1, *the conclusion of the theorem still holds.*

Proof. See Mundici [1982c, II and 198?b]. □

It remains an open problem whether or not the main results of Chapter XVIII—notably, the implication AP $\Rightarrow$ compactness—or even Theorem 1.3 above can be extended to logics over arbitrary $\mathscr{C}$. With the help of such axioms as $V = L$ or $\neg O^{\#}$ we can strengthen Theorem 1.1 to the effect that if $\mathscr{L}$ is not compact, then there is a proper class of regular cardinals κ such that $\mathscr{L}$ is not $[\kappa]$-compact. By using Theorem 2.4, the proof of this fact for $\mathscr{O}$ can be extended to arbitrary $\mathscr{C}$. Hence, we have

2.5 Theorem ($V = L$, *or even* $\neg O^{\#}$). *For $\mathscr{C}$ an arbitrary semantic domain with substructures and $\mathscr{L}$ a regular logic over $\mathscr{C}$, assume that $\mathscr{L}[\tau]$ is a set for every τ and that $|\tau_\varphi| < \omega$ for every sentence φ. Then if $\mathscr{L}$ has the Robinson property, $\mathscr{L}$ is compact.*

Proof. The reader is referred to Mundici [1982c, II and 198?b]. Actually, the theorem is proven there under the weaker assumption (denoted $\natural$) that for every infinite regular cardinal κ and for every uniform ultrafilter D over κ, D is λ-descendingly incomplete for all infinite $\lambda \leq \kappa$. For a proof that $\natural$ is weaker than $\neg O^{\#}$, the reader should consult D. Donder, R. B. Jensen, and B. J. Koppelberg; *Lecture Notes in Mathematics*, **872** (1981), p. 91. □

3. *Duality Between Logics and Equivalence Relations*

We now return to ordinary (first-order) structures. As we remarked in Section 1, the Robinson property of a logic $\mathscr{L}$ only depends on $\equiv_{\mathscr{L}}$. In general, for $\sim$ an arbitrary equivalence relation on the class of all structures, we say that $\sim$ has the *Robinson property* iff for every $\mathfrak{A}' \in \mathrm{Str}(\tau')$, $\mathfrak{A}'' \in \mathrm{Str}(\tau'')$, if $\mathfrak{A}' \upharpoonright \tau \sim \mathfrak{A}'' \upharpoonright \tau$ and $\tau = \tau' \cap \tau''$ then there is $\mathfrak{M} \in \mathrm{Str}(\tau' \cup \tau'')$ such that $\mathfrak{M} \upharpoonright \tau' \sim \mathfrak{A}'$ and $\mathfrak{M} \upharpoonright \tau'' \sim \mathfrak{A}''$. It is immediately seen that whenever $\sim \; = \; \equiv_{\mathscr{L}}$ for some logic $\mathscr{L}$, the relation $\sim$ has the Robinson property iff $\mathscr{L}$ satisfies the Robinson consistency theorem. Among the equivalence relations with the Robinson property, we mention elementary equivalence $\equiv$, isomorphism $\cong$, equality $=$ and $\equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}^{\mathrm{II}}_{\kappa\kappa}$ (see Example 1.9(b)). All the equivalence relations considered in this paper will satisfy a few natural prerequisites which the attentive reader may find reminiscent of the simplest axiomatic properties of logics.

3.1 Definition. Let $\sim$ be an equivalence relation on the class of all structures. Then $\sim$ is said to be *regular* iff $\sim$ satisfies the following conditions (for every two structures $\mathfrak{M}$ and $\mathfrak{N}$):

vocabulary: $\mathfrak{M} \sim \mathfrak{N}$ implies $\tau_{\mathfrak{M}} = \tau_{\mathfrak{N}}$;
renaming: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M}^\rho \sim \mathfrak{N}^\rho$ for any $\rho: \tau_{\mathfrak{M}} \to \tau'$;
reduct: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M} \restriction \tau \sim \mathfrak{N} \restriction \tau$ for any $\tau \subseteq \tau_{\mathfrak{M}}$;
isomorphism: $\mathfrak{M} \cong \mathfrak{N}$ implies $\mathfrak{M} \sim \mathfrak{N}$;
expressiveness: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M} \equiv \mathfrak{N}$.

Moreover, we say that $\sim$ is *bounded* iff for every vocabulary τ there is a *set* $S_\tau \subseteq \mathrm{Str}(\tau)$ such that for every $\mathfrak{A} \in \mathrm{Str}(\tau)$ there is $\mathfrak{B} \in S_\tau$ with $\mathfrak{B} \sim \mathfrak{A}$. Thus, all equivalence classes have a representative in S_τ. Observe that this has nothing to do with "bounded" logics. When $\sim$ is a regular equivalence relation on the class of all structures and has the Robinson property, then we simply say that $\sim$ is a *Robinson equivalence relation.* Of the four equivalence relations given above, $\equiv$ and $\equiv_{\mathscr{L}^{\mathrm{II}}_{\kappa\kappa}}$ are bounded Robinson equivalence relations. If $\mathscr{L}[\tau]$ is a set for every τ, then $\equiv_{\mathscr{L}}$ is (regular and) bounded. Conversely, if $\sim\ =\ \equiv_{\mathscr{L}}$ then $\mathscr{L}$ is (equivalent to) a logic where $\mathscr{L}[\tau]$ is a set for all τ, provided $\sim$ is bounded. Finally, we say that $\sim$ has the *finite vocabulary property* iff for every τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$, we have that $\mathfrak{A} \sim \mathfrak{B}$ iff $\mathfrak{A} \restriction \tau_0 \sim \mathfrak{B} \restriction \tau_0$ for each finite vocabulary $\tau_0 \subseteq \tau$.

As remarked in the introduction, an open problem of abstract model theory is whether $\equiv$ is the only bounded Robinson equivalence relation $\sim$ having the finite vocabulary property and satisfying $\sim\ =\ \equiv_{\mathscr{L}}$ for some logic $\mathscr{L}$. In the following pages we will see that if any such relation $\sim\ \neq\ \equiv$ exists, then that relation $\sim$ has many properties in common with $\equiv$.

3.2 Theorem (Relative Compactness Theorem). *Let $\sim$ be a Robinson equivalence relation having the finite vocabulary property. Let $\mathscr{L}' = \mathscr{L}(Q^i)_{i \in I}$ and $\mathscr{L}'' = \mathscr{L}(Q^j)_{j \in J}$ be logics, with $\equiv_{\mathscr{L}'}$ and $\equiv_{\mathscr{L}''}$ both coarser than $\sim$. Let $\psi \in \mathscr{L}''[\tau]$ and $\Gamma \subseteq \mathscr{L}'[\tau]$ be an arbitrary set. If $\Gamma \models \psi$, then $\Gamma_0 \models \psi$, for some finite $\Gamma_0 \subseteq \Gamma$, where $\Gamma \models \psi$ means* $\mathrm{Mod}_{\mathscr{L}'}\, \Gamma \subseteq \mathrm{Mod}_{\mathscr{L}''}\, \psi$.

Proof. Assume that $\Gamma \models \psi$ holds but for no finite $\Gamma_0 \subseteq \Gamma$ do we have $\Gamma_0 \models \psi$ (*absurdum* hypothesis). Since Γ is a set, we can write $\Gamma = \{\varphi_\alpha | \alpha < \kappa\}$. We can safely assume κ is minimal, so that each $T_\beta = \{\neg\psi\} \cup \{\varphi_\alpha | \alpha < \beta\}$ has a model $\mathfrak{A}_\beta$, for each $\beta < \kappa$. Now, construct the disjoint union $\mathfrak{A}$ of the $\mathfrak{A}_\beta$ and let $\mathfrak{M} = [\mathfrak{A}, \langle \kappa, <, c_\beta \rangle_{\beta < \kappa}, f]$ exactly as in the proof of Theorem 1.3 (here we use the hypothesis that all sentences in $\mathscr{L}'$ and $\mathscr{L}''$ have a finite vocabulary). The claim in the proof of Theorem 1.3 now reads as follows:

Whenever $\mathfrak{N} \sim \mathfrak{M}$, the $\{c_\beta^{\mathfrak{N}}\}_{\beta<\kappa}$ are unbounded in the order $<^{\mathfrak{N}}$.

To prove the present claim, for each $\beta < \kappa$, let ψ'_β and ψ''_β be defined by

$$\psi'_\beta \underset{\mathrm{def}}{=} \forall z(c_\beta < z \to \varphi_\beta^{\{x | f(x) = z\}}), \qquad \psi''_\beta \underset{\mathrm{def}}{=} \forall z(c_\beta < z \to (\neg\psi)^{\{x | f(x) = z\}}).$$

Observe that since $\mathfrak{M} \models_{\mathscr{L}'} \psi'_\beta$ and $\mathfrak{M} \models_{\mathscr{L}''} \psi''_\beta$, then so does $\mathfrak{N} \sim \mathfrak{M}$, since $\equiv_{\mathscr{L}'}$ and $\equiv_{\mathscr{L}''}$ are assumed to be coarser than $\sim$. Therefore, if $\mathfrak{N}$ were a counterexample to the claim, i.e., for some $n \in N$, $\langle \mathfrak{N}, n\rangle \models c_\beta < n$ (for all $\beta < \kappa$), then

$$\mathfrak{N} | \{x \in N | \langle \mathfrak{N}, x\rangle \models f(x) = n\}$$

would provide a model of $\Gamma \cup \{\neg\psi\}$. But this is impossible and our claim is thus established.

