FINITE GROUPS HAVING AN INVOLUTION CENTRALIZER WITH A 2-COMPONENT OF DIHEDRAL TYPE, I[1]

BY

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1. Introduction and statement of results

All groups considered in this paper are finite.

In current standard terminology, a group L such that $L = L'$ and $L/O(L)$ is quasisimple is said to be 2-quasisimple. Also any subnormal 2-quasisimple subgroup of a group G is called a 2-component of G.

Recently, a great deal of progress has been made on the fundamental problem of classifying all finite groups G such that $O(G) = 1$ and such that G contains an involution t such that $H = C_G(t)$ has a 2-component L (cf., [2, Theorem 1], [3], [4], and [18]). These results suggest the importance of investigating such groups G in which $C_H(L/O(L))$ has 2-rank 1. Of particular interest is the case where L is of dihedral type.

We shall now state the first main result of this paper.

Theorem 1. *Let G be a finite group with $O(G) = 1$. Suppose the involution $t \in G - Z(G)$ is such that $H = C_G(t)$ contains a 2-component L such that a Sylow 2-subgroup of L is dihedral, $m_2(C_H(L/O(L))) = 1$ and such that $N_H(L)/(LC_H(L/O(L)))$ is cyclic. Let $S \in Syl_2(N_G(L))$ be such that $t \in S$ and let $D = S \cap L$. Then the following conditions hold:*

(i) *$L/O(L)$ is isomorphic to $\mathscr{A}_7$ or to $PSL(2, q)$ for some odd prime power q with $q > 3$, $N_G(L) = O(N_G(L))H$ and $S \in Syl_2(H)$.*

(ii) *$O_2(G) = F(G) = C_G(E(G)) = 1$ and $F^*(G) = E(G)$.*

(iii) *If $F^*(G)$ is not simple, then $F^*(G) = R \times R^t$ where R is simple and $L = \langle rr^t \mid r \in R\rangle \cong R$.*

(iv) *If $F^*(G)$ is simple and $r_2(F^*(G)) \leq 4$, then the possibilities for $F^*(G)$ and G can be obtained from [7, Main Theorem].*

(v) *If $F^*(G)$ is simple and $r_2(F^*(G)) > 4$, then*

$$\langle t\rangle \in Syl_2(C_G(L/O(L))$$

and $H = C_G(t)$ contains a normal subgroup K such that $H = \langle t\rangle \times K$, $K^{(\infty)} = H^{(\infty)} = L$, $C_K(L/O(L)) = O(H) = O(K)$, $K/O(K)$ is isomorphic to a subgroup of Aut $(L/O(L))$ *containing* Inn $(L/O(L))$ *properly with $(LO(K))/O(K)$ corresponding to* Inn $(L/O(L))$ *and such that $K/(LO(K))$ is cyclic. Also if $L/O(L) \cong \mathscr{A}_7$,*

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then $K/O(K) \cong \Sigma_7$ and if $L/O(L) \cong PSL(2, q)$ for some odd prime power q with $q > 3$, then q is a square and $K/O(K)$ contains an involution that acts as a "field automorphism" of order 2 on $(LO(K))/O(K)$.

The second main result of this paper treats the open case of Theorem 1(v) in which $|D|$ is minimal.

THEOREM 2. *Let G, t, H, L, S, and D be as in Theorem 1. Assume that $F^*(G)$ is simple, $r_2(F^*(G)) > 4$ and $|D| = 2^3$. Then $|F^*(G)|_2 \leq 2^{10}$ and exactly one of the following two conclusions holds:*

(i) *$L/O(L) \cong \mathscr{A}_7$ and G is isomorphic to* Aut $(\mathscr{H}e)$;
(ii) *$L/O(L) \cong \mathscr{A}_6 \cong PSL(2, 9)$ and G is isomorphic to* Aut $(Sp(4, 4))$, Aut $(SL(5, 2))$, *or* Aut $(PSU(5, 2))$.

Before presenting a corollary of our results and its proof, we give some definitions.

A subgroup K of G is tightly embedded in G if $|K|$ is even and K intersects its distinct conjugates in a subgroup of odd order. A *standard subgroup* of G is a quasisimple subgroup A of G such that $K = C_G(A)$ is tightly embedded in G, $N_G(A) = N_G(K)$ and A commutes with none of its conjugates. (The importance of these concepts for the classification of simple groups is described in [2, Section 1].)

COROLLARY. *Let G be a finite group with $O(G) = 1$ and assume that A is a standard subgroup of G such that $|Z(A)|$ is odd and $A/Z(A) \cong \mathscr{A}_7$. Set $X = \langle A^G \rangle$. Then exactly one of the following holds:*

(1) *$X = A$ and $Z(A) = 1$;*
(2) *$X \cong \mathscr{A}_{11}$ and $Z(A) = 1$;*
(3) *$X \cong \mathscr{A}_7 \times \mathscr{A}_7$ and $Z(A) = 1$;*
(4) *$G \cong \Sigma_9$, $X = G'$, and $Z(A) = 1$;*
(5) *$G \cong$ Aut $(\mathscr{H}e)$, $X = G'$ and $|Z(A)| = 3$.*

Proof. Assume that (1) does not hold and set $K = C_G(A)$. If $m_2(K) \geq 2$, then [4, Theorem] yields (2). Suppose that $m_2(K) = 1$ and let $t \in I(K)$. Then $H = C_G(t) \leq N_G(K) = N_G(A)$ and hence $A \lhd H$. Thus $H \neq G$, $t \notin Z(G)$, and $m_2(C_H(A/O(A))) = 1$. Applying Theorem 1, we conclude that $F^*(G) = E(G)$ and $O_2(G) = 1$. Also if $F^*(G)$ is not simple, then clearly (3) holds. Suppose that $F^*(G)$ is simple. If $r_2(G) \leq 4$, then [7, Main Theorem] implies that (4) holds. Finally suppose that $r_2(G) > 4$. Then Theorem 2 yields (5). This completes the proof of the corollary.

Actually the same argument can be applied to any finite group G with $O(G) = 1$ and such that G contains a standard subgroup A of type D_8 such that $N_G(A)/AC_G(A)$ is cyclic.

The outline of the paper is as follows. Section 2 contains a collection of 2-group lemmas which are utilized at various points in the later sections. In Section 3, we prove Theorem 1. In the remainder of the paper (Sections 4–12), we prove Theorem 2.

The analyses of Sections 8–12 are primarily due to the first author.

Our notation is fairly standard and tends to follow the notation of [6] and [7]. In particular, if n is a positive integer, then $\mathscr{A}_n$ and Σ_n respectively denote the alternating and symmetric groups of degree n. Moreover, for any finite group J and any 2-power n, $\mathscr{E}_n(J)$ denotes the set of elementary abelian subgroups of J of order n and E_n denotes an elementary abelian subgroup of order n. Also for any finite group J, $m_2(J)$ denotes the 2-rank of J, $r_2(J)$ denotes the sectional 2-rank of J and $I(J)$ denotes the set of involutions of J.

2. Preliminary results

In this section, we present several results on 2-groups that are required at various points in our proofs of Theorems 1 and 2.

By surveying all groups of order 2^4, the following result is easily verified:

LEMMA 2.1. *If X is a group of order 2^4 with $|\mathrm{Aut}\,(X)|_2 \neq 1$, then X is isomorphic to E_{16}, $Z_4 \times Z_4$, $Z_4 \times Q_8$, or $Z_4 * Q_8$.*

LEMMA 2.2. *Let $\mathscr{Q} = \langle y, x \mid y^2 = x^{2^{n-1}} = t$ and $x^y = x^{-1}\rangle$ be a generalized quaternion group of order 2^{n+1} with $n \geq 2$. Assume that $\mathscr{Q}$ is a maximal subgroup of the 2-group S and that $Z(S) = Z(\mathscr{Q}) = \langle t\rangle$. Then exactly one of the following conditions holds:*

(i) *S is generalized quaternion.*

(ii) *S is semidihedral.*

(iii) *$n > 2$ and there is an element $v \in \mathscr{Q} - \langle x\rangle$ (of order 4) and an involution $\tau \in S - \mathscr{Q}$ such that $x^\tau = xt$, $v^\tau = v$, and $X = \langle \mathscr{Q}, \tau\rangle$. Also t, τ, and $xv\tau$ are representatives for the 3 conjugacy classes of involutions of X.*

Proof. Clearly $C_S(\mathscr{Q}) = \langle t\rangle = Z(\mathscr{Q})$ and we may assume that $\langle x\rangle \lhd S$. If S contains a cyclic maximal subgroup, then [6, Theorems 5.4.3 and 5.4.4] imply that (i) or (ii) holds. Thus we may assume that no maximal subgroup of S is cyclic. Suppose that $\langle x\rangle < C_S(\langle x\rangle)$. Then $C_S(\langle x\rangle) = \langle x\rangle \times \langle\tau\rangle$ for some involution $\tau \in S - \mathscr{Q}$ and $\tau^y = \tau t$ since $[\tau, \mathscr{Q}] \neq 1$. Then

$$y^{\tau x^{2^{n-2}}} = (yt)^{x^{2^{n-2}}} = y$$

and hence $\tau x^{2^{n-2}} \in C_S(\mathscr{Q}) - \mathscr{Q}$ which is impossible. Thus $C_S(\langle x\rangle) = \langle x\rangle$, $n > 2$ and there is an involution $\tau \in S - \mathscr{Q}$ such that $x^\tau = xt$ and $M = \langle x, \tau\rangle$ is a modular maximal subgroup of S. If $\tau^y = \tau$, set $v = y$. If $\tau^y = \tau t$, then $(xy)^\tau = xtyt = xy$ and $v = xy$ does the job. In this last case, $S' = \langle x^2\rangle = \Phi(S)$ and hence $S/\Phi(S) \cong E_8$, so that exactly one of conditions (i)–(iii) hold. The proof is complete.

A proof similar to that just above yields:

LEMMA 2.3. *Let* $\mathscr{D} = \langle y, x \mid y^2 = x^{2^n} = 1$ *and* $x^y = x^{-1}\rangle$ *be a dihedral group of order* 2^{n+1} *with* $n \geq 2$ *and let* $x^{2^{n-1}} = t$. *Assume that* $\mathscr{D}$ *is a maximal subgroup of the* 2-*group* S *and that* $Z(S) = Z(\mathscr{D}) = \langle t\rangle$. *Then exactly one of the following conditions holds:*

(i) S *is dihedral.*
(ii) S *is semidihedral.*
(iii) $n > 2$ *and there is an involution* $v \in \mathscr{D} - \langle t\rangle$ *and an involution* $\tau \in S - \mathscr{D}$, *such that* $x^\tau = x\tau$, $v^\tau = v$, *and* $S = \langle \mathscr{D}, \tau\rangle$. *Also* t, v, vx, τ, *and* $v\tau$ *are representatives for the* 5 *conjugacy classes of involutions of* S.

LEMMA 2.4. *Let* $\mathscr{S} = \langle y, x \mid y^2 = x^{2^n} = 1$ *and* $x^y = x^{-1}t$ *where* $t = x^{2^{n-1}}\rangle$ *be a semidihedral group of order* 2^{n+1} *with* $n \geq 3$. *Assume that* $\mathscr{S}$ *is a maximal subgroup of the* 2-*group* S *and that* $Z(S) = Z(\mathscr{S}) = \langle t\rangle$. *Then there is an involution* $\tau \in S - \mathscr{S}$ *such that* $S = \langle \mathscr{S}, \tau\rangle$ *and* $x^\tau = xt$ *and exactly one of the following conditions holds:*

(i) $y^\tau = y$ *and* $I(S) = I(\mathscr{S}) \cup \{t, \tau, t\tau\} \cup \{x^j y\tau \mid j \in Z\}$.
(ii) $y^\tau = yt$ *and* $I(S) = I(\mathscr{S}) \cup \{t, \tau, t\tau\}$.

Proof. As above, we may assume that S contains no cyclic maximal subgroup, $C_S(\langle x\rangle) = \langle x\rangle \lhd S$ and that S contains an involution $\tau \in S - \mathscr{S}$ such that $x^\tau = xt$ and $[y, \tau] \in \langle t\rangle$. If $[y, \tau] = 1$, then it is easy to see that (i) holds. If $[y, \tau] = t$, it is easy to see that (ii) holds and the proof is complete.

LEMMA 2.5. *Let* $B \cong Z_4 \times Z_4 \times Z_4$, $G = \operatorname{Aut}(B)$, *and let* $t \in G$ *be such that* $\beta^t = \beta^{-1}$ *for all* $\beta \in B$. *Then* t *is not a square in* G.

Proof. Let $X = \Omega_1(B)$ and $H = C_G(X)$. Then $t \in H$, H is an elementary abelian 2-group, $H = O_2(G)$ and $G/H \hookrightarrow \operatorname{Aut}(B/X) \cong GL(3, 2)$ since $H = C_G(B/X)$. Assume that $\tau \in G$ is such that $\tau^2 = t$. Then $\tau^2 \in H = C_G(B/X)$ and $\tau \notin H$ since H is an elementary abelian 2-group. Let $X < B_0 < B$ be such that $C_{B/X}(\tau) = B_0/X$. Then $|B_0| = 2^5$ and $\Omega_1(B_0) = X$.

Let $b \in B_0 - X$. Then $b^\tau = bx$ for some $x \in X$ and hence $b^t = b^{-1} = bxx^\tau$. Thus $b^2 \in [X, \tau]$ and $E_4 \cong \mho^2(B_0) \leq [X, \tau]$. On the other hand, $\tau^2 \in H = C_G(X)$. This implies that $|[X, \tau]| = 2$. This contradiction proves the lemma.

LEMMA 2.6. *Let* T *be a* 2-*group and let* $\langle t\rangle \times \langle \rho\rangle$ *be a subgroup of* $\operatorname{Aut}(T)$ *such that* $|t| = 2$, $|\rho| = 3$, *and* $|C_T(t)| = 4$. *Then* $|C_T(\rho)| = 1$ *and precisely one of the following holds:*

(i) $T \cong Z_{2^n} \times Z_{2^n}$ *for some integer* $n \geq 1$ *and* t *inverts* $\mho^1(T)$.
(ii) $T \cong E_{16}$.
(iii) T *is isomorphic to a Sylow* 2-*subgroup of* $L_3(4)$, $C_T(t) = \Phi(T) = T' = Z(T)$ *and the inverse image in* T *of* $C_{T/\Phi(T)}(t)$ *is isomorphic to* $Z_4 \times Z_4$ *and is inverted by* t.