At this point, we consider $\mathfrak{M}'$ and $\mathfrak{M}''$, of vocabulary τ' and τ'', respectively, exactly as in the final part of the proof of Theorem 1.3, where $\tau' = \tau_{\mathfrak{M}} \cup \tau_0$ and $\tau_0 = \{P_\beta\}_{\beta<\kappa}$, and $\tau'' = \tau_0 \cup \{c\}$. Using the assumed finite vocabulary property of $\sim$, we must have that $\mathfrak{M}' \restriction \tau_0 \sim \mathfrak{M}'' \restriction \tau_0$. By the Robinson property of $\sim$, there is $\mathfrak{D}$ with $\mathfrak{D} \restriction \tau' \sim \mathfrak{M}'$ and $\mathfrak{D} \restriction \tau'' \sim \mathfrak{M}''$. In particular, $\mathfrak{M}''$, $\mathfrak{D} \models \neg P_\beta c$, for every $\beta < \kappa$, so that $c^{\mathfrak{D}}$ is a strict upper bound for the set $\{c_\beta^{\mathfrak{D}}\}_{\beta<\kappa}$. In definitive, $\mathfrak{D} \restriction \tau_{\mathfrak{M}} \sim \mathfrak{M}$ is counterexample to our claim. Having thus obtained a contradiction, we conclude the proof of the theorem. □

3.3 Corollary. *Let $\sim$ be a Robinson equivalence relation with the finite vocabulary property. For a set I, let $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ be a logic with $\equiv_{\mathscr{L}}$ coarser than $\sim$. Then $\mathscr{L}$ is compact.*

Proof. The proof follows immediately from Theorem 3.2. □

The following corollary is a "unique representability" result:

3.4 Corollary. *Let $\sim$ be a bounded Robinson equivalence relation. Then there is at most one (up to equivalence) logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ such that $\equiv_{\mathscr{L}} = \sim$. Furthermore, if any such $\mathscr{L}$ exists, then $\mathscr{L}$ is compact and has the interpolation property.*

Proof. Let $\mathscr{L}' = \mathscr{L}(Q^j)_{j\in J}$ be such that $\equiv_{\mathscr{L}'} = \equiv_{\mathscr{L}} = \sim$. Observe that $\sim$ necessarily has the finite vocabulary property. Also, I and J may be assumed to be sets, for $\sim$ is bounded. For arbitrary φ in $\mathscr{L}[\tau_\varphi]$, φ has the same models as

$$\bigcup_{\mathfrak{A} \models_{\mathscr{L}} \varphi,\, \mathfrak{A} \in \mathrm{Str}(\tau_\varphi)} \ \bigcap_{\psi \in \mathrm{Th}_{\mathscr{L}'}(\mathfrak{A})} \mathrm{Mod}_{\mathscr{L}'} \psi.$$

Using Theorem 3.2 and noting that $\mathrm{Th}_{\mathscr{L}'}(\mathfrak{A})$ is a set, we see that for every $\mathfrak{A} \models_{\mathscr{L}} \varphi$ there is $\psi_{\mathfrak{A}} \in \mathrm{Th}_{\mathscr{L}'}(\mathfrak{A})$ such that $\mathrm{Mod}_{\mathscr{L}'}(\psi_{\mathfrak{A}}) \subseteq \mathrm{Mod}_{\mathscr{L}} \varphi$. Applying Theorem 3.2 to $\neg\varphi$, there exist $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that φ has the same models as $\psi_{\mathfrak{A}_1} \vee \cdots \vee \psi_{\mathfrak{A}_n}$. Therefore, φ is equivalent to some sentence in $\mathscr{L}'$. Finally, the fact that $\mathscr{L}$ is compact and has the interpolation property now follows from Theorem 1.3 and Corollary 1.4 (recall that I and J are sets). □

3.5 Corollary. *Up to equivalence, first-order logic is the only logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ such that $\equiv_{\mathscr{L}} = \equiv$.* □

For an alternative proof of Corollary 3.5, see Theorem III.2.1.4. Observe also that the (generalized downward) Löwenheim–Skolem theorem coupled with Lindström's theorem (see Theorem III.1.1.4) implies that $\mathscr{L}_{\omega\omega}$ is the only *countably* *compact* logic $\mathscr{L}$ with $\equiv_{\mathscr{L}} = \equiv$. The point of Corollary 3.5 is that $\equiv$ uniquely characterizes $\mathscr{L}_{\omega\omega}$ among *all* logics $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$.

Corollary 3.4 shows that *at most one* logic $\mathscr{L}$ exists with $\equiv_{\mathscr{L}} = \sim$, whenever $\sim$ is a bounded Robinson equivalence relation. The problem of whether *at least one* such $\mathscr{L}$ exists will be settled in the remainder of this section.

3.6 Notational Convention. If φ is a sentence in $\mathscr{L}$ of vocabulary $\tau \cup \{c_1, \ldots, c_n\}$ with $c_1, \ldots, c_n \notin \tau$, then for $\mathfrak{A} \in \mathrm{Str}(\tau)$ we define the set $\varphi^{\mathfrak{A}}$ by

$$\varphi^{\mathfrak{A}} \underset{\mathrm{def}}{=} \{(a_1, \ldots, a_n) \in A^n \mid \langle \mathfrak{A}, a_1, \ldots, a_n \rangle \models_{\mathscr{L}} \varphi\}.$$

3.7 Lemma. *Let $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$ be a logic such that $\equiv_{\mathscr{L}}$ is coarser than a Robinson equivalence relation $\sim$. Given $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ with $\mathfrak{A} \sim \mathfrak{B}$, and*

$$\varphi \in \mathscr{L}[\tau \cup \{c_1, \ldots, c_n\}],$$

let R be a new n-ary relation symbol and let structures $\mathfrak{A}^+, \mathfrak{B}^+ \in \mathrm{Str}(\tau \cup \{R\})$ be defined by $\mathfrak{A}^+ = \langle \mathfrak{A}, R^{\mathfrak{A}^+} \rangle$ and $\mathfrak{B}^+ = \langle \mathfrak{B}, R^{\mathfrak{B}^+} \rangle$, where $R^{\mathfrak{A}^+} = \varphi^{\mathfrak{A}}$ and $R^{\mathfrak{B}^+} = \varphi^{\mathfrak{B}}$. Then $\mathfrak{A}^+ \sim \mathfrak{B}^+$.

Proof. Let R_1, R_2 be new n-ary relation symbols, and let ρ_1 be the renaming on $\tau \cup \{R\}$ which maps R into R_1 and is equal to the identity on τ. Let ρ_2 similarly, map R into R_2. Let $\rho_1(\mathfrak{A}^+)$ and $\rho_2(\mathfrak{B}^+)$ be the correspondingly renamed structures (see Definition II.1.1.1). By the assumed Robinson property of $\sim$, there exists $\mathfrak{N} \in \mathrm{Str}(\tau \cup \{R_1, R_2\})$ such that

$$(1) \qquad \mathfrak{N} \upharpoonright \tau \cup \{R_1\} \sim \rho_1(\mathfrak{A}^+) \quad \text{and} \quad \mathfrak{N} \upharpoonright \tau \cup \{R_2\} \sim \rho_2(\mathfrak{B}^+).$$

Therefore, we have

$$(2) \qquad \begin{aligned} &\mathfrak{N}, \rho_1(\mathfrak{A}^+) \models_{\mathscr{L}} \forall c_1, \ldots, c_n(\varphi \leftrightarrow R_1 c_1, \ldots, c_n), \quad \text{and} \\ &\mathfrak{N}, \rho_2(\mathfrak{B}^+) \models_{\mathscr{L}} \forall c_1, \ldots, c_n(\varphi \leftrightarrow R_2 c_1, \ldots, c_n), \end{aligned}$$

whence $R_1^{\mathfrak{N}} = R_2^{\mathfrak{N}}$. Now using (1) and the renaming property of $\sim$ we get

$$(3) \qquad \mathfrak{A}^+ \sim \rho_1^{-1}(\mathfrak{N} \upharpoonright \tau \cup \{R_1\}) = \rho_2^{-1}(\mathfrak{N} \upharpoonright \tau \cup \{R_2\}) \sim \mathfrak{B}^+. \quad \square$$

3.8 Corollary. *Let $\mathscr{L}' = \mathscr{L}(Q^i)_{i \in I}$ and $\mathscr{L}'' = \mathscr{L}(Q^j)_{j \in J}$, where $I \cap J = \varnothing$, be logics such that $\equiv_{\mathscr{L}'}$ and $\equiv_{\mathscr{L}''}$ are both coarser than a Robinson equivalence relation $\sim$. Let $\mathscr{L} = \mathscr{L}(Q^k)_{k \in I \cup J}$. Then $\equiv_{\mathscr{L}}$ is coarser than $\sim$.*

Proof. Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ with $\mathfrak{A} \sim \mathfrak{B}$. We must prove that $\mathfrak{A} \models_{\mathscr{L}} \varphi$ iff $\mathfrak{B} \models_{\mathscr{L}} \varphi$, for every φ in $\mathscr{L}$. To this purpose, it suffices to show that for every ψ in $\mathscr{L}$ of vocabulary $\tau \cup \{c_1, \ldots, c_n\}$ we have $\langle \mathfrak{A}, \psi^{\mathfrak{A}} \rangle \sim \langle \mathfrak{B}, \psi^{\mathfrak{B}} \rangle$, as structures of vocabulary $\tau \cup \{R\}$ (recall the notational convention (3.6) and the notation of Lemma 3.7). We proceed by induction on the complexity (quantifier rank) of ψ. The only nontrivial step is when, say,

$$\psi \underset{\mathrm{def}}{=} Q^t x_0 \vec{x}_1, \ldots, \vec{x}_m\ \varphi_0(x_0), \varphi_1(\vec{x}_1), \ldots, \varphi_m(\vec{x}_m) \quad \text{for some} \quad t \in I.$$

By the induction hypothesis, we have

$$\langle \mathfrak{A}, \varphi_0^{\mathfrak{A}}, \varphi_1^{\mathfrak{A}}, \ldots, \varphi_m^{\mathfrak{A}} \rangle \sim \langle \mathfrak{B}, \varphi_0^{\mathfrak{B}}, \varphi_1^{\mathfrak{B}}, \ldots, \varphi_m^{\mathfrak{B}} \rangle,$$

as structures of vocabulary $\{R_0, R_1, \ldots, R_m\} \cup \tau$. By Lemma 3.7, after noting that $Q^t x_0 \vec{x}_1, \ldots, \vec{x}_m R_0 x_0 R_1 \vec{x}_1, \ldots, R_m \vec{x}_m$ is a sentence in $\mathscr{L}'$, we have

$$\langle \mathfrak{A}, \varphi_0^{\mathfrak{A}}, \varphi_1^{\mathfrak{A}}, \ldots, \varphi_m^{\mathfrak{A}}, \psi^{\mathfrak{A}} \rangle \sim \langle \mathfrak{B}, \varphi_0^{\mathfrak{B}}, \varphi_1^{\mathfrak{B}}, \ldots, \varphi_m^{\mathfrak{B}}, \psi^{\mathfrak{B}} \rangle$$

as structures of vocabulary $\tau \cup \{R_0, R_1, \ldots, R_m, R\}$. Finally, by the reduct property of $\sim$, we have the desired conclusion. $\square$

3.9 Definition. We say that a regular equivalence relation $\sim$ is *separable by quantifiers* iff whenever $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and not-$\mathfrak{A} \sim \mathfrak{B}$, then there is a quantifier Q such that $\equiv_{\mathscr{L}(Q)}$ is coarser than $\sim$, and $\mathfrak{A} \not\equiv_{\mathscr{L}(Q)} \mathfrak{B}$.

Notice that if $\sim$ is representable as $\sim \; = \; \equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$, then $\sim$ is separable by quantifiers. The next theorem shows that separability is not only necessary, but also sufficient for the representability of $\sim$, provided $\sim$ has the Robinson property and is bounded.

3.10 Theorem. *Let $\sim$ be an arbitrary bounded Robinson equivalence relation. Let $\mathscr{L}^* = \mathscr{L}\{Q \mid \equiv_{\mathscr{L}(Q)}$ is coarser than $\sim\}$. Then we have:*

(i) *$\mathscr{L}^*$ is the strongest logic $\mathscr{L}$ of the form $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$ such that $\equiv_{\mathscr{L}}$ is coarser than $\sim$.*
(ii) *The identity $\sim \; = \; \equiv_{\mathscr{L}^*}$ holds iff $\sim$ is separable by quantifiers. If this is the case, then $\mathscr{L}^*$ is uniquely determined by $\sim$ (up to equivalence) and is a compact logic with the interpolation property.*

Proof. The assertion in (i) is immediate from Corollary 3.8. As for (ii) clearly, if $\sim$ is separable by quantifiers, then $\sim$ is coarser than $\equiv_{\mathscr{L}^*}$. Hence, $\sim \; = \; \equiv_{\mathscr{L}^*}$. Conversely, if $\sim \; = \; \equiv_{\mathscr{L}^*}$, not-$\mathfrak{A} \sim \mathfrak{B}$, and $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$, then $\mathfrak{A} \not\equiv_{\mathscr{L}^*} \mathfrak{B}$ so that $\mathfrak{A} \models_{\mathscr{L}^*} \psi$ and $\mathfrak{B} \models_{\mathscr{L}^*} \neg \psi$ for some ψ in $\mathscr{L}^*$. Let Q_ψ be the quantifier given by the class $\mathrm{Mod}_{\mathscr{L}^*}(\psi)$. Then $\mathscr{L}(Q_\psi) \leq \mathscr{L}^*$ by the regularity properties of logics generated by quantifiers (see Section II.4.1), hence $\equiv_{\mathscr{L}(Q_\psi)}$ is coarser than $\sim$ and $\mathfrak{A} \not\equiv_{\mathscr{L}(Q_\psi)} \mathfrak{B}$. Therefore, $\sim$ is separable by quantifiers. To conclude the proof, the uniqueness of $\mathscr{L}^*$ follows from Corollary 3.4, while the compactness and interpolation properties of $\mathscr{L}^*$ follow from Theorem 1.3 and Corollary 1.4 upon noting that since $\sim$ is bounded, then $\mathscr{L}^*[\tau]$ is a set for all τ. $\square$

Can the duality given by (Corollary 3.4 and) Theorem 3.10 be extended beyond the realm of logics and equivalence relations with the Robinson property? The answer is partially affirmative. As a matter of fact, using the equivalence between JEP and compactness (see Chapter XVIII), we have that the bijection given by Theorem 3.10 can be extended to an injection from compact logics into equivalence relations via the following generalization of Corollary 3.4:

3.11 Theorem. *Let $\sim$ be an arbitrary regular equivalence relation such that $\sim \; = \; \equiv_{\mathscr{L}^*}$ for some logic $\mathscr{L}^* = \mathscr{L}(Q^i)_{i \in I}$, where I is a set. If $\mathscr{L}^*$ is compact (or,* *equivalently, if $\mathscr{L}^*$ has the* JEP) *then $\mathscr{L}^*$ is uniquely determined by $\sim$ up to equivalence.*

Proof. First observe that the JEP is indeed equivalent to compactness (see Examples 4.2 below and Theorem XVIII.3.3.3). We now prove the following lemma, which is of independent interest:

3.11.1 Lemma. *Let $\mathscr{L}' = \mathscr{L}(Q^j)_{j\in J}$, $\mathscr{L}'' = \mathscr{L}(Q^k)_{k\in K}$, J, K disjoint sets; let $\mathscr{L}$ be the weakest logic closed under existential quantification and boolean operations with $\mathscr{L} \geq \mathscr{L}'$ and $\mathscr{L} \geq \mathscr{L}''$. If $\equiv_{\mathscr{L}'}$ is finer than $\equiv_{\mathscr{L}''}$, then $\equiv_{\mathscr{L}} = \equiv_{\mathscr{L}'}$.*

Proof of Lemma. Assume $\equiv_{\mathscr{L}} \neq \equiv_{\mathscr{L}'}$, so that for some $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M} \equiv_{\mathscr{L}'} \mathfrak{N}$, we have $\mathfrak{M} \models_{\mathscr{L}} \psi$ and $\mathfrak{N} \models_{\mathscr{L}} \neg\psi$ for some ψ in $\mathscr{L}$. It is easy to see that ψ can be written in the form

$$\psi = Q_1 y_1, \ldots, Q_r y_r\, B(\varphi'_1, \ldots, \varphi'_p, \varphi''_1, \ldots, \varphi''_q),$$

where $Q_n \in \{\exists, \forall\}$ for each $n = 1, \ldots, r$, B is a boolean function, that is, a finite composition of $\wedge$, $\vee$, $\neg$, each φ'_i is a sentence in $\mathscr{L}'$ and each φ''_j is in $\mathscr{L}''$. Let $R_1, \ldots, R_p$ be new r-ary relation symbols, and let $\mathfrak{M}^+ = \langle \mathfrak{M}, R_1, \ldots, R_p \rangle$, $\mathfrak{N}^+ = \langle \mathfrak{N}, R_1, \ldots, R_p \rangle$ be given by

$$\text{(1)} \qquad \mathfrak{N}^+, \mathfrak{M}^+ \models_{\mathscr{L}'} \bigwedge_{i=1}^{p} \forall \vec{y}(\varphi'_i \leftrightarrow R_i \vec{y}), \quad \text{with} \quad \vec{y} = (y_1, \ldots, y_r).$$

Let sentence δ in $\mathscr{L}''$ be defined by $\delta \underset{\text{def}}{=} Q_1 y_1, \ldots, Q_r y_r\, B(R_1, \ldots, R_p, \varphi''_1, \ldots, \varphi''_q)$, and observe that these substitutions are legitimate and $\mathfrak{M}^+ \models_{\mathscr{L}''} \delta$, $\mathfrak{N}^+ \models_{\mathscr{L}''} \neg\delta$. Since $\equiv_{\mathscr{L}'}$ is finer than $\equiv_{\mathscr{L}''}$ and $\mathfrak{M}^+ \not\equiv_{\mathscr{L}''} \mathfrak{N}^+$, then, for some sentence χ in $\mathscr{L}'$, we have $\mathfrak{M}^+ \models_{\mathscr{L}'} \chi$ and $\mathfrak{N}^+ \models_{\mathscr{L}'} \neg\chi$. Define sentence θ in $\mathscr{L}'$ by

$$\theta \underset{\text{def}}{=} \chi(\varphi'_1/R_1, \ldots, \varphi'_p/R_p);$$

that is, θ is obtained from χ by replacing each occurrence of R_i in χ by φ'_i. Again, these substitutions are allowed in $\mathscr{L}'$. In conclusion, recalling (1), we have $\mathfrak{M}^+ \models_{\mathscr{L}'} \theta$ and $\mathfrak{N}^+ \models_{\mathscr{L}'} \neg\theta$. Whence $\mathfrak{M} \models_{\mathscr{L}'} \theta$ and $\mathfrak{N} \models_{\mathscr{L}'} \neg\theta$, which contradicts $\mathfrak{M} \equiv_{\mathscr{L}'} \mathfrak{N}$. ☐

3.11.2 *End of Proof of Theorem* 3.11. Assume that both $\mathscr{L}^*$ and $\mathscr{L}'' = \mathscr{L}(Q^k)_{k\in K}$ have $\equiv_{\mathscr{L}^*} = \equiv_{\mathscr{L}''} = \sim$. Let $\mathscr{L}$ be as in Lemma 3.11.1 (with regard to $\mathscr{L}^*$ and $\mathscr{L}''$). Using this lemma twice, we get $\equiv_{\mathscr{L}} = \sim$. Now, $\mathscr{L}[\tau]$ is a set for every τ, as can be seen by examining the form of any sentence ψ in $\mathscr{L}$, according to the proof of Lemma 3.11.1. Moreover, $\mathscr{L}$ is closed under relativizations to boolean combinations of atomic sentences, and functions can be replaced by relations in $\mathscr{L}$. Now the fact that $\mathscr{L}$ has the joint embedding property is enough to prove that $\mathscr{L}$ is compact (our assertions in Remarks 1.5 can be extended to the present case, to the effect that the results in Theorem XVIII.3.3.3 can be applied to $\mathscr{L}$). By a familiar finite cover argument such as the one given in Theorem III.1.1.5 we finally conclude that $\mathscr{L}$, $\mathscr{L}''$, and $\mathscr{L}^*$ are equivalent. ☐

3.12 Corollary. *Let $\mathscr{L}$ be an arbitrary logic with $\mathscr{L} \leq \Delta\mathscr{L}(Q^{\mathrm{cf}\,\omega})$. Then $\equiv_{\mathscr{L}'} = \equiv_{\mathscr{L}}$ iff $\mathscr{L}'$ is equivalent to $\mathscr{L}$.*

Proof. The Δ-closure of any compact logic is still compact (see Proposition II.7.2.5), and sublogics of compact logics are compact; $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ is compact (see Theorem II.3.2.3). □

3.13 Notes and Remarks. Regular equivalence relations in abstract model theory were introduced in Nadel [1980a]. In Theorem 7 of his paper, we proves that whenever $\sim \; = \; \equiv_{\mathscr{L}_0}$, for some logic $\mathscr{L}_0$ (i) if $\sim$ is bounded, then there is a strongest logic $\mathscr{L}$ with $\sim \; = \; \equiv_{\mathscr{L}}$ and which is closed under negation, conjunction and disjunction. By constrast, he also shows (ii) that no such strongest $\mathscr{L}$ exists if the transitive closures $\langle \bar{x}, \in \rangle$ and $\langle \bar{y}, \in \rangle$ of any two sets $x \neq y$ are never $\sim$-equivalent. Nadel's logics are systems of sentences obeying only the basic axioms given in Definition II.1.1.1. He also has a number of results about logics closed under Scott sentences, that is, logics $\mathscr{L}$ in which each $\equiv_{\mathscr{L}}$-equivalence class of structures is $\mathrm{EC}_{\mathscr{L}}$.