(iv) *T is isomorphic to a Sylow 2-subgroup of $U_3(4)$, $C_T(t) = \Phi(T) = T' = Z(T)$ and the inverse image in T of $C_{T/\Phi(T)}$ is isomorphic to $Z_4 \times Z_4$ and is inverted by t.*

Proof. By [6, Theorem 5.3.4], ρ acts nontrivially on $C_T(t)$. Hence $C_T(t) \cong E_4$ and the result follows from [15, Theorem B].

LEMMA 2.7. *Let T be a 2-group with an involution t such that $|C_T(t)| = 8$ and such that $C_T(t)$ is not quaternion. Assume that T has an automorphism ρ of order 3 such that $t \in C_T(\rho)$. Let $T_1 = [T, \rho]$. Then $C_T(\rho) = \langle t \rangle \not\leq T_1$, $T = T_1\langle t \rangle$ and precisely one of the following holds:*

(i) *$T_1 \cong E_4$ and $T \cong E_8$;*
(ii) *$T_1 \cong Z_{2^n} \times Z_{2^n}$ for some integer $n \geq 2$ and T_1 char T;*
(iii) *$T_1 \cong E_{16}$ and T_1 char T;*
(iv) *T_1 is isomorphic to a Sylow 2-subgroup of $L_3(4)$ and T_1 char T;*
(v) *T_1 is isomorphic to a Sylow 2-subgroup of $U_3(4)$ and T_1 char T.*

Proof. Clearly $\langle t \rangle \times \langle \rho \rangle$ acts on T. Then [6, Theorem 5.3.4] implies that ρ is nontrivial on $C_T(t)$. Thus $C_T(t) = \langle t \rangle \times [C_T(t), \rho]$ where $[C_T(t), \rho] \cong E_4$. Hence $C_T(\rho) = \langle t \rangle$.

Let T be a minimal counterexample to the lemma. Then $t \notin Z(T)$ and $Z(T) = [C_T(t), \rho] \cong E_4$. Let $X = C_T(t)$. Then $X < T$ and ρ acts on $N_T(X) > X$. If $u \in N_T(X) - X$, then $t^u = t\tau$ for some $\tau \in Z(T)^{\#}$ and hence $Z(T) = [N_T(X), t] = [N_T(X), X]$. Letting $\bar{T} = T/Z(T)$, we have

$$C_{\bar{T}}(\bar{t}) = \overline{N_T(X)}, \quad |C_{\bar{T}}(\bar{t})| = 8, \quad \text{and} \quad C_{\bar{T}}(\bar{t}, \rho) = \langle \bar{t} \rangle.$$

Since t is not a square in T, neither is any element of $tZ(T)$. Thus $C_{\bar{T}}(\bar{t})$ is not quaternion. Since $|\bar{T}| < |T|$, we conclude that $\bar{T} = [\bar{T}, \rho]C_{\bar{T}}(\rho)$ where $C_{\bar{T}}(\rho) = \langle \bar{t} \rangle \not\leq [\bar{T}, \rho]$ and hence $T = T_1\langle t \rangle$ where $C_T(\rho) = \langle t \rangle \not\leq T_1 = [T, \rho] \lhd T$. Clearly $|C_{T_1}(t)| = 4$. Then Lemma 2.6 implies that T_1 has the required isomorphism type. In all cases, $\bar{T}_1$ is abelian and $C_{T_1}(t) \cong E_4$. Thus, if $\bar{T}_1$ is not isomorphic to E_4, we have $\bar{T}_1 = J_0(\bar{T})$ char $\bar{T}$ and hence T_1 char T. If $\bar{T}_1 \cong E_4$, then again $T_1 = J_0(T)$ char T and we are done.

LEMMA 2.8. *Let T be a 2-group that is isomorphic to a Sylow 2-subgroup of $U_3(4)$. Then* Aut (T) *does not contain a subgroup isomorphic to Σ_3.*

Proof. Let $\langle \rho, x \mid |\rho| = 3,\ |x| = 2,$ and $\rho^x = \rho^{-1} \rangle \leq \text{Aut}\,(T)$. Since $Z(T) \cong E_4$, it follows from [7, VI, Lemma 2.5(vii)–(viii)] that $[x, Z(T)] = 1$. Thus $[\rho, Z(T)] = 1$ which is false by [7, VI, Lemma 2.5(ii)] and we are done.

LEMMA 2.9. *Let T be a nonabelian 2-group of order 2^6 such that $\langle \rho, x \mid |\rho| = 3, |x| = 2, \rho^x = \rho^{-1} \rangle \leq$ Aut (T) with $C_T(\rho) = 1$. Then T is isomorphic to a Sylow 2-subgroup of $L_3(4)$ and $C_T(x)$ is isomorphic to D_8 or Q_8.*

Proof. By Lemma 2.8 and [7, VI, Lemma 2.18], it follows that T is isomorphic to a Sylow 2-subgroup of $L_3(4)$. Hence $Z(T) = T' = \Phi(T) \cong E_4$, $T/T' \cong E_{16}$, $C_{T/T'}(x) \cong E_4$, and $|C_{Z(T)}(x)| = 2$. Let $\bar{T} = T/T'$. Then $C_{\bar{T}}(x)$ is not ρ-invariant. Also $\bar{T}$ has exactly five ρ-invariant subgroups isomorphic to E_4, say $\bar{T}_i$ for $1 \leq i \leq 5$, such that $\bar{T}^{\#} = \bigcup_1^5 \bar{T}_i^{\#}$ where the union is disjoint. Thus we may assume that x fixes $\bar{T}_1$, $\bar{T}_2$, and $\bar{T}_3$ and $x\colon \bar{T}_4 \leftrightarrow \bar{T}_5$. Let T_i denote the inverse image in T of $\bar{T}_i$ for $1 \leq i \leq 5$. Three of the T_i are isomorphic to $Z_4 \times Z_4$ and two of the T_i are isomorphic to E_{16}. Thus we may assume that $T_1 \cong Z_4 \times Z_4$. Thus $C_{T_1}(x) = \langle \gamma_1 \rangle \cong Z_4$ where $\gamma_1^2 = z$ generates $C_{Z(T)}(x)$. Thus $z \in C_{T_i}(x) \lhd C_T(x)$ and $|C_{T_i}(x)| = 4$ for $i = 1, 2, 3$. Hence

$$|C_{T_1}(x)C_{T_2}(x)| = 8, \ \langle z \rangle = Z(T) \cap (C_{T_1}(x)C_{T_2}(x)), \text{ and } C_{T_1}(x)C_{T_2}(x) \trianglelefteq C_T(x).$$

Clearly $C_{T_3}(x) \leq C_{T_1}(x)C_{T_2}(x)$ and hence $C_T(x) = C_{T_1}(x)C_{T_2}(x)$. As $C_T(\gamma_1) = T_1$, the lemma follows.

LEMMA 2.10. *Let T be a group of order 2^5 with*

$$\Sigma_3 \cong \langle \rho, x \mid |\rho| = 3, |x| = 2, \text{ and } \rho^x = \rho^{-1} \rangle \leq \operatorname{Aut}(T).$$

Assume also that $Z(T) \cong E_8$ and $|C_T(\rho)| = 2$. Then exactly one of the following two conditions holds:

(i) *There is a $\langle \rho, x \rangle$ invariant subgroup Q of T with $Q \cong Q_8$, $Q\langle \rho, x \rangle \cong GL(2, 3)$, $Q' = C_V(\rho)$ and with $T = [Z(T), \rho] \times Q$.*

(ii) $T' = C_T(\rho) < Z(T) = \Phi(T) = \mho^1(T) = C_T(\rho) \times [Z(T), \rho], \Omega_1(T) = Z(T)$, $\exp(T) = 4$ *and for any $\alpha \in T - Z(T)$, one has $|\alpha| = 4$, $\alpha^2 \notin C_T(\rho)$, $\alpha^2 \notin [Z(T), \rho]$, and $C_T(\alpha) = \langle \alpha, Z(T) \rangle$. Also*

$$T/T' \cong Z_4 \times Z_4, \quad T/[Z(T), \rho] \cong Q_8, \quad \text{and} \quad T\langle \rho, x \rangle/[Z(T), \rho] \cong GL(2, 3).$$

Proof. Clearly $T/Z(T) \cong E_4$ and hence $|T'| = 2$ and $T' \leq \Phi(T) \leq Z(T)$. Since $C_T(\rho) \leq Z(T)$, it follows that $T' = C_T(\rho) = \langle u \rangle$ for some involution $u \in T$. Thus $C_{T/T'}(\rho) = 1$ and $T/T' \cong E_{16}$ or $Z_4 \times Z_4$. Setting $F = [Z(T), \rho]$, we have $E_4 \cong F \lhd T\langle \rho, x \rangle$ and $Z(T) = \langle u \rangle \times F$. Note that $|C_F(x)| = 2$.

Suppose that $T/T' \cong E_{16}$. Since $Z(T)/T' \cong F$ over $T\langle \rho, x \rangle$ and $|C_{T/T'}(x)| = 4$, it follows that x fixes some ρ-irreducible subspace Q/T' with $Q \neq Z(T)$, $T' < Q \lhd T\langle \rho, x \rangle$, and $Q/T' \cong E_4$. Clearly (i) holds in this case.

Suppose that $T/T' \cong Z_4 \times Z_4$. Then $T' = \langle u \rangle < \Phi(T) = Z(T) = \mho^1(T)$, $\Omega_1(T) = Z(T)$, and $\exp(T) = 4$. Also $T/F \cong Q_8$ and $T\langle \rho, x \rangle/F \cong GL(2, 3)$. Letting $\alpha \in T - Z(T)$, we have $|\alpha| = 4$, $C_T(\alpha) = \langle \alpha, Z(T) \rangle$, $T = \langle \alpha, \alpha^\rho \rangle$, and $u = [\alpha, \alpha^\rho]$. If $\alpha^2 = u$, then $(\alpha^\rho)^2 = u$ and $\mho^1(T/T') = 1$ which is false. Thus $\alpha^2 \notin T' = C_T(\rho)$. Since $T/F \cong Q_8$ and $\Omega_1(T/F) = \langle uF \rangle$, it follows that $\alpha^2 \notin F$ and we are done.

LEMMA 2.11. *Let T and $\langle \rho, x \rangle \leq \mathrm{Aut}(T)$ be as in Lemma* 2.10 *and assume that T satisfies conclusion* (ii) *of Lemma* 2.10. *Let τ be an involution in $C_{\mathrm{Aut}(T)}(\langle \rho, x \rangle)$. Then the following conditions hold:*

(i) $[Z(T), \tau] = 1$;
(ii) *τ either inverts or acts trivially on T/T'*;
(iii) $m_2(C_{\mathrm{Aut}(T)}(\langle \rho, x \rangle)) = 1$.

Proof. Since $|C_{Z(T)}(\tau)| \geq 4$ and $C_{Z(T)}(\tau)$ is ρ-invariant, (i) holds. Let $T' = \langle u \rangle$, $F = [Z(T), \rho]$, $C_F(x) = \langle z \rangle$, and $F = \langle y, z \rangle$ for some involution $y \in F - \langle z \rangle$. Let $W = \langle v \in T \mid v^2 \in \langle uz \rangle \rangle$. Then W is a maximal subgroup of T, W is abelian, $W \cong Z_4 \times E_4$, and W is $\langle \tau, x \rangle$-invariant. Also $\mho^1(W) = \langle uz \rangle$ and hence there is a unique maximal subgroup Y of W such that $uz \in Y$ and $Y/\langle uz \rangle = C_{W/\langle uz \rangle}(x)$. Then $W = Y \times \langle y \rangle$, $Y \cong Z_4 \times Z_2$, $\Omega_1(Y) = \langle u, z \rangle$, $\mho^1(Y) = \langle uz \rangle$, and Y is τ-invariant. Hence $Y = \langle q \rangle \times \langle z \rangle$ for some element $q \in Y - \langle u, z \rangle$ such that $q^2 = uz$ and $q^\tau \in q\langle u, z \rangle$. Since $T = qZ(T) \cup q^\rho Z(T) \cup q^{\rho^2} Z(T)$, it follows that $q^\tau \notin \{q, q^{-1} = quz\}$. Thus $q^\tau \in \{qu, qz = q^{-1}u\}$ and (ii) holds. Suppose that $E_4 \cong \langle \tau, \tau_1 \rangle \leq C_{\mathrm{Aut}(T)}(\langle \rho, x \rangle)$. Then we may assume that $q^\tau = qu$ and $q^{\tau_1} = q^{-1}u$. Hence $q^{\tau\tau_1} = q^{-1}$ and $\tau\tau_1$ inverts T which is impossible and we are done.

Our final result of this section is:

LEMMA 2.12. *Let T be a 2-group such that $T = R * Q_1 * Q_2$ where Q_1, Q_2 are quaternion of order* 8, *$|R| \geq 2^3$ and R is dihedral or generalized quaternion. Let $\tau \in I(T)$. Then $|C_T(\tau)| \geq 2^6$.*

Proof. Clearly we may assume that $\tau \notin Z(T)$ and let $Z(T) = \langle u \rangle$ where u is an involution and $Q_1' = Q_2' = \Omega_1(R') = \langle u \rangle$.