Corollaries 3.4 and 3.5, and the duality theorem (Theorem 3.10) of this section were orginally proved in Mundici [1982a]. The assumption used there that there are no uncountable measurable cardinals is unnecessary and was subsequently dropped (see Mundici [1982e, Section 1.1]). In Mundici [1982c, II and 198?b], Theorem 3.10 is extended to logics and equivalence relations for enriched structures (see Section 2). For instance, it is proved that topological, monotone, uniform logics are uniquely determined by their own elementary equivalence relations. The proof of Theorem 3.10 given here depends on Theorem 3.2, Corollary 3.3, Lemma 3.7 and Corollary 3.8, which were given by Flum in a private communication. Theorem 3.11 is due to Lipparini [1982].

4. *Duality Between Embedding and Equivalence Relations*

The notion of $\mathscr{L}$-(elementary) equivalence is generalized in Definition 3.1; the notion of $\mathscr{L}$-(elementary) embedding is generalized in the following:

4.1 Definition. An arbitrary binary relation $\rightarrow$ on the class of all structures is called an (abstract) *embedding relation* iff $\rightarrow$ satisfies the following axioms (for every two structures $\mathfrak{M}, \mathfrak{N}$):

vocabulary:	$\mathfrak{M} \rightarrow \mathfrak{N}$ implies $\tau_{\mathfrak{M}} \subseteq \tau_{\mathfrak{N}}$;
	$\mathfrak{M} \rightarrow \mathfrak{N} \upharpoonright \tau_{\mathfrak{M}}$ iff $\mathfrak{M} \rightarrow \mathfrak{N}$;
renaming:	$\mathfrak{M} \rightarrow \mathfrak{N}$ implies $\mathfrak{M}^{\rho} \rightarrow \mathfrak{N}^{\rho}$ for any renaming ρ of $\tau_{\mathfrak{N}}$;
reduct:	$\mathfrak{M} \rightarrow \mathfrak{N}$ implies $\mathfrak{M} \upharpoonright \tau \rightarrow \mathfrak{N} \upharpoonright \tau$ for all $\tau \subseteq \tau_{\mathfrak{M}}$;
isomorphism:	$\mathfrak{M} \cong \mathfrak{N}$ implies $\mathfrak{M} \rightarrow \mathfrak{N}$;
expressiveness:	$\mathfrak{M} \rightarrow \mathfrak{N}$ implies $\mathfrak{M}_M \equiv \mathfrak{N}^{+}$ for some expansion $\mathfrak{N}^{+}$ of $\mathfrak{N} \upharpoonright \tau_{\mathfrak{M}}$;
transitivity:	$\mathfrak{M} \rightarrow \mathfrak{N}$ and $\mathfrak{N} \rightarrow \mathfrak{B}$ implies $\mathfrak{M} \rightarrow \mathfrak{B}$.

Recall that $\mathfrak{M}_M$ denotes the diagram expansion of $\mathfrak{M}$. An embedding relation $\to$ has the *expanded amalgamation property*, denoted by AP^+ (resp., the *amalgamation property*, denoted by AP) iff whenever $\mathfrak{A} \leftarrow \mathfrak{N} \to \mathfrak{B}$ and $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}} = \tau_{\mathfrak{N}}$ (resp., $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}} = \tau_{\mathfrak{N}}$), then $\mathfrak{A} \to \mathfrak{M} \leftarrow \mathfrak{B}$ for some structure $\mathfrak{M}$. Given an embedding $\to$ and an equivalence relation $\sim$, we say that the pair $(\sim, \to)$ has the *joint embedding property*, denoted as before by JEP, iff whenever $\mathfrak{A} \sim \mathfrak{B}$ then $\mathfrak{A} \to \mathfrak{M} \leftarrow \mathfrak{B}$ for some $\mathfrak{M}$. When $\sim\ =\ \equiv_{\mathscr{L}}$ this agrees with Section 1. If $\sim$ is a regular equivalence relation (on the class of all structures), then $\sim$ *generates* an embedding relation $\to$ by stipulating that $\mathfrak{A} \to \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \sim \mathfrak{B}^+$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. We denote by $\sim^*$ the embedding relation generated by $\sim$. Conversely, any embedding relation $\to$ *generates* a regular equivalence relation $\sim$ by stipulating that $\mathfrak{A} \sim \mathfrak{B}$ iff $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and there is a finite path:

$$\mathfrak{A} = \mathfrak{N}_0 \underset{1}{\text{---}} \mathfrak{N}_1 \underset{2}{\text{---}} \cdots \underset{k}{\text{---}} \mathfrak{N}_k = \mathfrak{B},$$

with $\tau_{\mathfrak{N}_0} = \cdots = \tau_{\mathfrak{N}_k}$ and $\underset{i}{\text{---}}$ being either $\to$ or $\leftarrow$, depending on i ($i = 1, \ldots, k$). We denote by $\to^*$ the regular equivalence relation generated by $\to$.

4.2 Examples. (a) If $\mathscr{L}$ is a logic, define $\to_{\mathscr{L}}$ by stipulating that $\mathfrak{A} \to_{\mathscr{L}} \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \equiv_{\mathscr{L}} \mathfrak{B}^+$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. Then $\to_{\mathscr{L}}$ is an embedding relation, called *$\mathscr{L}$-embedding*. Observe that $\to_{\mathscr{L}} = (\equiv_{\mathscr{L}})^*$. For the particular case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, we have that $\mathfrak{A} \to_{\mathscr{L}} \mathfrak{B}$ iff $\mathfrak{A} \precsim \mathfrak{B} \restriction \tau_{\mathfrak{A}}$; that is, $\mathfrak{A}$ is elementarily embedded into $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. Returning now to the general case, assume that $\to\ =\ \to_{\mathscr{L}}$, for $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$, where I is a set. Let $\sim\ =\ \equiv_{\mathscr{L}}$ (so that $\to\ =\ \sim^*$). Then $\mathscr{L}$ is compact iff $\to$ has the AP, iff the pair $(\sim, \to)$ has the JEP. For a proof of this fact see Theorem XVIII.3.3.3. The above equivalences—originally proved in Mundici [1982b] (compactness = JEP) and, independently, in Makowsky–Shelah [1983] (compactness = AP = JEP)—enable us to regard the notion of compactness as an algebraic property of embedding or equivalence relations in much the same way as compactness + interpolation is algebraized via the Robinson property. The latter, in turn, has an equivalent counterpart for embeddings in terms of the AP^+, as will be shown in Theorem 4.8.

(b) If $\mathscr{L}$ is a logic, define $\to^{\#}_{\mathscr{L}}$ by stipulating that $\mathfrak{A} \to^{\#}_{\mathscr{L}} \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}^{\#} \equiv_{\mathscr{L}} \mathfrak{B}''$ for some expansion $\mathfrak{B}''$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$, where $\mathfrak{A}^{\#}$ denotes the complete expansion of $\mathfrak{A}$ (see Section XVIII.1.2). Then $\to^{\#}_{\mathscr{L}}$ is an embedding relation, called the *$\mathscr{L}$-complete embedding relation.* In case $\mathscr{L} = \mathscr{L}_{\omega\omega}$ it is well known that $\to^{\#}_{\mathscr{L}}$ has AP^+. Indeed, $(\equiv, \to^{\#}_{\mathscr{L}})$ has the JEP; also, $(\to^{\#}_{\mathscr{L}})^* = \equiv$.

We now begin consideration of the (preservation) properties of the map $*$.

4.3 Proposition. *Let $\sim$ be a regular equivalence relation. Let $\to\ =\ \sim^*$, and $\approx\ =\ \to^*$. Then $\approx$ is finer than $\sim$.*

Proof. First observe that if $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and $\mathfrak{A} \sim^* \mathfrak{B}$ then $\mathfrak{A} \sim \mathfrak{B}$. As a matter of fact, $\mathfrak{A} \sim^* \mathfrak{B}$ means that $\mathfrak{A}_A \sim \mathfrak{B}^+$, for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \upharpoonright \tau_{\mathfrak{A}}$ ($=\mathfrak{B}$, in the present case). Therefore, by the reduct axiom, $\mathfrak{A} = \mathfrak{A}_A \upharpoonright \tau_{\mathfrak{A}} \sim \mathfrak{B}^+ \upharpoonright \tau_{\mathfrak{A}} = \mathfrak{B}$. Now, to conclude the proof of our proposition, if $\mathfrak{M} \approx \mathfrak{N}$, then by definition there is a path

$$\mathfrak{M} = \mathfrak{A}_0 \underset{1}{-} \mathfrak{A}_1 \underset{2}{-} \cdots \underset{k}{-} \mathfrak{A}_k = \mathfrak{N},$$

with $\tau_{\mathfrak{A}_0} = \cdots = \tau_{\mathfrak{A}_\kappa}$ and $\underset{i}{-} = \rightarrow$ or $\underset{i}{-} = \leftarrow$; by the above initial remark we have that $\mathfrak{A}_0 \sim \cdots \sim \mathfrak{A}_k$, as required. $\square$

4.4 Proposition. *Let $\rightarrow$ be an embedding relation with* AP$^+$. *Let $\sim\, = \,\rightarrow^*$; then we have*:

(i) *the pair* $(\rightarrow^*, \rightarrow)$ *has the* JEP;
(ii) $\sim$ *is a regular Robinson equivalence relation.*

Proof. For (i), we assume $\mathfrak{M} \sim \mathfrak{N}$, and let $\tau = \tau_{\mathfrak{M}} = \tau_{\mathfrak{N}}$. By definition there is a path:

$$(+) \qquad \mathfrak{M} = \mathfrak{A}_0 \underset{1}{-} \mathfrak{A}_1 \underset{2}{-} \cdots \underset{n}{-} \mathfrak{A}_n = \mathfrak{N},$$

with $\tau_{\mathfrak{A}_i} = \tau$, for each $i = 0, \ldots, n$, and $\underset{i}{-}$ being either $\rightarrow$ or $\leftarrow$. If $n = 1$, then let $\mathfrak{D} = \mathfrak{N}$ or $\mathfrak{D} = \mathfrak{M}$, according to whether $\underset{1}{-} = \rightarrow$ or $\underset{1}{-} = \leftarrow$ is the case; then $\mathfrak{M} \rightarrow \mathfrak{D} \leftarrow \mathfrak{N}$, and we are done. Proceeding now by induction on n, we obtain from $(+)$:

$$(++) \qquad \mathfrak{M} \rightarrow \mathfrak{B} \leftarrow \mathfrak{A}_{n-1} \underset{n}{-} \mathfrak{N}.$$

Now, if $\underset{n}{-} = \leftarrow$, then by transitivity we see that $\mathfrak{M} \rightarrow \mathfrak{B} \leftarrow \mathfrak{N}$. If $\underset{n}{-} = \rightarrow$, then by the AP$^+$ (actually only the AP is needed here) we have

$$(+++) \qquad \begin{tikzcd} \mathfrak{M} \arrow[r] & \mathfrak{B} \arrow[dr] & \mathfrak{A}_{n-1} \arrow[l] \arrow[r] & \mathfrak{N}; \arrow[dl] \\ & & \mathfrak{D} & \end{tikzcd}$$

hence $\mathfrak{M} \rightarrow \mathfrak{D} \leftarrow \mathfrak{N}$, as required.

As for (ii), we see that the regularity of $\rightarrow^*$ is an immediate consequence of Definition 4.1. Let $\mathfrak{M} \upharpoonright \tau \sim \mathfrak{N} \upharpoonright \tau$, where $\tau = \tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}}$. From (i) above and the regularity properties of $\sim$, we must have, for some $\mathfrak{D} \in \mathrm{Str}(\tau)$, that

$$\mathfrak{M} \leftarrow \mathfrak{M} \upharpoonright \tau \rightarrow \mathfrak{D} \leftarrow \mathfrak{N} \upharpoonright \tau \rightarrow \mathfrak{N}.$$

By repeated application of the AP$^+$, we obtain, for some $\mathfrak{A} \in \mathrm{Str}(\tau_{\mathfrak{M}})$, $\mathfrak{B} \in \mathrm{Str}(\tau_{\mathfrak{N}})$ and $\mathfrak{S} \in \mathrm{Str}(\tau_{\mathfrak{M}} \cup \tau_{\mathfrak{N}})$:

$$\begin{tikzcd} \mathfrak{M} \arrow[r] & \mathfrak{A} \arrow[dr] & \mathfrak{D} \arrow[l] \arrow[r] & \mathfrak{B} \arrow[dl] & \mathfrak{N} \arrow[l] \\ & & \mathfrak{S} & & \end{tikzcd}$$

We thus conclude that $\mathfrak{M} \to \mathfrak{S} \restriction \tau_{\mathfrak{M}}$ and $\mathfrak{N} \to \mathfrak{S} \restriction \tau_{\mathfrak{N}}$. By definition of $\sim$, we finally obtain that $\mathfrak{M} \sim \mathfrak{S} \restriction \tau_{\mathfrak{M}}$ and $\mathfrak{N} \sim \mathfrak{S} \restriction \tau_{\mathfrak{N}}$, thus showing that $\sim$ has the Robinson property. □

4.5 Proposition. *Let $\sim$ be a regular Robinson equivalence relation. Let $\to \; = \; \sim^*$. Then we have:*

(i) *the pair $(\sim, \sim^*)$ has the* JEP;
(ii) *$\to$ is an embedding relation with the* AP^+.

Proof. If $\mathfrak{M} \sim \mathfrak{N}$, let $\mathfrak{M}_M$ and $\mathfrak{N}_N$ be obtained by using different constants so that $\tau = \tau_{\mathfrak{M}} = \tau_{\mathfrak{N}} = \tau_{\mathfrak{M}_M} \cap \tau_{\mathfrak{N}_N}$ and $\mathfrak{M}_M \restriction \tau \sim \mathfrak{N}_N \restriction \tau$. Using the Robinson property of $\sim$, we let $\mathfrak{A}$ be such that $\mathfrak{A} \restriction \tau_{\mathfrak{M}_M} \sim \mathfrak{M}_M$ and $\mathfrak{A} \restriction \tau_{\mathfrak{N}_N} \sim \mathfrak{N}_N$. By definition of $\to$, $\mathfrak{M} \to \mathfrak{A} \leftarrow \mathfrak{N}$. Turning now to (ii) Assume $\mathfrak{M} \leftarrow \mathfrak{B} \to \mathfrak{N}$ with $\tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}} = \tau_{\mathfrak{B}}$. By the initial remark in the proof of Proposition 4.3 we automatically have that $\mathfrak{M} \restriction \tau_{\mathfrak{B}} \sim \mathfrak{B} \sim \mathfrak{N} \restriction \tau_{\mathfrak{B}}$. If different constants are used in the diagram expansions of $\mathfrak{M}$ and $\mathfrak{N}$, we also have that $\mathfrak{M}_M \restriction \tau_{\mathfrak{B}} \sim \mathfrak{B} \sim \mathfrak{N}_N \restriction \tau_{\mathfrak{B}}$, and, by the Robinson property of $\sim$, there is some $\mathfrak{D}$ such that $\mathfrak{D} \restriction \tau_{\mathfrak{N}_N} \sim \mathfrak{N}_N$ and $\mathfrak{D} \restriction \tau_{\mathfrak{M}_M} \sim \mathfrak{M}_M$. From the definition of $\to$, we obtain $\mathfrak{M} \to \mathfrak{D} \leftarrow \mathfrak{N}$, which establishes the desired AP^+ property for $\to$. □

4.6 Proposition. (i) *If $\sim$ is a regular Robinson equivalence relation then $(\sim^*)^* = \sim$; and*

(ii) *If $\sim_1$ and $\sim_2$ are different regular Robinson equivalence relations, then $\sim_1^*$ is different from $\sim_2^*$.*

Proof. For (i), we observe that in view of Proposition 4.3, it suffices to show that $\sim$ is finer than $\sim^{**}$. Now, if $\mathfrak{M} \sim \mathfrak{N}$, then for some $\mathfrak{D}$ we have $\mathfrak{M} \to \mathfrak{D} \leftarrow \mathfrak{N}$, by Proposition 4.5(i), where $\to \; = \; \sim^*$. From the definition of $\to^*$, we thus have $\mathfrak{M} \sim^{**} \mathfrak{N}$, as required.

Turning now to (ii), we assume $\mathfrak{M} \sim_1 \mathfrak{N}$ and not-$\mathfrak{M} \sim_2 \mathfrak{N}$. Let $\to_1 \; = \; \sim_1^*$ and $\to_2 \; = \; \sim_2^*$. We also that assume $\to_1 \; = \; \to_2$ (*absurdum* hypothesis). By Proposition 4.5(i), for some $\mathfrak{D}$, we have $\mathfrak{M} \to_1 \mathfrak{D} \; {}_1\!\leftarrow \mathfrak{N}$. Hence, $\mathfrak{M} \to_2 \mathfrak{D} \; {}_2\!\leftarrow \mathfrak{N}$, whence it follows that $\mathfrak{N} \sim_2 \mathfrak{M}$ (by the first remark in Proposition 4.3). This contradicts our assumption. □

4.7 Remark. The counterpart of Proposition 4.6(i) and (ii) does not hold for embeddings with the AP^+ in place of Robinson equivalence relations. For example, the complete embedding relation $\to_{\mathscr{L}}^{\#}$ arising from $\mathscr{L} = \mathscr{L}_{\omega\omega}$ (see Example 4.2(b)) generates $\equiv$, and $\equiv$ in turn generates $\to_{\mathscr{L}}$, which is different from $\to_{\mathscr{L}}^{\#}$. To obtain the analogue of Proposition 4.6, we must restrict attention to *involutive* embedding relations $\to$ with AP^+ (where $\to$ is involutive iff $\to \; = \; \to^{**}$). An example of involutive embedding relation with AP^+ is $\to_{\mathscr{L}_{\omega\omega}}$. Indeed, we have the following quite general fact:

4.8 Theorem. *Let $\mathscr{R}$ be the family of all regular Robinson equivalence relations; let $\mathscr{A}$ be the family of all involutive embedding relations with* AP^+. *Then $*$ maps $\mathscr{A}$ one-one onto $\mathscr{R}$, and vice versa. Furthermore, $**$ is the identity function on $\mathscr{A} \cup \mathscr{R}$.*

Proof. Map $*$ sends elements of $\mathscr{R}$ into elements of $\mathscr{A}$ by Proposition 4.5(ii), and by noting that $(\sim^*)^{**} = (\sim^{**})^* = \sim^*$, see Proposition 4.6(i). Also, $*$ is injective from $\mathscr{R}$ into $\mathscr{A}$ by Proposition 4.6(ii). Map $*$ sends elements of $\mathscr{A}$ into elements of $\mathscr{R}$ by Proposition 4.4(ii) and is injective from $\mathscr{A}$ into $\mathscr{R}$. As a matter of fact, if $\to_1$ and $\to_2$ are in $\mathscr{A}$ and $\to_1^* = \to_2^*$, then also $\to_1^{**} = \to_2^{**}$. Whence it follows that $\to_1 = \to_2$, by definition of $\mathscr{A}$. Map $**$ is the identity on $\mathscr{A}$ by definition, and is the identity on $\mathscr{R}$ by Proposition 4.6(i). Finally, $*$ maps $\mathscr{A}$ onto $\mathscr{R}$, and $\mathscr{R}$ onto $\mathscr{A}$, because every element in $\mathscr{A} \cup \mathscr{R}$ is the $*$-image of its own $*$-image. ☐

From Example 4.2(a) we now recall the definition of $\mathscr{L}_{\omega\omega}$-embedding, $\to_{\mathscr{L}_{\omega\omega}}$ in terms of $\precsim$:

4.9 Theorem. *First-order logic is the only (up to equivalence) logic* $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$ *such that* $\to_{\mathscr{L}} = \to_{\mathscr{L}_{\omega\omega}}$.