Suppose that $|R| = 2^3$. Then $T' = \langle u \rangle$ and $\tau^T = \{\tau, \tau u\}$. Hence $|C_T(\tau)| = 2^6$ since $|T| = 2^7$. Suppose that $|R| = 2^a \geq 2^4$. Let $\langle \gamma \rangle$ denote the cyclic maximal subgroup of R, let $\langle \omega \rangle = \Omega_2(\langle \gamma \rangle)$ and let $U = \langle \omega \rangle * Q_1 * Q_2$. Then $U \lhd T$, $C_T(\omega) = C_T(\gamma) = \langle \gamma \rangle * Q_1 * Q_2$ and $U = \Omega_1(C_T(\omega))$. Suppose that $\tau \in C_T(\omega)$. Then, since $|C_T(\omega)| \geq 2^7$ and $C_T(\omega)' = \langle u \rangle$, we have $|C_{C_T(\omega)}(\tau)| \geq 2^6$. Suppose that $\tau \notin C_T(\omega)$; then $\tau \notin U$ and $T_0 = \langle U, t \rangle$ has order 2^7. But $T_0 = (T_0 \cap R) * Q_1 * Q_2$ where $|T_0 \cap R| = 2^3$, $\langle \omega \rangle \leq T_0 \cap R$ and $\langle \omega \rangle \not\leq Z(T_0)$. Thus $T_0 \cap R$ is dihedral or quaternion and since $\tau \in T_0$, we have $|C_{T_0}(\tau)| = 2^6$. This completes the proof of the lemma.

3. The proof of Theorem 1

In this section, we present our proof of Theorem 1.

PROPOSITION 3.1. *Let G be a finite group with $O(G) = 1$. Assume that $G - Z(G)$ contains an involution t such that $H = C_G(t)$ contains a 2-component L such that $m_2(L) > 1$ and $m_2(C_H(L/O(L))) = 1$. Then the following conditions hold:*

(i) $m_2(C_G(L/O(L))) = 1$, $L \lhd H$, $N_G(L) = O(N_G(L))H$, *and* $C_G(L/O(L))$ *is tightly embedded in* G;

(ii) $O_2(G) = F(G) = C_G(E(G)) = 1$ *and* $F^*(G) = E(G)$;

(iii) *either* $F^*(G)$ *is simple of* $F^*(G) = R \times R^t$ *where* R *is simple and* $L = \langle rr^t \mid r \in R\rangle \cong R$.

Proof. Set $N = N_G(L)$, $M = C_G(L/O(L))$, and $\bar{N} = N/O(N)$. Note that if K is a 2-component of H and $K \neq L$, then $K \leq C_H(L/O(L))$ and hence $m_2(K) = 1$. Thus $L \trianglelefteq H \leq N$. Also $t \in C_G(L) \leq M \trianglelefteq N$. Choose $S \in Syl_2(N)$ such that $t \in S$ and let $T = S \cap M \in Syl_2(M)$. Then $t \in T$ and hence $t \in Z(T)$ and $\langle t\rangle = \Omega_1(T)$. Thus $N = MH$ by the Frattini argument and $m_2(M) = 1$. Thus T is cyclic or generalized quaternion, $M = O(M)C_M(t)$ and $N_G(L) = O(N_G(L))H$. Next suppose that $g \in G - N_G(M)$ is such that $|M \cap M^g|$ is even. Then there are elements $m_1, m_2 \in M$ such that $t^{m_1} = t^{m_2 g}$. Hence $m_2 g m_1^{-1} \in H \leq N \leq N_G(M)$ which implies that $g \in N_G(M)$, a contradiction. Thus (i) holds.

Suppose that $Q = O_2(G) \neq 1$. Then $1 \neq C_Q(t) \trianglelefteq C_H(L/O(L))$ and hence $t \in Q$. Then $t \in Z(Q)$ and $\Omega_1(Q) = \langle t\rangle \leq Z(G)$, which is false. Thus (ii) holds.

Assume that $E(G) = R_1 \times R_2 \times \cdots \times R_r$ where $r \geq 2$ and R_i is simple for all $1 \leq i \leq r$. Note that $L_{2'}(H) = L_{2'}(C_{E(G)}(t))$ by [9, Corollary 3.2]. Suppose that t normalizes R_i with $1 \leq i \leq r$. Then, [9, Lemma 2.18] implies that $L_{2'}(H) = L_{2'}(C_{E(G)}(t)) = L_2(C_{R_1}(t)) \times \cdots \times L_{2'}(C_{R_r}(t))$ and hence we may assume that L is a 2-component of $C_{R_1}(t)$. But $|C_{R_j}(t)|_2 \geq 2$ for all $j \neq 1$. Thus $r = 2$, $t \in R_2$, $R_1 \leq H$, and $L = R_1$ since $m_2(R_1) \geq 2$. Choose $U \in Syl_2(R_2)$ with $t \in U$. Then $C_U(t) \leq C_H(L)$ and hence $\Omega_1(U) = \langle t\rangle$. Then R_2 is not simple, a contradiction, consequently we may assume that $R_1^t = R_2$. Then

$$D = \langle r_1 r_1^t \mid r_1 \in R_1\rangle = C_{R_1 \times R_2}(t) \trianglelefteq \trianglelefteq H$$

and $D \cong R_1 \cong R_2$. Since D is simple, $m_2(D) \geq 2$ and $D = L$. Thus t normalizes R_j for all $j \geq 3$. If $r \geq 3$, we proceed as above to obtain a contradiction. Thus (iii) holds and the proof of the proposition is complete.

Thus, under the hypotheses of this lemma, if $F^*(G) = E(G)$ is not simple and the structure of $L/O(L)$ is given, then the possibilities for G are determined by the structure of Aut $(L/O(L))$. Also when a Sylow 2-subgroup of M is not cyclic and $F^*(G) = E(G)$ is simple, the possibilities for G are completely determined by [3].

Combining [8, Theorem] and [7, Main Theorem], it follows that conditions (i)–(iv) of Theorem 1 hold. Next we complete the proof of Theorem 1 by proving the following result.

PROPOSITION 3.2. *Let* G *be a finite group with* $O(G) = 1$, $F^*(G)$ *simple and with* $r_2(G) > 4$. *Suppose that* G *contains an involution* t *such that* $H = C_G(t)$ *contains a 2-component* L *with* $L/O(L) \cong A_7$ *or* $PSL(2, q)$ *for some odd prime* *power* q, $m_2(C_H(L/O(L))) = 1$ *and with* $N_H(L)/(LC_H(L/O(L)))$ *cyclic. Let* $S \in Syl_2(N_G(L))$ *be such that* $t \in S$ *and let* $D = S \cap L \in Syl_2(L)$. *Then the following conditions hold:*

(i) $\langle t \rangle = S \cap C_G(L/O(L)) \in Syl_2(C_G(L/O(L)))$.

(ii) $S \in Syl_2(H)$ *and* $L = H^{(\infty)}$.

(iii) *H contains a normal subgroup K such that* $H = \langle t \rangle \times K$ *where* $K^{(\infty)} = H^{(\infty)} = L$, $C_K(L/O(L)) = O(H) = O(K)$, $K/O(K)$ *is isomorphic to a subgroup of* Aut $(L/O(L))$ *containing* Inn $(L/O(L))$ *properly with* $LO(K)/O(K)$ *corresponding to* Inn $(L/O(L))$ *and such that* $K/LO(K)$ *is cyclic. Also if* $L/O(L) \cong A_7$, *then* $K/O(K) \cong \Sigma_7$ *and if* $L/O(L) \cong PSL(2, q)$ *for some odd prime power q, then q is a square and* $K/O(K)$ *contains an involution that acts as a "field automorphism" of order* 2 *on* $LO(K)/O(K)$.

Proof. Let $Q = S \cap C_G(L/O(L))$. Then $Q \lhd S$, $D \lhd S$, $\Omega_1(Q) = \langle t \rangle \leq Z(S)$, Q is cyclic or generalized quaternion, D is dihedral, $[Q, D] = [Q \cap D] = 1$, $QD = Q \times D$, and $S/(Q \times D)$ is cyclic.

Let $H = C_G(t)$, $N = N_G(L)$, $M = C_G(L/O(L))$, and $\bar{N} = N/M$. Then $S \in Syl_2(H)$, $L \cap M = O(L) \lhd N$, $Q \in Syl_2(M)$, $L/O(L) \cong \bar{L} \trianglelefteq \bar{N}$, $\bar{N}/\bar{L}$ is cyclic, and $\bar{N}$ is isomorphic to a subgroup of Aut $(\bar{L})$ containing Inn $(\bar{L})$ with $\bar{L}$ corresponding to Inn $(\bar{L})$. Also S/Q is isomorphic to a Sylow 2-subgroup of $\bar{N}$. Setting $\tilde{S} = S/Q$, we conclude that exactly one of the following holds: (α) $\tilde{S}$ is dihedral; (β) $\tilde{S}$ is semidihedral; (γ) $\tilde{S}/\tilde{D}$ is cyclic and if $\tilde{D} < \tilde{U} \leq \tilde{S}$ with $\tilde{U}/\tilde{D} = \Omega_1(\tilde{S}/\tilde{D})$, then $\tilde{U} = \tilde{D} \times \langle \tilde{\tau} \rangle$ where $\tilde{\tau}$ is an involution. Also if $L/O(L) \cong PSL(2, q)$ for some odd prime power q, then q is a square and $\tilde{\tau}$ acts like a "field automorphism" of order 2 on $L/O(L)$.

We shall assume that G is a counterexample to the proposition and shall proceed in a series of three steps to a contradiction.

(1) $S \notin Syl_2(G)$.

Proof. Assume that $S \in Syl_2(G)$. If Q is cyclic, then

$$r_2(S) \leq r_2(S/(Q \times D)) + r_2(Q \times D) \leq 4,$$

which is false. Thus Q is generalized quaternion and $r_2(S/Q) > 2$. Hence $S \neq Q \times D$ and (γ) holds. Let U denote the inverse image of $\tilde{U}$ in S. Thus U contains a subgroup $V \lhd U$ such that $V = C_U(D)$, $V \cap (Q \times D) = Q$, $|V/Q| = 2$, and $U = V \times D$. Also V/Q induces an outer automorphism of order 2 on $L/O(L)$ that centralizes $DO(L)/O(L)$. Hence $|D| \geq 2^3$.

Let $D = \langle y, d \mid y^2 = d^{2^n} = 1, d^y = d^{-1} \rangle$ for some integer $n \geq 2$ and let $z = d^{2^{n-1}}$. Note that $Z(D) = \langle z \rangle$, $\Omega_1(S) \leq U$, $C_S(D) = V \times \langle z \rangle$, and $U' = V' \times \langle d^2 \rangle$ where $V' \leq Q'$ and V' is cyclic by Lemma 2.2.

Suppose that D is strongly involution closed in S with respect to G. Then, since $U = V \times D$, [11, Theorem 3.1] implies that D is strongly closed in S with respect to G. Hence [11, Theorem] implies that $F^*(G)$ is isomorphic to

$U_3(4)$, A_7, or $PSL(2, q_1)$ for some odd prime power $q_1 \geq 5$. Since $C_G(F^*(G)) = 1$, we conclude that $r_2(G) \leq 4$ which is a contradiction. Thus $z^G \cap S = z^G \cap U \nsubseteq D$.

Let $\sigma \in I(S) - (Q \times D) = I(U) - (Q \times D)$. Then $U = (Q \times D)\langle\sigma\rangle$ and $D\langle\sigma\rangle \in Syl_2(L\langle\sigma\rangle)$. Since $U/Q \cong D\langle\sigma\rangle$ is neither dihedral nor semidihedral and $C_{L\langle\sigma\rangle}(L/O(L)) = O(L)$, it follows from the structure of Aut $(L/O(L))$ that there is an $l \in L$ such that $\sigma^l \in Z(D\langle\sigma\rangle)$. Then $D\langle\sigma\rangle = D \times \langle\sigma^l\rangle$ and $U = Q\langle\sigma^l\rangle \times D$.

Suppose that $t \nsim_G z \nsim_G zt$. Then there is an involution $\sigma \in z^G \cap (S - (Q \times D))$. By the above, we may assume that $D \leq C_S(\sigma)$. Since $\langle t, z\rangle \leq Z(S)$, there is an element $g \in G$ such that $\sigma^g = z$ and $C_S(\sigma)^g \leq S$. Then $D^g \leq (\Omega_1(C_S(\sigma))^g \leq \Omega_1(S) \leq U$ and $(D^g)' \leq U'$. Thus $z^g = z$ since $\Omega_1(U') = \langle t, z\rangle$ and we have a contradiction. Hence we may assume that $t \sim_G z$ or $zt \sim_G z$. Since $\langle z, t\rangle \leq Z(S)$, this fusion must take place in $N_G(S)$.

Suppose that $\Omega_1(S) \leq Q \times D$. Then $\Omega_1(S) = \langle t\rangle \times D$ and $\langle z\rangle = \Omega_1(\Omega_1(S)') \trianglelefteq N_G(S)$ which is false. Thus $I(S) = I(U) \nsubseteq Q \times D$ and $U/D \cong V$ is not generalized quaternion.

Suppose that $C_V(Q) > Z(Q) = \langle t\rangle$. Then $|C_V(Q)| = 4$. If $C_V(Q) = \langle u\rangle$ where $u^2 = t$, then $\Omega_1(S) = U$, $Z(U) = \langle u\rangle \times \langle z\rangle$, $\langle u, t\rangle$ char S and $\langle t\rangle$ char S. Thus $\langle z\rangle \trianglelefteq N_G(S)$ which is false. If $C_V(Q) = \langle t, u\rangle$ where $u^2 = 1$, then $\Omega_1(S) = \langle u, t\rangle \times D$ and again $\langle z\rangle \trianglelefteq N_G(S)$, a contradiction. Thus $C_V(Q) = Z(Q) = \langle t\rangle$ and Lemma 2.2 applies.

Suppose that V is semidihedral. Then $r_2(U) = r_2(V \times D) = 4$ and $U < S$. Since $S/(Q \times D)$ is cyclic, Lemma 2.3 yields a contradiction. Thus V satisfies (iii) of Lemma 2.2. Then $|Q| > 2^3$ and $Q_1 = Q \cap \Omega_1(V)$ is a maximal subgroup of both $\Omega_1(V)$ and Q. Also Q_1 is generalized quaternion and $Z(\Omega_1(V)) \cong Z_4$. Letting $Z(\Omega_1(V)) = \langle u\rangle$ where $u^2 = t$, we have $Z(\Omega_1(S)) = \langle u\rangle \times \langle z\rangle$ and we obtain a contradiction as above. Thus (1) holds.