Proof. Assume $\mathscr{L}$ is a logic with $\to_{\mathscr{L}} = \to_{\mathscr{L}_{\omega\omega}}$. By definition of $\to_{\mathscr{L}}$, we have that $\equiv_{\mathscr{L}}^* = \equiv^*$. Hence $\equiv_{\mathscr{L}}^{**} = \equiv^{**} = \equiv$ (the fact that $\equiv^{**} = \equiv$ is a consequence of Proposition 4.6(i), since $\equiv$ has the Robinson property). By Proposition 4.3, $\equiv_{\mathscr{L}}$ is coarser than $\equiv_{\mathscr{L}}^{**} = \equiv$. Conversely, $\equiv_{\mathscr{L}}$ is finer than $\equiv$, as $\mathscr{L} \geq \mathscr{L}_{\omega\omega}$. Therefore, we have $\equiv_{\mathscr{L}} = \equiv$. We now apply Corollary 3.5 to conclude that $\mathscr{L}$ is equivalent to first-order logic. ☐

4.10 Remarks. Abstract embedding relations were introduced in Mundici [1982d, 1983a and 198?a]. The results of the present section are extracted from the last paper. Notice that if we delete the expressiveness axiom from both definitions of $\sim$ and $\to$, the duality between (the resulting, weaker) embedding and equivalence relations can still be shown to hold exactly as in Theorem 4.8. In Mundici [198?a], Theorem 4.8 is partially extended, replacing the Robinson (or the AP^+) assumption by the weaker requirement that $(\sim, \sim^*)$ has the JEP.

5. *Sequences of Finite Partitions, Global and Local Back-and-Forth Games*

The separability assumption in Theorem 3.10(ii) can be neglected in the important case of equivalence relations associated with countably generated compact logics with interpolation. In general, countably generated logics are given by sequences of finite partitions on structures; and these are, in turn, related to the back-and-forth games for $\mathscr{L}$-elementary equivalence. Throughout this section, the vocabularies will only contain relation and constant symbols, for the sake of simplicity.

5.1 Definition. A *back-and-forth system* is a function $\simeq$ assigning to every finite vocabulary τ a sequence $\{\simeq_\tau^n\}_{n<\omega}$, with $\simeq_\tau^n$ a finite partition on $\mathrm{Str}(\tau)$, that is, an equivalence relation with finitely many classes, coarser than isomorphism and satisfying the following conditions, for every $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$:

renaming: $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ implies $\mathfrak{A}^\rho \simeq_{\tau'}^n \mathfrak{B}^\rho$ for any $\rho: \tau \to \tau'$;

reduct: $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ implies $\mathfrak{A} \restriction \tau' \simeq_{\tau'}^n \mathfrak{B} \restriction \tau'$ for any $\tau' \subseteq \tau$;

atomic: $\mathfrak{A} \simeq_\tau^0 \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic sentences of vocabulary τ;

diagram: $\mathfrak{A} \simeq_\tau^{n+1} \mathfrak{B}$ implies $\forall a \in A\, \exists b \in B$ with $\langle \mathfrak{A}, a \rangle \simeq_{\tau'}^n \langle \mathfrak{B}, b \rangle$, where τ' is obtained from τ by adding one constant symbol;

substructure: $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ implies

$$\mathfrak{A} | \{a \in A | \langle \mathfrak{A}, a \rangle \models \alpha(a)\} \simeq_\tau^n \mathfrak{B} | \{b \in B | \langle \mathfrak{B}, b \rangle \models \alpha(b)\},$$

whenever $\alpha(x)$ is a boolean combination of atomic sentences of vocabulary $\tau_\alpha \subseteq \tau \cup \{x\}$, $x \notin \tau$. $\mathfrak{A}|A'$ is the substructure of $\mathfrak{A}$ generated by $A' \subseteq A$.

Note that the diagram condition together with the reduct axiom imply that $\simeq_\tau^{n+1}$ is finer than $\simeq_\tau^n$.

5.2 Examples. In Theorem 5.3 we will see that every countably generated logic $\mathscr{L} = \mathscr{L}(Q^i)_{i<\omega}$ determines a back-and-forth system $\simeq$, if we let $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ mean that $\tau_\mathfrak{A} = \tau_\mathfrak{B} = \tau$ finite, and $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathscr{L}$ of vocabulary τ and quantifier rank $\leq n$. In the particular case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, we get the *Fraïssé–Ehrenfeucht* back-and-forth system, which can be equivalently obtained by writing "iff" instead of "implies" in the diagram axiom above; and, if this is done, the substructure axiom becomes superfluous. Back-and-forth systems are a natural generalization of the familiar games for $\mathscr{L}$-equivalence. For the case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, see Section II.4.2 and Section IX.4. For many other $\mathscr{L}$'s the reader should consult Weese [1980], Caicedo [1979], Makowsky–Shelah [1981], Flum–Ziegler [1980]. In a final subsection we shall relate the back-and-forth games existing in the literature to our present back-and-forth systems. Given a logic $\mathscr{L}$, the question of the existence and uniqueness of a back and forth system characterizing $\equiv_\mathscr{L}$ arises. In Theorem 5.4 we will use the Robinsion assumption to establish a one–one correspondence between back-and-forth systems and countably generated logics. Before defining the proper uniqueness notion for back-and-forth systems, however, let us remark that any such system $\simeq$ *generates* a bounded regular equivalence relation $\sim$ on the class of all structures by letting $\mathfrak{A} \sim \mathfrak{B}$ mean that $\tau_\mathfrak{A} = \tau_\mathfrak{B}$ and $\mathfrak{A} \restriction \tau \simeq_\tau^n \mathfrak{B} \restriction \tau$ for every finite $\tau \subseteq \tau_\mathfrak{A}$ and all $n < \omega$. Now, given two back-and-forth systems $\simeq'$ and $\simeq''$, we say that $\simeq''$ is *finer* than $\simeq'$ iff for every finite τ and $n < \omega$, there is $m < \omega$ such that $\simeq''^m_\tau$ is finer than $\simeq'^n_\tau$. In case $\simeq'$ is finer than $\simeq''$ and vice versa, we say that $\simeq'$ and $\simeq''$ are *equivalent*.

The great generality of the notion of a back-and-forth system is shown by the following result.

5.3 Theorem. *Let $\mathscr{L}$ be a countably generated logic. Then $\equiv_\mathscr{L}$ is generated by some back-and-forth system.*

Proof. Write $\mathscr{L}$ as $\mathscr{L}(Q^i)_{i<\omega}$ and assign a rank $r_i = 2 + i$ to each Q^i, the rank 1 being assigned to $\exists$ and to $\forall$. Then the sentences of $\mathscr{L}$ inherit a quantifier rank as in Definition II.4.2.5. Notice that for any finite τ and $n < \omega$, there are in $\mathscr{L}[\tau]$ only a finite number of pairwise inequivalent sentences with quantifier rank $\leq n$. Define $\simeq_\tau^n$ by

$\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ iff $\tau_\mathfrak{A} = \tau_\mathfrak{B} = \tau$, τ finite, and $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathscr{L}[\tau]$ with quantifier rank $\leq n$.