Let $S < T \in Syl_2(G)$ and let $x \in N_T(S) - S$ be such that $x^2 \in S$. Note that $t^x \neq t$.

(2) Q is cyclic, $|Q| \geq 4$, and $S \neq Q \times D$.

Proof. Suppose that Q is generalized quaternion. Then $Q^x \lhd S$ and $Q^x \cap Q = [Q, Q^x] = 1$ since $t^x \neq t$. Hence $Q \cong (Q^x)^\sim \lhd \tilde{S} = S/Q$, $U = (Q \times D)Q^x = QC_S(Q)$ where $C_S(Q) = (\langle t\rangle \times D)Q^x$. Since $Q^x \cap (\langle t\rangle \times D)$ is a maximal subgroup of Q^x, we have $t^x = z$ where $Z(D) = \langle z\rangle$. But $Q \cong (Q^x)^\sim \lhd \tilde{S}$ and $\tilde{U} = \tilde{D}(Q^x)^\sim$. Thus (β) holds and $\tilde{S}$ is semidihedral. Then $\Omega_1(\tilde{S}) = \tilde{D}$ and $\Omega_1(S) \leq D \times Q$. Hence $\Omega_1(S) = D \times \langle t\rangle$ and $\langle z\rangle = \Omega_1(\Omega_1(S)')$ char X. Thus $z^x = z$, a contradiction. It follows that Q is cyclic. Suppose that $Q = \langle t\rangle$. If S/D is cyclic, then $\Omega_1(S) = \langle t\rangle \times D$, $z^x = z$ and $C_S(\Omega_1(S)) = \langle y, z\rangle$ where $y^2 \in \{t, tz\}$. Then $\langle z\rangle$ char S, $\langle y^2\rangle$ char S and hence $\langle t\rangle$ char S, which is impossible. Thus $tD \notin \mho^1(S/D)$ and there is an $f \in S$ such that $S/D = \langle tD\rangle \times \langle fD\rangle$. Therefore $t \notin \Phi(S)$ and hence $S = \langle t\rangle \times X$

for some maximal subgroup X of S. Now [14, I, 17.4] implies that $H = \langle t \rangle \times K$ for some normal subgroup K of H. Then $K^{(\infty)} = H^{(\infty)} = L$ since $H \leq N$. Also $C_K(L/O(L)) = O(K) = O(H)$. Note that if $S/\langle t \rangle$ is dihedral or semidihedral, then so is X and [12, Theorem 2] yields a contradiction. Hence (γ) holds and (iii) of the proposition holds. Thus Q is cyclic and $|Q| \geq 4$. If $S = Q \times D$, then $\Omega_1(\mho^1(Z(S))) = \langle t \rangle$ char S, which is impossible. Consequently (2) holds.

(3) $S \cong D \times D$, $T \cong D \operatorname{wr} Z_2$, and $r_2(G) \leq 4$.

Proof. Suppose that $C_S(D) = Q \times Z(D)$. Then $\tilde{S} = S/Q$ is dihedral or semidihedral and $\tilde{D}$ is a maximal subgroup of $\tilde{S}$. If $\Omega_1(S) = \Omega_1(Q \times D) = \langle t \rangle \times D$, then $C_S(\Omega_1(S)) = Q \times Z(D)$ and $\langle t \rangle$ char S which is false. Then $S = (Q \times D)\langle \sigma \rangle$ for some involution σ and $\tilde{S}$ is dihedral. Since $\langle t \rangle$ is not characteristic in S, $Z(S) = \langle t, z \rangle$ and hence σ acts dihedrally or semidihedrally on Q. Thus $C_S(\sigma) = C_T(\langle t, \sigma \rangle) = \langle t, z, \sigma \rangle$ and [12, Theorem 2] yields a contradiction. Hence $Q \times Z(D)$ is a maximal subgroup of $C_S(D)$ and (γ) holds. Thus $C_S(D) = V \times Z(D)$ for some subgroup V containing Q as a maximal subgroup. Also $U = VD = V \times D$. Suppose that $\Omega_1(S) \leq Q \times D$. Then $\Omega_1(S) = \langle t \rangle \times D$ and $C_S(\Omega_1(S)) = V \times Z(D)$ where $\Omega_1(V) = \langle t \rangle$. Hence

$$\Omega_1(\mho^1(C_S(\Omega_1(S)))) = \langle t \rangle \text{ char } S,$$

a contradiction. Thus $\langle t \rangle \times D < \Omega_1(S) \leq U$ and there is an involution $\tau \in V - Q$ such that $V = Q\langle \tau \rangle$. If V is abelian, then $\Omega_1(S) = \langle \tau, t \rangle \times D$, $C_S(\Omega_1(S)) = V \times Z(D)$, and

$$\Omega_1(\mho^1(C_S(\Omega_1(S)))) = \langle t \rangle \text{ char } S,$$

a contradiction. A similar argument applies if V is modular. Thus V is dihedral or semidihedral. Setting $\bar{S} = S/D$, we have $C_{\bar{S}}(\bar{Q}) = \bar{Q}$ since $\bar{S}/\bar{Q}$ is cyclic. Hence [14, I, 13.19] implies that $S = U = V \times D$. Now $V \times D = S = S^x = V^x \times D^x$ and $t^x \neq t$. If $V \ncong D$, then [14, I, 12.5] implies that there is a normal automorphism α of S such that $V^{x\alpha} = V$ and $D^{x\alpha} = D$. Since α is normal, α acts trivially on S' and hence $t^x = t$, a contradiction. Thus $S \cong D \times D$ and $V \cong D$. Suppose that $Y = V \cap V^x \neq 1$. Then $Y \lhd V$ and $Y \lhd V^x$. Since $t^x \in \{z, zt\}$, this is impossible. Thus $S = V \times V^x$ and $\langle S, x \rangle \cong D \operatorname{wr} Z_2$. But $J_e(\langle S, x \rangle) = S$ and hence S char $\langle S, x \rangle$. Since $Z(S) = \langle t, z \rangle$, we have $N_T(S) = \langle S, x \rangle$, $T = \langle S, x \rangle$, and $r_2(T) \leq 4$. This contradiction concludes the proof of Proposition 3.2 and of Theorem 1.

4. Beginning the proof of Theorem 2

We now commence our proof of Theorem 2.

Let G, t, H, L, S, and D be as in Theorem 1 and assume that $F^*(G)$ is simple, that $r_2(F^*(G)) > 4$ and that $|D| = 2^3$.

Observe that if $|F^*(G)|_2 \leq 2^{10}$, then [5] determines the structure of $F^*(G)$ and the conclusion of Theorem 2 follows. Consequently we may assume that

$|F^*(G)|_2 > 2^{10}$ and we shall obtain a contradiction by showing that $|O^2(G)|_2 \leq 2^{10}$.

Applying Theorem 1(v), we have $S \cong E_4 \times D_8$, $|S'| = 2$, $t \notin S'$, and $S \notin Syl_2(G)$. Hence we may choose involutions $u, z \in S$ such that

(4.1) $$S' = \langle z\rangle, \quad Z(S) = \langle t, u, z\rangle, \quad \text{and} \quad S \notin Syl_2(G).$$

Moreover we may choose involutions x, y of S such that $S = \langle t, u\rangle \times \langle x, y\rangle$, $\langle x, y\rangle \cong D_8$, and $\langle x, y\rangle' = \langle z\rangle$, and:

(4.2) The elements of $Z(S)^{\#}$ are representatives for the distinct H-conjugacy classes of $I(S)$, $u \sim x \sim xz$ in H, $z \sim y \sim yz \sim xu \sim xuz$ in H, $uz \sim yu \sim yuz$ in H, $tu \sim tx \sim txz$ in H, $tz \sim ty \sim tyz \sim txu \sim txuz$ in H, $tuz \sim tyu \sim tyuz$ in H, and $D = \langle y, xu\rangle$.

Since $S \in Syl_2(C_G(t))$ and $S' = \langle z\rangle$, we have:

(4.3) $t \nsim z$ in G and t is not a square in G.

Set $A = \langle t, u, z, y\rangle$ and $B = \langle t, u, z, x\rangle$.

(4.4) $\mathscr{E}_{16}(S) = \{A, B\}$, $I(S) \subseteq A \cup B$, and every elementary abelian subgroup of S is contained in A or in B.

Also we have

(4.5) $$C_G(A) = C_H(A) = O(C_G(A)) \times A,$$
$$C_G(B) = C_H(B) = O(C_G(B)) \times B, \quad C_H(z) = O(C_H(z))S.$$

Since $u^H \cap A = \{u\}$, $u^H \cap B = \{u, x, xz\}$, $(uz)^H \cap B = \{uz\}$, and $(uz)^H \cap A = \{uz, yu, yuz\}$, we have

(4.6) $$A \nsim B \text{ in } H, \quad \langle u\rangle \lhd N_H(A), \quad \text{and} \quad \langle uz\rangle \lhd N_H(B).$$

Also we have

(4.7) $$C_G(S) = C_H(S) = O(C_G(S)) \times Z(S), \quad N_H(S) = O(C_G(S)) \times S,$$
$$C_G(Z(S)) = C_H(Z(S)) = O(C_H(Z(S)))S = O(C_G(Z(S)))S.$$

Setting $\bar{H} = H/O(H)$, we conclude:

(4.8) There is a 3-element $\rho \in C_H(u) \cap N_H(A)$ such that $\rho^x = \rho^{-1}$, $C_A(\rho) = \langle t, u\rangle$, $[A, \rho] = \langle y, z\rangle$, and $N_{\bar{H}}(\bar{A}) = \langle \bar{t}, \bar{u}\rangle \times \langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x}\rangle$ with $\langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x}\rangle \cong \Sigma_4$.

(4.9) There is a 3-element $\rho_1 \in C_H(uz) \cap N_H(B)$ such that $\rho_1^y = \rho_1^{-1}$, $C_B(\rho_1) = \langle t, uz\rangle$, $[B, \rho_1] = \langle z, ux\rangle$, and $N_{\bar{H}}(\bar{B}) = \langle \bar{t}, \bar{u}\bar{z}\rangle \times \langle \bar{z}, \bar{x}\bar{u}, \bar{\rho}_1, \bar{y}\rangle$ with $\langle \bar{z}, \bar{x}\bar{u}, \bar{\rho}_1, \bar{y}\rangle \cong \Sigma_4$.

Thus $N_H(A) \leq O(H)A\langle \rho, x\rangle$ and hence

(4.10) $N_H(A) = (O(H) \cap N_H(A))A\langle \rho, x\rangle$

where $O(C_G(A)) = O(N_G(A)) = O(H) \cap N_H(A)$.

Similarly for B, we have

(4.11) $N_H(B) = (O(H) \cap N_H(B))B\langle \rho_1, y\rangle$

where $O(C_G(B)) = O(N_G(B)) = O(H) \cap N_H(B)$.

Suppose that $\bar{L} \cong \mathscr{A}_7$. Then $C_{\bar{H}}(\bar{u}) = \langle \bar{t}, \bar{u}\rangle \times \mathfrak{A}$ for some subgroup $\mathfrak{A}$ of $\bar{H}$ with $\langle y, z, \rho\rangle \leq \mathfrak{A}$ and $\mathfrak{A} \cong \Sigma_5$ or Σ_4. Suppose that $\bar{L} \cong PSL(2, q)$ where $q = p^{2n}$ for some odd prime integer p and integer $n \geq 1$. Then $C_{\bar{H}}(\bar{u}) = \langle \bar{t}, \bar{u}\rangle = \mathfrak{A}$ for some subgroup $\mathfrak{A}$ of $\bar{H}$ such that $\mathfrak{A}' \cong PSL(2, p^n)$, $O^{2'}(\mathfrak{A}) \cong PGL(2, p^n)$, and $\langle \bar{y}, \bar{z}, \bar{\rho}\rangle \leq \mathfrak{A}$.

Hence $S \cap O^2(C_G(t, u)) = \langle y, z\rangle \in Syl_2(O^2(C_G(t, u)))$ in all cases and:

(4.12) If $\tau \in t^G$ and $\lambda \in I(C_G(\tau))$ with $\lambda \neq \tau$, then $O^2(C_G(\lambda, \tau))$ is of odd order or has Sylow 2-subgroups of type E_4.

Clearly:

(4.13) $N_G(S)$ controls the G-fusion of element of $t^G \cap Z(S)$ and $N_G(S) \cap C_G(Z(S)) = O(C_G(S)) \times S$.

Thus, since $S \notin Syl_2(G)$, we have

$$N_G(S)/(N_G(S) \cap C_G(Z(S))) \hookrightarrow \mathrm{Aut}\,(Z(S)) \cong GL(3, 2)$$

and $2 \mid |N_G(S)/(N_G(S) \cap C_G(Z(S)))|$. Set

$$\gamma = |N_G(S)/(N_G(S) \cap C_G(Z(S))|$$

and note that $\gamma \leq 6$ since $t \not\sim z$ in G. Thus $\gamma \in \{2, 4, 6\}$.

Suppose that $\gamma = 6$. Let $P \in Syl_3(N_G(S))$. Then $t \sim tu \sim tz \sim tuz \sim u \sim uz$ in $N_G(S)$ and $X = (N_G(S) \cap C_G(Z(S))P \trianglelefteq N_G(S)$. Since

$$Z(S) = C_{Z(S)}(X) \times [Z(S), X]$$

where $C_{Z(S)}(X) = \langle z\rangle \lhd N_G(S)$ and $E_4 \cong [Z(S), X] \trianglelefteq N_G(S)$, we conclude that $N_G(S)$ has 3 orbits on $Z(S)^{\#}$. This contradiction implies that $\gamma \neq 6$.

In the next section, we shall examine the case when $\gamma = 2$ and the remainder of the paper will be concerned with the case $\gamma = 4$.

5. The case $|N_G(S)/(N_G(S) \cap C_G(Z(S)))| = 2$

Throughout this section, we assume that $\gamma = 2$ and we choose $S_1 \in Syl_2(N_G(S))$. Then $|S_1/S| = 2$ and $t^{S_1} = \{t, \alpha\}$ where $\alpha \in \{u, tu, uz, tuz, tz\}$. We shall now proceed to prove that $|G|_2 \leq 2^9$ in a series of lemmas.