Then the equivalence relation $\simeq_\tau^n$ on $\mathrm{Str}(\tau)$ has finitely many equivalence classes and is coarser than isomorphism. Moreover, the reduct, renaming and atomic properties follow immediately from the basic closure properties of $\mathscr{L}$. As to the diagram axiom, let $\mathfrak{A} \simeq_\tau^{n+1} \mathfrak{B}$ and $a \in A$; let $T = \{\psi_1, \ldots, \psi_t\}$ display, without repetitions of equivalent sentences, the finitely many sentences of $\mathscr{L}[\tau]$ having quantifier rank $\leq n$, and which are satisfied by $\langle \mathfrak{A}, a\rangle$. Since $\langle \mathfrak{A}, a\rangle \models_\mathscr{L} \psi_1 \wedge \cdots \wedge \psi_t$, then $\mathfrak{A} \models_\mathscr{L} \exists a(\psi_1 \wedge \cdots \wedge \psi_t)$; since the quantifier rank of this latter sentence is $\leq n + 1$, then by assumption, $\mathfrak{B}$ is among its models. Hence $\langle \mathfrak{B}, b\rangle \models_\mathscr{L} \psi_1 \wedge \cdots \wedge \psi_t$, for some $b \in B$, whence $\langle \mathfrak{B}, b\rangle \simeq_{\tau'}^n \langle \mathfrak{A}, a\rangle$, thus establishing the diagram property of $\simeq$ (τ' is given by τ plus one constant). Concerning the substructure axiom, let $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$. Assume further that $\varphi \in \mathscr{L}[\tau]$ is an arbitrary sentence with quantifier rank $\leq n$, such that $\mathfrak{A}_0 = \mathfrak{A} | \{a \in A | \langle \mathfrak{A}, a\rangle \models \alpha(a)\} \models_\mathscr{L} \varphi$. It then follows that $\mathfrak{A} \models_\mathscr{L} \varphi^{\{x|\alpha(x)\}}$. But the latter sentence has the same quantifier rank as φ: to see this, we first note that τ-closure amounts to saying that all the constants of τ satisfy α, and this can be expressed by an atomic sentence in light of the finiteness of τ and of our assumption that τ has no function symbols. Moreover, as $\mathscr{L}$ is generated by quantifiers, we see that writing down $\varphi^{\{x|\alpha(x)\}}$ involves the conjunction of α with the sentences in the scope of Q^i (if φ is of the form $Q^i\chi$). Thus, the quantifier rank is not increased; in case φ is of the form $\neg\chi$, or $\chi \vee \psi$, or $\chi \wedge \psi$, the quantifier rank of the relativization to α is still not increased. In definitive, $\varphi^{\{x|\alpha(x)\}}$ has quantifier rank $\leq n$. Hence, by assumption, $\mathfrak{B} \models_\mathscr{L} \varphi^{\{x|\alpha(x)\}}$ so that $\mathfrak{B}_0 = \mathfrak{B} | \{b \in B | \langle \mathfrak{B}, b\rangle \models \alpha(b)\} \models_\mathscr{L} \varphi$. Since φ is arbitrary, we have proven that $\mathfrak{A}_0 \simeq_\tau^n \mathfrak{B}_0$, which yields the substructure property of $\simeq$. Finally, it is clear that $\simeq$ generates $\equiv_\mathscr{L}$, for two structures $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathscr{L}$-equivalent iff they satisfy the same sentences of quantifier rank $\leq n$ and vocabulary τ for all $n < \omega$ and all finite $\tau \subseteq \tau_\mathfrak{M} = \tau_\mathfrak{N}$. $\square$

When $\sim$ has the Robinson property we have a strong converse of the above theorem, as follows.

5.4 Theorem. *For $\sim$ an arbitrary Robinson equivalence relation the following are equivalent*:

(i) $\sim \, = \, \equiv_\mathscr{L}$ *for some countably generated logic* $\mathscr{L}$;
(ii) $\sim \, = \, \equiv_\mathscr{L}$ *for a unique (up to equivalence) countably generated logic* $\mathscr{L}$; *further,* $\mathscr{L}$ *is compact and has the interpolation property*;
(iii) $\sim$ *is generated by some back-and-forth system*;
(iv) $\sim$ *is generated by precisely one (up to equivalence) back-and-forth system.*

Proof. The implications (ii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (iii) are trivial. The implication (i) $\Rightarrow$ (iii) has been shown in Theorem 5.3. In order to prove that (iii) implies (ii) we proceed as follows: In the light of Theorem 1.3 and Corollary 3.4, it suffices to prove that (iii) implies (i). To this purpose, let $\simeq$ be a back-and-forth system generating $\sim$.

Define $[\mathscr{L}, \models_{\mathscr{L}}]$ by

(∗) $\varphi \in \mathscr{L}[\tau]$ iff φ is a union of equivalence classes of $\simeq^n_{\tau_\varphi}$ for some $n < \omega$ and some (necessarily unique and finite) vocabulary $\tau_\varphi \subseteq \tau$; and,

($\overset{*}{*}$) $\mathfrak{A} \models_{\mathscr{L}} \varphi$ iff $\varphi \in \mathscr{L}[\tau_{\mathfrak{A}}]$ and $\mathfrak{A} \restriction \tau_\varphi \in \varphi$.

Then clearly $\mathscr{L}$ satisfies the isomorphism, (finite) occurrence, renaming, reduct axioms for logics (the reader is referred to Definition II.1.1.1), and $\mathscr{L}$ contains the classes of models of atomic sentences and is closed under the boolean operations. To prove that $\mathscr{L}$ is closed under $\exists$, we assume that φ is a union of equivalence classes of $\simeq^n_{\tau_\varphi}$. It now suffices to prove that $\exists c\varphi$ is also a union of equivalence classes of $\simeq^{n+1}_\tau$, where $\tau = \tau_\varphi \backslash \{c\}$. Here we pose a denial (*absurdum* hypothesis) so that for some $\mathfrak{A}$ and $\mathfrak{B}$, with $\mathfrak{A} \simeq^{n+1}_\tau \mathfrak{B}$ we have that $\mathfrak{A} \in \exists c\varphi$ and $\mathfrak{B} \notin \exists c\varphi$. Now $\langle \mathfrak{A}, a\rangle \in \varphi$ for some $a \in A$. The assumed diagram property of $\simeq$ assures us that $\langle \mathfrak{B}, b\rangle \simeq^n_{\tau_\varphi} \langle \mathfrak{A}, a\rangle$ for some $b \in B$. Hence $\langle \mathfrak{B}, b\rangle \in \varphi$, whence we have that $\mathfrak{B} \in \exists c\varphi$—a contradiction. To prove that $\mathscr{L}$ is closed under relativization, we first show that $\mathscr{L}$ is closed under relativization to any boolean combination α of atomic sentences. Assume then that φ is a union of equivalence classes of $\simeq^n_{\tau_\varphi}$, then it suffices to show that $\varphi^{\{x \mid \alpha(x)\}}$ is also a union of equivalence classes of $\simeq^n_\tau$, with $\tau = \tau_\varphi \cup (\tau_\alpha \backslash \{x\})$. Again, we pose a denial (*absurdum* hypothesis) so that for some $\mathfrak{A}$ and $\mathfrak{B}$, with $\mathfrak{A} \simeq^n_\tau \mathfrak{B}$, we have $\mathfrak{A} \in \varphi^{\{x \mid \alpha(x)\}}$ and $\mathfrak{B} \notin \varphi^{\{x \mid \alpha(x)\}}$. By definition of relativization, $\mathfrak{A}_0 \in \varphi$ and $\mathfrak{B}_0 \notin \varphi$, where $\mathfrak{A}_0 = \mathfrak{A} \mid \{a \in A \mid \langle \mathfrak{A}, a\rangle \models \alpha(a)\} \restriction \tau_\varphi$, and $\mathfrak{B}_0 = \mathfrak{B} \mid \{b \in B \mid \langle \mathfrak{B}, b\rangle \models \alpha(b)\} \restriction \tau_\varphi$. In contrast, however, the substructure together with the reduct axiom for $\simeq$ are to the effect that $\mathfrak{A}_0 \simeq^n_{\tau_\varphi} \mathfrak{B}_0$; hence, $\mathfrak{A}_0 \in \varphi$ iff $\mathfrak{B}_0 \in \varphi$. But this is a contradiction, which proves that $\mathscr{L}$ is closed under relativization to α, as required. By conditions (∗) and ($\overset{*}{*}$), $\equiv_{\mathscr{L}}$ is coarser than $\sim$. On the other hand, if $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and not-$\mathfrak{A} \sim \mathfrak{B}$, then, since $\sim$ is generated by $\simeq$, there is a finite $\tau \subseteq \tau_{\mathfrak{A}}$ and $n < \omega$ such that not-$\mathfrak{A} \restriction \tau \simeq^n_\tau \mathfrak{B} \restriction \tau$. Thus, there is a $\varphi \in \mathscr{L}[\tau]$ such that $\mathfrak{A} \models_{\mathscr{L}} \varphi$ and $\mathfrak{B} \models_{\mathscr{L}} \neg\varphi$, whence it follows that $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{B}$ and $\equiv_{\mathscr{L}} = \sim$. We now prove that $\mathscr{L}$ is countably generated. Hence, let ψ be an $\mathscr{L}[\tau]$-sentence, with $\tau = \{R_1, \ldots, R_n\}$ (without constants for the sake of notational simplicity). Also, let Q_ψ be the quantifier given by $\mathrm{Mod}_{\mathscr{L}}(\psi)$, and let θ be the sentence of $\mathscr{L}^+ = \mathscr{L}(Q_\psi) \cup \mathscr{L}$ given by

$$\theta \underset{\mathrm{def}}{=} Q_\psi x_0 \vec{x}_1, \ldots, \vec{x}_n\ \varphi_0(x_0), \varphi_1(\vec{x}_1), \ldots, \varphi_n(\vec{x}_n),$$

where the φ_i are arbitrary sentences in $\mathscr{L}$. By definition of Q_ψ, we have $\mathfrak{M} \models_{\mathscr{L}^+} \theta$ iff $\mathfrak{M}$ has an expansion $\mathfrak{N} = [\mathfrak{M}, \langle s, R_1, \ldots, R_n\rangle, f]$ with the following properties: s is a new sort, $\langle s, R_1, \ldots, R_n\rangle \models_{\mathscr{L}} \psi$, f maps sort s one–one onto $\varphi_0^{\mathfrak{M}}$ (recalling

Notational Convention 3.6), and $\mathfrak{N} \models_{\mathscr{L}} \eta \wedge \delta$, where

$$\delta \underset{\text{def}}{=} \bigwedge_{i=1}^{n} \forall \vec{y}_i R_i(\vec{y}_i) \leftrightarrow \varphi_i(f(\vec{y}_i)), \quad \text{and}$$

$$\eta \underset{\text{def}}{=} \bigwedge_{i=1}^{n} \forall \vec{x}_i(\varphi_i(\vec{x}_i) \rightarrow \text{the coordinates of } \vec{x}_i \text{ satisfy } \varphi_0).$$

To conform to our stipulation that function symbols are absent in this section, we regard f as a binary relation symbol. The above shows that $\mathrm{Mod}_{\mathscr{L}^+}(\theta)$ is $\mathrm{RPC}_{\mathscr{L}}$ (see Definition II.3.1.1). Similarly, we prove that $\mathrm{Mod}_{\mathscr{L}^+}(\neg\theta)$ is also $\mathrm{RPC}_{\mathscr{L}}$. Now Corollary 1.6 and Theorem 1.3 can be applied to $\mathscr{L}$, since $\mathscr{L}$ is closed under relativization to atomic sentences (and there are no function symbols whatsoever). Therefore, $\mathscr{L}$ is compact; whence the Robinson property also implies that $\mathscr{L}$ obeys Craig's interpolation theorem (see Corollary 1.4), and so, *a fortiori*, $\mathscr{L}$ is Δ-closed (see Section II.3.1). In particular, θ must be a sentence of $\mathscr{L}$, which shows that application of Q_ψ in $\mathscr{L}$ does not lead beyond $\mathscr{L}$; in short, $\mathscr{L}(Q_\psi) \leq \mathscr{L}$. Observe also that as a Δ-closed logic $\mathscr{L}$ is closed under full relativization and substitution. Now let ψ range over all sentences of $\mathscr{L}[\tau]$. Because of the finiteness of each partition $\simeq^n_\tau$, there exists a countable set Z_τ of quantifiers such that every sentence of $\mathscr{L}[\tau]$ can be written down using only the quantifiers in Z_τ. By the renaming and reduct properties of $\simeq$, we are now able to exhibit a countable set Z of quantifiers such that every sentence of $\mathscr{L}$ (no matter the τ involved) can be expressed using only the quantifiers in Z. In other words, $\mathscr{L}$ has been shown to be countably generated, as was required to complete the proof that (iii) implies (ii).