LEMMA 5.1. *If* $\alpha \neq tz$, *then* $|G|_2 \leq 2^7$.

Proof. Assume that $\alpha \neq tz$ and $|G|_2 \geq 2^8$.

Clearly $S_1 \in Syl_2(N_G(\langle t, \alpha\rangle))$, $S \in Syl_2(C_G(\langle t, \alpha\rangle))$, and $N_G(\langle t, \alpha\rangle) = C_G(\langle t, \alpha\rangle)S_1$.

Suppose that $\alpha = u$. Then $t^G \cap S = \{t, u, x, xz\}$. Since $S_1 \leq N_G(A) \cap N_G(B)$, we conclude that S_1 leaves $\{x, xz\}$ invariant.

Set $M = N_G(A)$ and $\bar{M} = M/O(M)$. Now

$$C_M(\langle t, u\rangle) = (O(C_G(A)) \times A)\langle \rho, x\rangle$$

where $\rho^3 \in O(C_G(A)) = O(M)$, $C_{\bar{M}}(\langle \bar{t}, \bar{u}\rangle) = \langle \bar{t}, \bar{u}\rangle \times \langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x}\rangle$, and $C_{\bar{M}}(\langle \bar{t}, \bar{u}\rangle)$ is $\bar{S}_1$ invariant. Let $\bar{R} = C_{\bar{M}}(\langle \bar{t}, \bar{u}\rangle)\bar{S}_1$. Then $C_{\bar{M}}(\langle \bar{t}, \bar{u}\rangle)$ is of index 2 in $\bar{R}$ and $O^2(\bar{R}) = \langle \bar{y}, \bar{z}, \bar{\rho}\rangle$. Thus $\bar{X} = \langle \bar{y}, z, \bar{\rho}, \bar{x}\rangle \lhd \bar{R}$. Since $\bar{X} \cong \Sigma_4$, we have $\bar{R} = C_{\bar{R}}(\bar{X}) \times \bar{X}$ where $\langle \bar{t}, \bar{u}\rangle$ is of index 2 in $C_{\bar{R}}(\bar{X})$. Hence there is an involution $\tau \in (S_1 - S) \cap C_G(\langle x, y\rangle)$ such that $S_1 = \langle \tau, t\rangle \times \langle x, y\rangle$ with $\langle \tau, t\rangle \cong D_8$ and $t^\tau = u$. Hence

$$S_0 = \langle \tau, t\rangle \times \langle x, z\rangle \in Syl_2(C_G(x))$$

and $t^G \cap S_0 = \{t, u, x, xz\}$ since $|t^G \cap S| = 4$. Since

$$I(S_1 - S_0) = yI(\langle \tau, t\rangle) \cup yzI(\langle \tau, t\rangle) \quad \text{and} \quad t^G \cap (zI(\langle \tau, t\rangle)) = \emptyset,$$

it follows that $t^G \cap S_1 = \{t, u, x, xz\}$. Also $|S_1| = 2^6$ and hence there is a 2 group T containing S_1 with $|T : S_1| = 2$. Thus there is an element $\omega \in T - S_1$ such that ω: $\{t, u\} \leftrightarrow \{x, xz\}$ and hence T is transitive on $t^G \cap S_1$. If S_1 char T, then $T \in Syl_2(G)$ and we are done. Thus, S_1 is not characteristic in T.

Now $S_1 = \Omega_1(S_1) = \langle D \mid D \in E_{16}(S_1)\rangle$. Thus there is an involution in $T - S_1$. Suppose that $\lambda \in I(T - S_1)$. Then λ leaves $t^G \cap S_1$ and $S_1 - (t^G \cap S_1)$ invariant. Thus λ normalizes $Y = \langle \tau, tu\rangle \times \langle y, z\rangle$ and $B = \langle t, u, x, z\rangle$ and hence λ: $C_Y(t, u) = \langle tu, y, z\rangle \leftrightarrow C_Y(x, xz) = \langle \tau, tu, z)$. Also λ normalizes $Z(S_1) = \langle tu, z\rangle = S_1'$ and hence $y^\lambda \in \tau\langle tu, z\rangle$. Thus implies that $C_{S_1/B}(\lambda) = \langle yy^\lambda B\rangle$. Since $C_B(\lambda) = \langle tt^\lambda, uu^\lambda\rangle$, we conclude that $C_{S_1}(\lambda) \cong D_8$. Hence $J_e(T) = S_1$ char T and we have a contradiction. Thus $\alpha \neq u$.

Applying similar arguments when $\alpha \in \{tu, uz, tuz\}$, we obtain Lemma 5.1.

LEMMA 5.2. *If* $\alpha = tz$, *then* $|G|_2 \leq 2^9$.

Proof. Assume that $t^{S_1} = \{t, tz\}$ and that $|G|_2 \geq 2^{10}$. Then

$$t^G \cap S = \{t, tz, ty, tyz, txu, txuz\}.$$

We shall proceed to a contradiction via a series of lemmas.

LEMMA 5.3. $A \not\sim B$ *in* G *and* $N_G(S) = N_G(A) \cap N_G(B)$.

Proof. Assume that $A \sim B$ in $N_G(S) = O(C_G(S))S_1$ and let $v \in S_1 - S$. Then $A^v = B$ and $t^v = tz$. Hence $\langle I(C_S(v))\rangle = \langle C_{A \cap B}(v)\rangle = C_{\langle t, u, z\rangle}(v) \cong$

E_4 since $t^v = tz$ and $v^2 \in S \leq C_G(t, u, z)$. Thus $J_e(S_1) = S$ char S_1 and hence $S_1 \in Syl_2(G)$. Since $|S_1| = 2^6$, this is impossible and hence $A \lhd N_G(S)$ and $B \lhd N_G(S)$. Then $\langle S_1, \rho\rangle$ is transitive on $t^G \cap A = t\langle y, z\rangle$ and $N_G(A)$ is transitive on $t^G \cap A$. Similarly $N_G(B)$ is transitive on $t^G \cap B$. Since $A \sim B$ in H, we conclude that $A \sim B$ in G and Lemma 5.3 follows.

Next we investigate the subgroup $M = N_G(A)$. Similar considerations will also clearly apply to the subgroup $N_G(B)$.

Set $\overline{M} = M/O(M)$ and $F = \langle y, z\rangle$. Then $F \cong E_4$ and $C_{\overline{M}}(\overline{A}) = \overline{A}$ since $C_G(A) = O(M) \times A$ and $\overline{M}/\overline{A} \hookrightarrow \mathrm{Aut}\,(A) \cong GL(4, 2) \cong \mathscr{A}_8$. Also $\overline{M}$ acts transitively on $t^G \cap A = tF$, $C_{\overline{M}}(\bar{t}) = \overline{A}\langle\bar{\rho}, \bar{x}\rangle$ and $C_{\overline{M}}(\bar{t})/\overline{A} \cong \Sigma_3$. Hence $|\overline{M}/\overline{A}| = 3\cdot 2^3$. Since $C_A(\bar{\rho}) = \langle t, u\rangle$ is not normal in M, we conclude that $O_3(\overline{M}/\overline{A}) = 1$ and hence $\overline{M}/\overline{A} \cong \Sigma_4$. Note also that $t^M = tF$ and hence $F \lhd M$.

Set $W = O_{2',2}(M)$, $V = O(M)[W, \rho]$, and $\tilde{M} = M/(O(M) \times F)$. Thus $O(M) \times F \leq V$.

Lemma 5.4. (i) *$\overline{M} = \overline{W}\langle\bar{\rho}, \bar{x}\rangle$ and $\langle\bar{\rho}, \bar{x}\rangle \cong \Sigma_3$;*
(ii) *$C_{\overline{W}}(\bar{\rho}) = \langle\bar{t}, \bar{u}\rangle$;*
(iii) *$V \lhd M$, $C_{\overline{V}}(\bar{\rho}) = 1$ and $\overline{V} \cong Z_4 \times Z_4$ or E_{16}.*

Proof. Clearly (i) holds and $\langle\tilde{t}, \tilde{u}\rangle \leq \tilde{W} \lhd \tilde{M} = \tilde{W}\langle\tilde{\rho}, \tilde{x}\rangle$ with $\langle\tilde{\rho}, \tilde{x}\rangle = \Sigma_3$. Also $|\tilde{W}| = 2^4$ and $E_4 \cong \langle\tilde{t}, \tilde{u}\rangle \leq C_{\tilde{W}}(\tilde{\rho}) < \tilde{W}$. Thus $C_{\tilde{W}}(\rho) = \langle\tilde{t}, \tilde{u}\rangle$ and $\tilde{W} \cong E_{16}$ or $Z_2 \times Q_8$ by Lemma 2.1. Hence (ii) holds and $V \lhd M$ since $\overline{M} = \overline{W}\langle\bar{\rho}, \bar{x}\rangle$.

Suppose that $\tilde{W} \cong Z_2 \times Q_8$. Then $\Omega_1(\tilde{W}) = \langle\tilde{t}, \tilde{u}\rangle = \tilde{A}$. Let $\mathscr{U}$ be a Sylow 2-subgroup of $M = N_G(A)$ such that $S < S_1 \leq \mathscr{U}$ and let $\mathscr{V} = \mathscr{U} \cap W$. Then $\mathscr{V} \lhd \mathscr{U}$, $\mathscr{U} = \mathscr{V}\langle x\rangle$, $x \notin \mathscr{V}$, and $|\mathscr{U}| = 2|\mathscr{V}| = 2^7$. Hence $\mathscr{U} \notin Syl_2(G)$ and there is a 2-element $s \in N_G(\mathscr{U}) - \mathscr{U}$ such that $s^2 \in \mathscr{U}$. Then $A \neq A^s \lhd \mathscr{U}$ and $A^s \cap \mathscr{V} \nleq A$ since $\mathscr{U}/A \cong D_8$ and $|\mathscr{V}/A| = 4$. However $\Omega_1(\tilde{W}) = A$ implies that $A/F = \Omega_1(\mathscr{V}/F)$. This contradiction implies that $\tilde{W} \cong E_{16}$. Since $\overline{V} = [\overline{W}, \bar{\rho}] \geq \overline{F}$ and $|\overline{V}| = 2^4$, (iii) follows from Lemma 2.1.

Lemma 5.5. *Assume that $\tilde{V} \cong E_{16}$ and let $\mathscr{U} \in Syl_2(M)$ be such that $S < S_1 \leq \mathscr{U}$. Then $\mathscr{E}_{32}(\mathscr{U})$ contains a unique element E such that:*

(i) *$C_E(t) = \langle\tau, F\rangle$ for a unique $\tau \in \{u, tu\}$;*
(ii) *$\mathscr{U} = E\langle x, t\rangle$;*
(iii) *$|C_E(x)| = |C_E(t)| = 2^3$;*
(iv) *$t^G \cap tE = t^E = tF$.*

Proof. Clearly we may assume that $O(M) = 1$. Then $E_{16} \cong V = [W, \rho] \lhd M$, $W = O_2(M) = V\langle t, u\rangle$, $F = \langle y, z\rangle = C_V(t)$, and $M = W\langle\rho, x\rangle$. Thus $\langle t, u\rangle \times \langle\rho, x\rangle$ acts on V with $C_V(\rho) = 1$ and $C_V(t) = F$. Thus $\langle z\rangle = C_V(x) \cap C_V(xt)$ and $|C_V(x)| = |C_V(xt)| = 4$. Also $\langle t, u\rangle$ centralizes $C_V(t) = F$ and $\langle t, u\rangle$ acts on $C_V(x) \neq F$. Since $C_V(x) \cong E_4$, there is a unique $\tau \in \{u, tu\}$ such that τ centralizes $C_{V_1}(x)$. Thus $|C_V(\tau)| \geq 2^3$ and since $C_V(\tau)$ is $\langle\rho\rangle$-invariant, we have $C_V(\tau) = V$. Set $E = V \times \langle\tau\rangle$. Then $E \in \mathscr{E}_{32}(\mathscr{U})$, $E \lhd \mathscr{U}$ and (i)–(iii) hold. Since $I(tE) = tF \cup t\tau F$, (iv) also holds. Now (iii) and (iv) imply that $\mathscr{E}_{32}(\mathscr{U}) = \{E\}$ and we are done.

LEMMA 5.6. $\bar{V} \cong Z_4 \times Z_4$.

Proof. Assume that $\bar{V} \cong E_{16}$ and choose $\mathscr{U}$ as in Lemma 5.5. Let $\mathscr{E}_{32}(\mathscr{U}) = \{E\}$ and set $N = N_G(E)$. Also choose $\mathscr{N} \in Syl_2(N)$ such that $\mathscr{U} \leq \mathscr{N}$. Thus $C_{\mathscr{N}}(t) = S$ and $C_S(E) = S \cap E = C_E(t) = \langle\tau, F\rangle$. Suppose that $f \in \mathscr{N}$ is such that $[t, f] \in E$. Then $t^f \in t^G \cap tE = t^E$ and hence $f \in ES = \mathscr{U}$. We conclude that $C_{\mathscr{N}}(E) = E$. Also setting $\bar{\mathscr{N}} = \mathscr{N}/E$, we have $\bar{\mathscr{N}} \hookrightarrow \mathrm{Aut}\,(E) \cong GL(5, 2)$ and $C_{\bar{\mathscr{N}}}(\bar{t}) = \bar{S} = \langle\bar{t}, \bar{x}\rangle \cong E_4$. Thus $\bar{\mathscr{N}}$ is dihedral or semidihedral by [14, III, 11.9(b) and 14.23]. Since the 2-exponent of $GL(5, 2)$ is 2^3, we have $|\mathscr{N}| \leq 2^9$. Thus there is a 2-element $s \in N_G(\mathscr{N}) - \mathscr{N}$ such that $s^2 \in \mathscr{N}$. Then $E \neq E_1 = E^s \lhd \mathscr{N}$, $\bar{E}_1 \lhd \bar{\mathscr{N}}$, and $\bar{t} \notin \bar{E}_1$ since $\bar{E}_1 \not\leq C_{\bar{\mathscr{N}}}(\bar{t}) = \bar{S} = \bar{\mathscr{U}}$. Thus $\bar{x} \in \bar{E}_1$ or $\bar{x}\bar{t} \in \bar{E}_1$. Then Lemma 5.5(iii) implies that $|\bar{E}_1| = 4$, $\bar{\mathscr{N}} \cong D_8$, $|E_1 \cap E| = 2^3$, and $E_1 \cap C_E(t) = \langle\tau, z\rangle$. Let $\{\bar{x}_1\} = \{\bar{x}, \bar{x}\bar{t}\} \cap \bar{E}_1$. Then $E_1 \cap E = C_E(\bar{x}_1)$ and hence $x \in E_1$ or $tx \in E_1$. Letting $x_1 = \{x, xt\} \cap E_1$, we have $E_1 = \langle E \cap E_1, x_1, v\rangle$ for some involution v. Then $v\colon tE \leftrightarrow tx_1E$ and $Z(\bar{\mathscr{N}}) = \langle\bar{x}_1\rangle$. Since $s\colon E \leftrightarrow E_1$, s normalizes $I(\mathscr{N} - (EE_1)) = I(tE) \cup I(tx_1E)$. But $I(tE) \cup I(tx_1E) = t^{\mathscr{N}} \cup (tu)^{\mathscr{N}}$. Since $t \nsim tu$ in G, it follows that $|C_{\mathscr{N}}(t)| = |S| < |C_{\mathscr{N}\langle s\rangle}(t)|$ which is impossible. Now Lemma 5.4(iii) yields Lemma 5.6.

Thus we have $\bar{V} \cong Z_4 \times Z_4$, $\Omega_1(\bar{V}) = \bar{F} = C_{\bar{V}}(\bar{t})$, and $\bar{V} \lhd \bar{M} = \bar{V}(\langle\bar{t}, \bar{u}\rangle \times \langle\bar{\rho}, \bar{x}\rangle)$. Since $\langle\bar{t}, \bar{u}\rangle$ normalizes $C_{\bar{V}}(\bar{x})$ and $C_{\bar{V}}(\bar{x}) \cong Z_4$, it follows that $\bar{t}$ inverts $\bar{V}$ and there is a unique involution $u_1 \in \{u, ut\}$ such that $\overline{u_1} \in C_{\bar{M}}(\bar{V})$. Hence $C_{\bar{M}}(\bar{V}) = \langle\overline{u_1}\rangle \times \bar{V} \lhd \bar{M}$.

Let $\mathscr{U} \in Syl_2(M)$ be such that $S < S_1 \leq \mathscr{U}$, set $\mathscr{W} = \mathscr{U} \cap W$, $\mathscr{V} = \mathscr{U} \cap V$, and $E = \mathscr{V}\langle u_1\rangle$. Then $E = \langle u_1\rangle \times \mathscr{V}$, $\mathscr{V} \cong Z_4 \times Z_4$, $\Omega_1(\mathscr{V}) = F$, $\mathscr{V} \lhd \mathscr{U}$, $E = C_{\mathscr{U}}(\mathscr{V}) \lhd \mathscr{U}$, t inverts $\mathscr{V}$, $\mathscr{W} = E\langle t\rangle$, and $\mathscr{U} = \langle u_1\rangle \times (\mathscr{V}\langle x, t\rangle)$. Also let $X = \langle u_1\rangle \times F = \Omega_1(E)$.

LEMMA 5.7. (i) *$\mathscr{V}\langle x, t\rangle$ is isomorphic to a Sylow 2-subgroup of M_{12}.*

(ii) *$Z(\mathscr{U}) = \langle u_1, z\rangle$, $\mathscr{U} = \Omega_1(\mathscr{U})$, and $\Omega_1(\mathscr{U}') = F$.*

(iii) *$X = Z(\mathscr{U})\Omega_1(\mathscr{U}')$ char $\mathscr{U}$.*

(iv) *$E = J_0(\mathscr{W})$ and $\mathscr{W} = C_{\mathscr{U}}(X)$ char $\mathscr{U}$.*

Proof. Clearly [7, II, Lemma 2.1(vi)] implies that $\bar{V}(\langle\bar{t}\rangle \times \langle\bar{\rho}, \bar{x}\rangle)$ has Sylow 2-subgroups of type M_{12} and hence (i) holds. Then (ii)–(iv) follow and we are done.

Since $\bar{\mathscr{W}} = \bar{W} \lhd \bar{M}$, we have $M = O(M)N_M(\mathscr{W})$, $t^{\mathscr{U}} = tF = t^{\mathscr{V}} = t^M$, and $\bar{M}/\bar{W} \cong \Sigma_3$. Thus $N_M(\mathscr{W}) = ((O(M) \cap N_M(\mathscr{V})) \times \mathscr{W})(N_M(\mathscr{W}) \cap H)$ and hence there is a 3-element $\kappa \in N_M(\mathscr{W}) \cap C_H(u_1)$ inverted by x such that $\kappa^3 \in O(M) \cap N_M(\mathscr{W})$.

Set $N = N_G(\mathscr{W})$ and $\bar{N} = N/O(N)$. Clearly $\langle \mathscr{U}, \kappa \rangle \leq N$ and $Z(\mathscr{W}) = X \leq C_G(\mathscr{W})$. Let $\mathscr{U} \leq \mathscr{N} \in Syl_2(N)$. Then $\mathscr{U} < \mathscr{N}$ since $|\mathscr{U}| = 2^7$ and $\mathscr{W}$ char $\mathscr{U}$. Let $f \in \mathscr{N}$ be such that $t^f \in tX$. Then $t^f \in t^G \cap (tX) = tF = t^{\mathscr{V}}$. Hence $f \in \mathscr{V}S = \mathscr{U}$. Noting that $C_{\mathscr{U}}(\mathscr{W}) = X$, we conclude:

LEMMA 5.8. (i) $X = Z(\mathscr{W}) \in Syl_2(C_G(\mathscr{W}))$.
(ii) $C_G(\mathscr{W}) = O(N) \times X$.

Next we prove:

LEMMA 5.9. (i) $|\mathscr{N}| = 2^9$.
(ii) $\mathscr{W}$ char $\mathscr{N}$.

Proof. Clearly we may assume that $O(N) = 1$. Thus $X \lhd N$ and $N/X \hookrightarrow \mathrm{Aut}\,(\mathscr{W})$. Hence $|N|_{2'} = 3$, $N = O_2(N)\langle \kappa, x \rangle$, and $\mathscr{N} = O_2(N)\langle x \rangle$. Clearly $C_{O_2(N)}(t) = A = \langle t, X \rangle$ and $C_{\mathscr{W}}(\kappa) = \langle u_1, t \rangle$. Since $t^G \cap \langle u_1, t \rangle = \{t\}$, we have $C_{O_2(N)}(\kappa) = \langle u_1, t \rangle$. Let $v_1 \in \mathscr{V}$ be such that $v_1^2 = y$ and set $v_2 = v_1^x$. Then $v_2^2 = yz$, $(v_1v_2)^2 = z$, and $C_{\mathscr{V}}(x) = \langle v_1v_2 \rangle$. Since $X \lhd N$, it follows that N permutes the sets $\{tX, tv_1X, tv_2X, tv_1v_2X\}$. Since $|t^G \cap tX| = |t^{\mathscr{V}}| = 4$, it follows that $|O_2(N)/\mathscr{W}| = 4$. Then (i) holds and $N/\mathscr{W} \cong \Sigma_4$.

Suppose that $\mathscr{W} < C_N(X) = O_2(N)$. Then $Z(N) = \langle u_1 \rangle$. Setting $\bar{N} = N/\langle u_1 \rangle$, it follows that

$$C_{\overline{O_2(N)}}(\kappa) = \langle \bar{t} \rangle \quad \text{and} \quad C_{\overline{O_2(N)}}(\bar{t}) = \overline{C_{O_2(N)}(t)} = \bar{A} \cong E_8$$

since $t^G \cap \langle t, u_1 \rangle = \{t\}$. Noting that $Z_4 \times Z_4 \cong \bar{E} \lhd N$ and $|\overline{O_2(N)}| = 2^7$, we conclude from Lemmas 2.7 and 2.8 that there is a subgroup J of $O_2(N)$ with $u_1 \in J$, $J \lhd N$, $t \notin J$, $|J| = 2^7$, $C_J(\kappa) = \langle u_1 \rangle$, $O_2(N) = J\langle t \rangle$, and with $\bar{J} \cong Z_8 \times Z_8$ or with $\bar{J}$ isomorphic to a Sylow 2-subgroup of $L_3(4)$. Letting $x_1 = x$ if $u_1 = u$ and $x_1 = xt$ if $u_1 = ut$, we have $x_1u_1 = xu$, $tx_1 \nsim t$ in G, $t \sim tx_1u_1 = txu$ in G, and $tx_1u_1 \nsim tx_1$ in G. Since $\mathscr{N} = J\langle x, t \rangle$, we have $|C_J(tx_1u_1)| \leq 2^3$. But $C_{\bar{J}}(\bar{t}\bar{x}_1\bar{u}_1) = C_{\bar{J}}(\bar{t}\bar{x}_1) = \overline{C_J(tx_1u_1)}$ since $tx_1u_1 \nsim tx_1$ in G. Thus $|C_{\bar{J}}(\bar{t}\bar{x}_1)| \leq 2^2$. But $\bar{t}\bar{x}_1$ inverts $\bar{\kappa}$ and hence $|C_{\bar{J}}(\bar{t}\bar{x}_1)| = 2^3$ in either case by Lemma 2.9. This contradiction implies that $\mathscr{W} = C_N(X)$ and hence $\Sigma_4 \cong N/\mathscr{W} \hookrightarrow \mathrm{Aut}\,(X) \cong GL(3, 2)$.

Clearly $X \lhd \mathscr{N} = O_2(N)\langle x \rangle$, $E \lhd N$, and hence $F = \mho^1(E) \lhd N$. Thus $Z(O_2(N)) = F$ and $Z(\mathscr{N}) = \langle z \rangle$. Suppose that $E_8 \cong Y \lhd \mathscr{N}$; then $t^G \cap Y = \emptyset$ since $4|S| < |\mathscr{N}| = 2^9$, $z \in C_Y(t)$, $|C_Y(t)| \geq 4$, and $C_Y(t) \leq A$ or $C_Y(t) \leq B$. Suppose that $C_Y(t) \nleq A$. Then there is an involution $\tau \in C_Y(t) \cap (x\langle u, z \rangle \cup tx\langle u, z \rangle)$. Since $|[\tau, v_1]| = 4$, this is impossible. Thus $C_Y(t) \leq A$ and $Y \leq N_{\mathscr{N}}(A) = \mathscr{U} = \mathscr{W}\langle x \rangle$ since Y normalizes X and $[t, Y] = C_Y(t) \leq A$. Utilizing v_1 as above, it follows that $Y \leq \mathscr{W} = E\langle t \rangle$. Assume that $Y \neq X$. Then $Y \nleq E$ and there is an involution $\tau \in (tu_1F) \cap Y$ since $O_2(N)$ is transitive on $\{tX, tv_1X, tv_2X, tv_1v_2X\}$. But $C_{O_2(N)}(\tau) = C_{\mathscr{W}}(\tau) = A$ and hence $|\tau^{O_2(N)}| = 2^4$ which is impossible. Thus $Y = X$ and X char $\mathscr{N}$. Then $C_{\mathscr{N}}(X) = \mathscr{W}$ char $\mathscr{N}$ and the lemma follows.

Thus $\mathcal{N} \in Syl_2(G)$, $|G|_2 = |\mathcal{N}| = 2^9$ and the proof of Lemma 5.2 is complete.

6. The case $|N_G(S)/(N_G(S) \cap C_G(Z(S)))| = 4$

As a result of Lemmas 5.1 and 5.2 we shall assume that

$$|N_G(S)/(N_G(S) \cap C_G(Z(S)))| = 4$$

throughout the remainder of the paper.

Let $S_1 \in Syl_2(N_G(S))$. Then $S \lhd S_1$, $C_{S_1}(t) = S$, $|S_1/S| = 4$, $|S_1| = 2^7$, and $N_G(S) = O(N_G(S))S_1$.

Suppose that $tz \notin t^{S_1}$; then $tz \notin t^{N_G(S)}$. Since

$$\langle t, u, z\rangle^{\#} - \{t, z, tz\} = \{u, uz, tu, tuz\}$$

and $|t^{S_1}| = 4$, we have $t \sim \alpha \sim \alpha z$ in S_1 for some $\alpha \in \{u, tu\}$. But $z \in Z(S_1)$ and hence $tz \sim \alpha z \sim \alpha$ in S_1. Thus $tz \in t^{S_1}$ and, by interchanging the roles of u and tu, if necessary, we have:

LEMMA 6.1. (i) $t^{S_1} = t^{N_G(S)} = t\langle u, z\rangle$.

(ii) $t^G \cap S = t\langle y, z\rangle \cup tu\langle y, z\rangle \cup \{tx, txz, tux, tuxz\}$.

(iii) $\langle I(S) - (t^G \cap S)\rangle = \langle u\rangle \times \langle x, y\rangle$ *and* $I(\langle u\rangle \times \langle x, y\rangle) = I(S) - (t^G \cap S)$.

(iv) $t^G \cap A = t\langle u, y, z\rangle$ *and* $t^G \cap B = t\langle u, x, z\rangle$.

(v) $\langle u, x, z\rangle$ *is strongly closed in A with respect to G and* $\langle u, x, z\rangle$ *is strongly closed in B with respect to G.*

Set $X = \langle u, y, z\rangle$, $M = N_G(A)$, and $\bar{M} = M/O(M)$. Since $C_G(A) = O(M) \times A$, we have $C_{\bar{M}}(\bar{A}) = \bar{A}$ and $\bar{M}/\bar{A} \hookrightarrow \mathrm{Aut}(A) \cong GL(4, 2)$. Also $C_M(t) = N_H(A) = O(M)A\langle \rho, x\rangle$ and $C_{\bar{M}}(\bar{t}) = \bar{A}\langle \bar{\rho}, \bar{x}\rangle$ by (4.10). Let $\bar{P}$ be a Sylow p-subgroup of $\bar{M}$ for some prime $p \neq 2$. Since $\bar{P}$ normalizes X, it centralizes an element of $A - X = t^G \cap A$. Then (4.10) and (4.11) imply that $|\bar{P}| = 3$. Thus $\langle \bar{\rho}\rangle \in Syl_3(\bar{M})$ and $\bar{M} = O_2(\bar{M})\langle \bar{\rho}, \bar{x}\rangle$.