Finally, we must prove that (iii) implies (iv). Assume that both $\simeq'$ and $\simeq''$ are back-and-forth systems generating $\sim$. Observe first of all that $\sim$ is a bounded Robinson equivalence relation. Now, as in the above proof of (iii) $\Rightarrow$ (ii), let $\mathscr{L}'$ and $\mathscr{L}''$ arise from $\simeq'$ and $\simeq''$, respectively, via definitions $(*)$ and $(\underset{*}{*})$. By Corollary 3.4 $\mathscr{L}'$ and $\mathscr{L}''$ are equivalent, since $\equiv_{\mathscr{L}'} = \equiv_{\mathscr{L}''} = \sim$. Now let ε be an equivalence class of $\sim'^n_\tau$. By clause $(*)$, ε is also a sentence of $\mathscr{L}'[\tau]$ and is (equivalent to) a sentence of $\mathscr{L}''[\tau]$. Whence it follows that ε is a union of equivalence classes of $\simeq''^{m_\varepsilon}_\tau$, for some $m_\varepsilon < \omega$. Letting ε range over all the equivalence classes of $\simeq'^n_\tau$, there will be a fixed $m < \omega$ providing an upper bound for the totality of the m_ε's. Indeed, $\simeq'^n_\tau$ has only finitely many equivalence classes. Therefore, $\simeq''^m_\tau$ is finer than $\simeq'^n_\tau$. Reversing the roles of $\simeq'$ and $\simeq''$, we finally establish that $\simeq'$ and $\simeq''$ are equivalent. This completes the proof of our theorem. $\square$

5.5 Corollary. *Elementary equivalence is generated by a unique (up to equivalence) back-and-forth system, namely the Fraïssé–Ehrenfeucht system of Example* 5.2. $\square$

5.6 Remarks. Abstract back-and-forth systems were introduced in Mundici [1982e], where Theorem 5.4 is also proven. We might wonder whether the above duality between countably generated logics and systems of sequences of finite partitions can be extended in the absence of the Robinson property. Lipparini [1982] considers special back-and-forth systems satisfying the following additional condition:

Expansion Axiom: For any finite τ, $\vec{c} = (c_1, \ldots, c_r)$, $R \notin \tau$ an r-ary relation symbol, φ a union of components of the partition $\simeq^n_{\tau \cup \{\vec{c}\}}$, if $\mathfrak{A} \simeq^{m+n}_\tau \mathfrak{B}$, then $\langle \mathfrak{A}, R^{\mathfrak{A}} \rangle \simeq^m_{\tau \cup \{R\}} \langle \mathfrak{B}, R^{\mathfrak{B}} \rangle$, where, e.g., $R^{\mathfrak{A}} = \{\vec{a} \in A^r \mid \langle \mathfrak{A}, \vec{a} \rangle \in \varphi\}$.

We then have the following converse of Theorem 5.3, namely

5.7 Theorem. *For every equivalence relation $\sim$ we have that $\sim\, = \,\equiv_{\mathscr{L}}$ for some countably generated logic $\mathscr{L}$ iff $\sim$ is generated by some back-and-forth system with the expansion property.*

Proof. See Lipparini [1982]. □

Using Theorems 3.11 and 5.7, Theorem 5.4 can be extended to yield a bijection between countably generated compact logics and back-and-forth systems with the expansion property such that $(\sim, \sim^*)$ has the JEP (where $\sim$ is the equivalence relation generated by $\simeq$, and $\sim^*$ is the embedding relation generated by $\sim$, see Definition 4.1). Thus the expansion property seems to be the right counterpart of the substitution axiom for logics in all general contexts where the latter property is not taken care of by the Robinson property.

5.8 Global Versus Local Versions of Back-and-Forth Games. The celebrated Fraïssé–Ehrenfeucht game G for elementary equivalence determines a sequence of finite partitions on $\mathrm{Str}(\tau)$, for each finite τ, as was remarked in Example 5.2. For more details the reader should consult Lemma II.4.2.6, where each partition is related to (the models of) the so-called Scott–Vaught–Hintikka sentences of the corresponding quantifier rank. We may regard this system of partitions as a *global* version of G, since each partition is defined over the whole of $\mathrm{Str}(\tau)$. On the other hand, given structures $\mathfrak{A}$ and $\mathfrak{B}$, G also determines a game $G(\mathfrak{A}, \mathfrak{B})$ or, equivalently, a decreasing sequence of sets of partial isomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$ (see also Section IX.4 for further information on this matter), and this may be regarded as a *local* version of G. Passing now to an arbitrary logic $\mathscr{L}(Q^i)_{i<\omega}$, we may fruitfully use the notion of back-and-forth system (see Definition 5.1) to study the global aspects of back-and-forth games in the general case. For example, Theorem 5.4 or Theorem 5.7 might be the starting point for investigating the abstract model-theoretical counterparts of the notion of *subformula*.

Is there a corresponding local version of back-and-forth game having the same degree of generality? To give an affirmative answer to this question we must first make the latter precise. We will restrict attention to $\mathscr{L}(Q)$ with Q an s-ary quantifier. Q determines a function which assigns to each structure $\mathfrak{A}$ a set $Q\mathfrak{A} \subseteq P(A^s)$ of s-ary relations on A; and (recalling Notational Convention 3.6) we have the familiar clause:

$$\mathfrak{A} \models Q\vec{x}\varphi(\vec{x}) \quad \text{iff} \quad \varphi^{\mathfrak{A}} \in Q\mathfrak{A}.$$

Now let $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}(\tau)$ with τ finite. We let A^* be the set of finite words over A, namely $A^* = \{\varnothing\} \cup A \cup A^2 \cup \cdots$. Arbitrary words over A will be denoted by a, x, t, and $|a|$ is the length of an arbitrary word a. Similarly, b, y, u will be arbitrary words over B, and w, w' arbitrary elements of $A^* \cup B^*$. Following Caicedo [1979], we give the following

5.9 Definition. With the above notation, a *back-and-forth game from* $\langle \mathfrak{A}, Q\mathfrak{A} \rangle$ *to* $\langle \mathfrak{B}, Q\mathfrak{B} \rangle$ is a sequence $\{\sim^p\}_{p<\omega}$, where each $\sim^p$ is a partition (i.e. an equivalence relation) on $A^* \cup B^*$ and, for all $p < \omega$, we have

(i) $w \sim^p w'$ implies $|w| = |w'|$;
(ii) $\varnothing \sim^p \varnothing$;
(iii) $a \sim^p b$ implies that the assignment $a_i \mapsto b_i$ is a partial isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ (as structures of vocabulary τ);
(iv) whenever $a \sim^{p+s} b$, there is a map $f: A^s \to B^s$ obeying conditions (iv′) and (iv″) below:
  (iv′) $ax \sim^p bf(x)$ for all $x \in A^s$, where ax denotes the juxtaposition of a and x;
  (iv″) for any $X \subseteq A^s$, if $\{t \in A^s \mid at \sim^p ax \text{ for some } x \in X\} \in Q\mathfrak{A}$, then $\{u \in B^s \mid bu \sim^p by, \text{ for some } y \in f(X)\} \in Q\mathfrak{B}$;
(v) same as (iv) with the roles of A and B interchanged.

5.10 Theorem. *For arbitrary* $\mathscr{L} = \mathscr{L}(Q)$, $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}(\tau)$, τ *finite, the following are equivalent*:

(i) $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$;
(ii) *there is a back-and-forth game from* $\langle \mathfrak{A}, Q\mathfrak{A} \rangle$ *to* $\langle \mathfrak{B}, Q\mathfrak{B} \rangle$.

Proof. See Caicedo [1979, Section 3.5]. □

Actually, Caicedo [1979] proves Theorem 5.10 for the general case $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$. Indeed, he also gives a back-and-forth characterization of $\mathscr{L}_{\infty\omega}(Q^i)_{i\in I}$, using the notion of a back-and-forth game from $\langle \mathfrak{A}, Q_i\mathfrak{A} \rangle_{i\in I}$ to $\langle \mathfrak{B}, Q_i\mathfrak{B} \rangle_{i\in I}$. The latter is still a sequence $\{\sim^p\}_{p<\omega}$ of equivalence relations on $A^* \cup B^*$ satisfying, roughly, the cartesian product of Definition 5.9 and I (see Caicedo [1979, Section 2.1]). Caicedo's (local) equivalence relations on $A^* \cup B^*$ generalize back-and-forth technology for specific quantifiers as developed by Fraïssé, Ehrenfeucht, Lipner, Brown, Vinner, Slomson, Krawczyk, Krynicki, Badger, Makowsky, Shelah, Tulipani, Kaufmann, and others. Weese [1980] proves an analogue of Theorem 5.10 for sets of monotone quantifiers (see Section II.4.2). Summing up the results of this section: Theorem 5.10 yields a map from logics onto (local) back-and-forth games for sets of quantifiers; with the help of Theorem 5.7 we now have a map from global onto local versions of back-and-forth games for countably generated logics.