Since $|t^G \cap A| = 2^3$ and $S < N_{S_1}(A) \le M$, we have:

LEMMA 6.2. $|\bar{M}/\bar{A}| \in \{12, 24, 48\}$.

We can easily eliminate one of these three cases.

LEMMA 6.3. $|\bar{M}/\bar{A}| \neq 12$.

Proof. Assume that $|\bar{M}/\bar{A}| = 12$. Then, since $\bar{M}/\bar{A}$ has a subgroup isomorphic to Σ_3, we have $\bar{M}/\bar{A} \cong Z_2 \times \Sigma_3$. Thus $C_A(O_3(\bar{M}/\bar{A})) = \langle t, u\rangle \lhd M$. Let $\mathcal{M} = N_{S_1}(A)$. Then $S \lhd_{\neq} \mathcal{M} \lhd_{\neq} S_1$, $\mathcal{M} \in Syl_2(M)$, $A^{S_1} = \{A, B\}$, and $\mathcal{M} \in Syl_2(N_G(B))$. Also $t^{\mathcal{M}} = \{t, tu\}$. Letting $\beta \in S_1 - \mathcal{M}$, we have $A^\beta = B$ and $M^\beta = N_G(B)$. But, by utilizing the element ρ_1 in (4.9), we have $\langle t, uz\rangle \lhd N_G(B)$. Hence $t^{\mathcal{M}} = \{t, tuz\}$. This contradiction establishes the lemma.

The remainder of this section is devoted to proving:

LEMMA 6.4. *If* $|\bar{M}/\bar{A}| = 24$, *then* $|G|_2 \le 2^9$.

Thus, throughout the rest of this section, we assume that $|\bar{M}/\bar{A}| = 24$ and that $|G|_2 \ge 2^{10}$ and we shall proceed to a contradiction.

Now ρ has the following orbits on $t^G \cap A = t\langle u, y, z\rangle$:

$$\{t\}, \quad \{tu\}, \quad \{ty, tz, tyz\}, \quad \text{and} \quad \{tuy, tuz, tuyz\}.$$

Since $|t^M| = 4$, we have:

(6.1) $t^M = t\langle y, z\rangle$ or $t^M = \{t, tuz, tuy, tuyz\}$, $t \not\sim tu$ in M, and $C_A(\rho) = \langle t, u\rangle \lhd M$.

Hence we have:

LEMMA 6.5. $\bar{M}/\bar{A} \cong \Sigma_4$ *and* $M = O_{2',2}(M)\langle \rho, x\rangle$.

Next we prove:

LEMMA 6.6. (i) $S_1/S \cong Z_4$ *and there is an element* $\tau \in S_1 - S$ *such that* $S_1 = \langle S, \tau\rangle$ *and* $\tau^2 \notin S$.

(ii) *If* $\tau \in S_1 - S$ *is such that* $\tau^2 \notin S$, *then* $\tau\colon A \leftrightarrow B$, $\tau^2\colon t \leftrightarrow tz$, *and* $\tau^2 \in N_G(A) \cap N_G(B)$.

(iii) $t^M = t\langle y, z\rangle$ *and* $t \not\sim tuz$ *in* $N_G(B)$.

Proof. Assume that $S_1/S \cong E_4$ and let $\omega \in S_1 - S$ be such that $t^\omega = tu$. Then $\omega\colon t \leftrightarrow tu$ and ω normalizes $O^2(C_G(t, tu)) \cap S = \langle y, z\rangle$. Thus $\omega \in N_G(\dot{A}) = M$ which contradicts (6.1) and (i) holds. Next, let $\tau \in S_1 - S$ be such that $\tau^2 \notin S$. Then $S_1 = \langle S, \tau\rangle$ and $\tau^2 \in N_G(A) \cap N_G(B)$. Since $t^{S_1} = t\langle u, z\rangle$ and $t \not\sim tu$ in M, it follows that $\tau\colon A \leftrightarrow B$. Thus $N_G(B)/C_G(B) \cong \Sigma_4$ and $t \not\sim tuz$ in $N_G(B)$ by the above argument applied to $N_G(B)$. Hence (ii) holds and (iii) follows from (6.1).

Fix $\tau \in S_1 - S$ such that $\tau^2 \notin S$ and set $\alpha = \tau^2$.

LEMMA 6.7. (i) $Z(S_1) = \langle z\rangle$, $\tau\colon u \leftrightarrow uz$, S_1/S *acts regularly on* $t\langle u, z\rangle$, *and* $\langle u, z\rangle \lhd S_1$.

(ii) $\langle S, \alpha\rangle = C_{S_1}(\langle u, z\rangle)$.

(iii) $t^{N_G(B)} = t\langle xu, z\rangle$.

(iv) $\Omega_1(S_1) = \langle S, \alpha\rangle$.

Proof. Clearly S_1/S acts regularly on $t\langle u, z\rangle$ and $\langle z\rangle \le Z(S_1) \le \langle u, z\rangle$. If $Z(S_1) = \langle u, z\rangle$, then $t^\tau \in \{tu, tuz\}$ implies that $t^{\tau^2} = t$, which is false and (i)–(ii) hold. Since $t^\alpha = tz$, $\alpha \in N_G(B)$, and $N_G(B)/C_G(B) \cong \Sigma_4$, the corresponding result for $N_G(B)$ in Lemma 6.6(iii) yields (iii). Finally $S \le \Omega_1(S_1) \le \langle S, \alpha\rangle$, $|S_1| = 2^7$ and the fact that S is not characteristic in S_1 yield (iv).

Let $S < \langle S,\alpha\rangle \le \mathscr{U} \in Syl_2(M)$, let $W = O_{2',2}(M)$, $\mathscr{W} = \mathscr{U} \cap W$, and $F = \langle y, z\rangle$. Then $A \le \mathscr{W} \lhd \mathscr{U}$, $|\mathscr{W}| = 2^6$, $\mathscr{W}/A \cong E_4$, $\langle t^G \cap A\rangle =$

$\langle t, F\rangle \lhd M$, and $\langle\langle t^G \cap A\rangle - t^G\rangle = F \lhd M$. Also $\mathscr{U} = \mathscr{W}\langle x\rangle$, $|\mathscr{U}| = 2^7$, and $M = O(M)\mathscr{W}\langle\rho, x\rangle$

LEMMA 6.8. *$Z(\mathscr{W}) = \langle u, y, z\rangle$, $Z(\overline{M}) = \langle\bar{u}\rangle$, and $O_2(Z(M)) = \langle u\rangle$.*

Proof. Clearly $O_2(Z(M)) \leq A$ and hence we may assume that $O(M) = 1$. Since $O_2(M) = \mathscr{W}$ is $\langle\rho\rangle$-invariant and $F = \langle y, z\rangle \lhd M$, we have $F \leq Z(\mathscr{W})$. Also $\langle u\rangle \leq C_M(\langle A, \rho, x, \alpha\rangle) = Z(M)$ and we are done.

We can now obtain fairly precise information about the structure of $\mathscr{U}$.

LEMMA 6.9. *$\mathscr{U}$ satisfies exactly one of the following two conditions:*

(i) *$\mathscr{U}$ contains a normal subgroup $\mathscr{V}$ inverted by t with $\mathscr{V} \cong Z_4 \times Z_4$, $\Omega_1(\mathscr{V}) = F$, $C_{\mathscr{U}}(\mathscr{V}) = \langle u\rangle \times \mathscr{V}$, and $\mathscr{U} = \langle u\rangle \times (\mathscr{V}\langle x, t\rangle)$ with $\mathscr{V}\langle x, t\rangle$ isomorphic to a Sylow 2-subgroup of M_{12}. Also $\mathscr{W} = \langle u\rangle \times (\mathscr{V}\langle t\rangle)$ and there is a 3-element $\kappa \in N_M(\mathscr{W}) \cap H$ such that $\kappa^x = \kappa^{-1}$, $C_{\mathscr{W}}(\kappa) = \langle t, u\rangle$, $[\mathscr{W}, \kappa] = \mathscr{V}$, and $\kappa^3 \in C_G(\mathscr{W})$.*

(ii) *$\mathscr{E}_{32}(\mathscr{U})$ contains a unique element E such that $\mathscr{U} = E\langle x, t\rangle$, $E \cap S = X = \langle u, y, z\rangle = C_E(t)$, $t^E = t\langle y, z\rangle$, $|C_E(x)| = |C_E(xt)| = 8$, $\mathscr{W} = E\langle t\rangle$, $I(tE) = t^E \cup (tu)^E$, $I(xE) = x^E \cup (xu)^E$, and $I(xtE) = (xt)^E \cup (xtu)^E$. Also there is a 3-element $\kappa \in N_M(\mathscr{W}) \cap N_M(E) \cap H$ such that $\kappa^x = \kappa^{-1}$, $C_E(\kappa) = \langle u\rangle$, $F \leq [E, \kappa]$, $|[E, \kappa]| = 16$, $\kappa^3 \in C_G(\mathscr{W})$, and $E = [\mathscr{W}, \kappa] \times \langle u\rangle$.*

Proof. Clearly $M = O(M)N_M(\mathscr{W})$ and, since $t^M = t^{\mathscr{W}}$, we have

$$M = O(M)\mathscr{W}(N_M(\mathscr{W}) \cap H).$$

Since $C_{\mathscr{W}}(t) = A$, we have $O(N_M(\mathscr{W}) \cap H) = O(M) \cap N_M(\mathscr{W}) \cap H$ and

$$O_{2',2}(N_M(\mathscr{W}) \cap H) = (O(M) \cap N_M(\mathscr{W}) \cap H) \times A.$$

Thus there is a 3-element $\kappa \in N_M(\mathscr{W}) \cap H$ such that $\kappa^x = \kappa^{-1}$ and $\overline{M} = \overline{\mathscr{W}}\langle\bar{\kappa}, \bar{x}\rangle$. It follows that we may assume that $O(M) = 1$. Set $\tilde{M} = M/F$ and let $\mathscr{V} = [\mathscr{W}, \kappa]$. Then $|\tilde{\mathscr{W}}| = 2^4$, $\tilde{\mathscr{W}} = O_2(\tilde{M})$, and $C_{\tilde{\mathscr{W}}}(\tilde{\kappa}) = \langle \tilde{t}, \tilde{u}\rangle = Z(\tilde{M})$ since $t^M = tF$.

Suppose that $\tilde{\mathscr{W}} \cong Z_2 \times Q_8$. Then there is a subgroup Y of A such that $F < Y < A$, $Y \lhd M$, $\mathscr{W}/Y \cong Q_8$, and $M/Y \cong GL(2, 3)$. Thus $\mathscr{U}/Y$ is semi-dihedral of order 16. Since $|\mathscr{U}| = 2^7$ and $\mathscr{U} \in Syl_2(N_G(A))$, it follows that $\mathscr{U}$ has a normal subgroup A^* with $A \neq A^*$ and $A \cong A^*$. Then $A^*Y/Y \lhd \mathscr{U}/Y$. and hence $A^*Y = A$. Since $A^* \neq A$, this is impossible. As $C_{\tilde{\mathscr{W}}}(\tilde{\kappa}) = \langle\tilde{t}, \tilde{u}\rangle$, Lemma 2.1 implies that $\tilde{\mathscr{W}} \cong E_{16}$ and hence $\mathscr{V} = [\mathscr{W}, \kappa]$ has order 16. Thus $\mathscr{V} \cong E_{16}$ or $\mathscr{V} \cong Z_4 \times Z_4$.

Suppose that $\mathscr{V} \cong Z_4 \times Z_4$. Then $\mathscr{V} \lhd M$, $\Omega_1(\mathscr{V}) = F$, $C_M(\mathscr{V}) = \langle u\rangle \times \mathscr{V}$, $\mathscr{W} = \langle u\rangle \times (\mathscr{V}\langle t\rangle)$, $M = \langle u\rangle \times \mathscr{V}(\langle t\rangle \times \langle\kappa, x\rangle)$, t inverts $\mathscr{V}$ and $\mathscr{V}(\langle t\rangle \times \langle\kappa, x\rangle)$ has Sylow 2-subgroups of type M_{12} by [7, II, Lemma 2.1(iv)–(vi)]. Thus (i) holds in this case.

Suppose that $\mathscr{V} \cong E_{16}$. Then $E = \langle u\rangle \times \mathscr{V} \in \mathscr{E}_{32}(\mathscr{U})$ and $\mathscr{W} = E\langle t\rangle$.

Since $\langle u \rangle = Z(M)$ and $C_{\mathscr{V}}(\kappa) = 1$, it is clear that $E \cap S = C_E(t) = X$, $C_E(x, t) = \langle u, z \rangle$, and $|C_E(x)| = |C_E(xt)| = 8$. Since $\mathscr{U} = E\langle x, t \rangle$, it follows that E is the unique element of $\mathscr{E}_{32}(\mathscr{U})$ and (ii) holds.

We shall now treat case (ii) of Lemma 6.9.

LEMMA 6.10. *Assume that* (ii) *of Lemma* 6.9 *holds. Then the following conditions hold:*

(i) $N_G(\mathscr{W}) \leq N_G(A) = M$ *and* $\mathscr{U} \in Syl_2(N_G(\mathscr{W}))$.
(ii) $t \sim tu$ *in* $N_G(E)$.
(iii) $E \in Syl_2(C_G(E))$ *and* $C_G(E) = O(C_G(E)) \times E$.

Proof. Since E char $\mathscr{W}$ and $\langle I(tE) \rangle = \langle I(\mathscr{W} - E) \rangle = A$, (i) follows. Suppose that $w \in N_G(E)$ is such that $t^w = tu$. Since $C_E(t) = C_E(tu) = X$, we have $w \in N_G(A) = M$ and (ii) holds. Then $t^{N_G(E)} \cap tE = t^E$. Since $\mathscr{U} = ES \leq N_G(E)$ and $C_{\mathscr{U}}(E) = E$, we have (iii).

LEMMA 6.11. *Lemma* 6.9(ii) *does not hold.*

Proof. Assume that Lemma 6.9(ii) holds and set $N = N_G(E)$ and $\bar{N} = N/O(N)$. Clearly $C_G(E) = O(N) \times E$ and $\bar{N}/\bar{E} \hookrightarrow \mathrm{Aut}\,(E) \cong GL(5, 2)$. Now Lemma 6.10(ii) implies that $C_{\bar{N}/\bar{E}}(\bar{t}\bar{E}) = \overline{C_N(t)}\bar{E}/\bar{E}$. However $C_N(t) \leq C_M(t) = O(M)A\langle \kappa, x \rangle$ and $A\langle \kappa, x \rangle \leq N$. Also $O(M) \cap N$ centralizes t and $C_E(t) = X$ and hence $O(M) \cap N \leq C_G(E)$ by [6, Theorem 5.3.4]. Thus $C_N(t) = (O(N) \cap H)A\langle \kappa, x \rangle$ and hence $C_{\bar{N}/\bar{E}}(\bar{t}\bar{E}) \times \langle \bar{\kappa}\bar{E}, \bar{x}\bar{E} \rangle \cong Z_2 \times \Sigma_3$. Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$. Clearly $\mathscr{U} < \mathscr{T}$ since E char $\mathscr{U}$. Also $\exp\,(\mathscr{T}/E) \leq 8$ and $\mathscr{T}/E$ is dihedral or semidihedral since $C_{\mathscr{T}/E}(tE) = \langle tE, xE \rangle$. Hence $|\mathscr{T}| \leq 2^9 < |G|_2$ and there is a 2-element $\tau \in (\mathscr{T}) - \mathscr{T}$ such that $\tau^2 \in \mathscr{T}$. Set $\mathscr{S} = \langle \mathscr{T}, \tau \rangle$ and let $E_1 = E^\tau$. Then $E_1 \neq E$, $E_1 \lhd \mathscr{T}$, and $t\colon E \leftrightarrow E_1$. Letting $\tilde{\mathscr{T}} = \mathscr{T}/E$, we conclude that $1 \neq \tilde{E}_1 \lhd \tilde{\mathscr{T}}$ and $Z(\tilde{\mathscr{T}}) \leq \langle \tilde{x}, \tilde{t} \rangle$. However $I(tE) \cup I(xtE) \subseteq t^G$ and $t^G \cap E_1 = \emptyset$. Thus $\langle \tilde{x} \rangle = Z(\tilde{\mathscr{T}}) \leq \tilde{E}_1$. Let $\mathscr{V} = [E, \kappa]$ and choose $\alpha \in \mathscr{V}^\#$ such that $C_{\mathscr{V}}(x) = \langle z, \alpha \rangle$. Then $\alpha^t = \alpha z$, $C_{\mathscr{V}}(xt) = \langle z, y\alpha \rangle$ and there is an element $\beta \in \mathscr{V} - \langle z, y, \alpha \rangle$ such that $\beta^t = \beta y$ and $\mathscr{V} = \langle z, y, \alpha, \beta \rangle$. Note that $N_E(S) = \langle z, u, \alpha \rangle$. Thus $C_E(xE) = \langle z, u, \alpha \rangle$. Thus $E \cap E_1 = C_E(x)$, $|\tilde{E}_1| = 4$, $\tilde{\mathscr{T}} \cong D_8$, and $|\mathscr{S}| = 2^9$. Also $I(xE) = x\langle z, u, \alpha \rangle$ so that $x \in E_1$. Thus $E_1 = \langle u, z, \alpha, x, \delta \rangle$ for some involution $\delta \in E_1 - \langle u, z, \alpha, x \rangle$. Now suppose that $E_2 \neq E$ and $E \cong E_2 \lhd \mathscr{T}$. Then the above argument implies that $\langle u, z, \alpha, x \rangle \leq E_2$ and $\tilde{E}_2 = \langle \tilde{x}, \tilde{\delta} \rangle$. Thus $E_2 = \langle u, z, \alpha, x, e\delta \rangle$ for some element $e \in E$. However $e \in C_E(x) = \langle u, z, \alpha \rangle$ and hence $E_2 = E_1$. Thus E and E_1 are the only two normal subgroups of $\mathscr{E}_{32}(\mathscr{T})$, $\mathscr{T} = EE_1\langle t \rangle$ where $t \notin EE_1$ and $Z(\mathscr{T}) = \langle u, z \rangle$. Note that $I(xE) \subseteq E_1$ and $E \cap E_1 \leq C_E(\delta)$. However, if $E \cap E_1 = C_E(\delta)$, $I(\delta E) \subseteq E_1$ and if $E \cap E_1 < C_E(\delta)$, then $\langle C_E(\delta), \tau \rangle$ is elementary abelian of order 2^5 for every $\tau \in I(\delta E)$. Since $t\colon \delta E \leftrightarrow \delta xE$, we have $t^G \cap (xE \cup \delta E \cup x\delta E) = \emptyset$ and hence $t^G \cap (EE_1) = \emptyset$.

On the other hand, $I(\mathscr{T}) = I(EE_1) \cup I(tE) \cup T(xtE)$ and $\delta\colon tE \leftrightarrow xtE$. It follows that we may assume that $t^\tau = tu$. Hence $S^\tau = C_{\mathscr{T}}(t)^\tau = C_{\mathscr{T}}(tu) = S$

and $\mathscr{S} = EE_1N_{\mathscr{S}}(S)$ where $\tau \in N_{\mathscr{S}}(S) - N_{\mathscr{T}}(S)$. Note that $N_{\mathscr{T}}(S) = \langle u, z, y, \alpha, x, t\rangle = N_{\mathscr{U}}(S) = \Omega_1(N_{\mathscr{S}}(S))$ by Lemmas 6.6 and 6.7. Set $Y = \langle u, z, \alpha, x\rangle = \langle u, \alpha\rangle \times \langle x, y\rangle$. Then

$$Y = (EE_1) \cap N_{\mathscr{T}}(S) \lhd N_{\mathscr{S}}(S).$$

Also $[x, E] \leq \langle u, z, \alpha\rangle \leq Y$ and hence E and $E_1 = E^{\tau}$ normalize Y. Thus $Y \lhd \mathscr{S}$. Similarly, since $Y\langle t\rangle = N_{\mathscr{T}}(S) \lhd N_{\mathscr{S}}(S)$ and $[t, E] \leq Y$, we conclude that $Y\langle t\rangle \lhd \mathscr{S}$.

Setting $\bar{\mathscr{S}} = \mathscr{S}/(Y\langle t\rangle)$, we have $\bar{\tau}: \bar{E} = \langle\bar{\beta}\rangle \leftrightarrow \bar{E}_1 = \langle\bar{\delta}\rangle$ and $\bar{\mathscr{S}} = \langle\bar{\beta}, \bar{\delta}, \bar{\tau}\rangle \cong D_8$. Suppose that $j \in I(\mathscr{S} - \mathscr{T})$. Then $\bar{j} \sim \bar{\tau}$ in $\bar{\mathscr{S}}$ and hence $j \in N_{\mathscr{S}}(S) - N_{\mathscr{T}}(S)$. Since $\Omega_1(N_{\mathscr{S}}(S)) = N_{\mathscr{T}}(S)$, this is impossible. Hence $\Omega_1(\mathscr{S}) = \mathscr{T}$ char $\mathscr{S}$ and $N_G(\mathscr{S})$ acts on $\{E, E_1\}$. Thus $\mathscr{S} \in Syl_2(G)$ and $|G|_2 = 2^9$ which is false and the proof of Lemma 6.11 is complete.

Thus, for the remainder of this section, we shall assume that Lemma 6.9(i) holds.

Let $\kappa_1 \in \mathscr{V}$ be such that $\kappa_1^2 = y$ and set $\kappa_2 = \kappa_1^x$. Then $\mathscr{V} = \langle\kappa_1\rangle \times \langle\kappa_2\rangle$ and $C_{\mathscr{V}}(x) = \langle\kappa_1\kappa_2\rangle$. Note that $\mathscr{U} = \langle u\rangle \times (\mathscr{V}\langle x, t\rangle)$, $Z(\mathscr{U}) = \langle u, z\rangle$, $\mathscr{U}' = \langle y, \kappa_1\kappa_2\rangle$, and $\Omega_1(\mathscr{U}') = F$. Thus $X = Z(\mathscr{U})\Omega_1(\mathscr{U}')$ char $\mathscr{U}$, $C_{\mathscr{U}}(X) = \langle u\rangle \times (\mathscr{V}\langle t\rangle)$ char $\mathscr{U}$ and $J_0(\langle u\rangle \times (\mathscr{V}\langle t\rangle)) = \langle u\rangle \times \mathscr{V}$ char $\mathscr{U}$.

Set $E = \langle u\rangle \times (\mathscr{V}\langle t\rangle)$, $N = N_G(E)$, and $\bar{N} = N/O(N)$. Then $\langle\mathscr{U}, \kappa\rangle \leq N$, $Z(E) = X \leq Z(C_G(E))$, and $Z(E) = X \lhd N \leq N_G(X)$. Also E char $\mathscr{U}$ and $|\mathscr{U}| = 2^7$ implies that $\mathscr{U} \notin Syl_2(N)$. Since $\bar{\mathscr{U}} \in Syl_2(N_{\bar{N}}(\bar{A}))$ and $C_{\bar{\mathscr{U}}}(\bar{E}) = \bar{X}$, it follows that $X \in Syl_2(C_G(E))$ and $C_G(E) = O(N) \times X$. Noting that $|\mathrm{Aut}\,(E)|_{2'} = 3$, we conclude that $\bar{N} = O_2(\bar{N})\langle\bar{\kappa}, \bar{x}\rangle$ where $\bar{\kappa}^3 = 1$.

Next we prove:

LEMMA 6.12. (i) $t^{N_G(X)} \cap (tuF) = t^N \cap (tuF) = \emptyset$.

(ii) *Every involution of $E - (\langle u\rangle \times \mathscr{V})$ is conjugate in N to t or tu.*

(iii) $|N|_2 = 2^9$.

(iv) $C_{O_2(\bar{N})}(\bar{\kappa}) = \langle\bar{t}, \bar{u}\rangle$.

Proof. Suppose that $n \in N_G(X)$ is such that $t^n = tu$. Then $n \in N_G(A) = M$. Since $t \nsim tu$ in M, this is impossible. Since $N = N_G(E) \leq N_G(X)$, (i) holds. Clearly $\langle\bar{t}, \bar{u}\rangle \leq C_{O_2(\bar{N})}(\bar{\kappa})$ and $C_{O_2(\bar{N})}(\bar{t}) = \bar{A}$. This implies (iv). Also $\langle u\rangle \times \mathscr{V} \lhd N$, if

$$\tau \in I(E - (\langle u\rangle \times \mathscr{V})) = I(t(\langle u\rangle \times \mathscr{V})),$$

then $\tau^E = \tau F$ and $\{t, tu, t\kappa_1, tu\kappa_1, t\kappa_2, tu\kappa_2, t\kappa_1\kappa_2, tu\kappa_1\kappa_2\}$ is a set of representatives for the E-conjugacy classes of involutions in $E - (\langle u\rangle \times \mathscr{V})$. Since $C_E(\kappa) = \langle t, u\rangle$ and κ is transitive on $\{\kappa_1F, \kappa_2F, \kappa_1\kappa_2F\}$, we conclude that $|O_2(\bar{N})/\bar{E}| = 4$, and we have (ii) and (iii).

LEMMA 6.13. *Let $\mathscr{U} < \mathscr{T}$ where $\mathscr{T}$ is a 2-group. Then:*

(i) *$X = \langle u, y, z\rangle$ is the unique normal element $\mathscr{E}_8(\mathscr{T})$ and $SCN_4(\mathscr{T}) = \emptyset$.*

(ii) $N_G(\mathscr{T}) \leq N_G(X)$.

Proof. Since $r_2(\mathcal{T}) \geq r_2(\mathcal{U}) \geq 1 + r_2(\mathcal{V}\langle x, t\rangle) = 5$, it follows from [16, Four Generator Theorem] that $\mathcal{T}$ contains a normal subgroup Y with $Y \in \mathcal{E}_8(\mathcal{T})$. Then $C_Y(t) \leq A$ or $C_Y(t) \leq B$, $|C_Y(t)| \geq 4$, and $t^G \cap Y = \emptyset$. Suppose that $C_Y(t) \nleq A$. Then there is an involution $\tau \in C_Y(t) \cap (x\langle u, z\rangle)$. Since $|[\kappa_1, \tau]| = 4$, this is impossible. Thus $C_Y(t) \leq A$ and hence $C_Y(t) \leq X = \langle u, y, z\rangle$. Suppose that $Y \neq X$. Then $C_Y(t) = A \cap Y$ is maximal in Y and hence $[Y, A] \leq A$. Thus $Y \leq N_{\mathcal{T}}(A) = \mathcal{U}$. As $t^G \cap Y = \emptyset$, there is an involution $\tau \in Y \cap \{x, xu\}$. Since $|[\kappa_1, \tau]| = 4$, this is impossible and hence $Y = X$. Next suppose that $Y \in \mathcal{E}_{16}(\mathcal{T})$ and $Y \lhd \mathcal{T}$. Then $X \leq Y$ and hence $Y \leq N_{\mathcal{T}}(A) = \mathcal{U}$. Then $Y \leq C_{\mathcal{U}}(X) = \langle u\rangle \times \mathcal{V}\langle t\rangle$ and Y is conjugate in N to A by Lemma 6.12(ii). Thus $t^G \cap Y \neq \emptyset$ and Y is conjugate in G to A. Since $|N_G(A)|_2 = |\mathcal{U}|$, this is impossible and the lemma holds.

We shall now conclude the proof of Lemma 6.4.

Clearly $C_N(X) \lhd N$ and $C_G(E)E \leq C_N(X) \leq O_{2',2}(N)$; thus $\bar{N}/C_{\bar{N}}(X) \hookrightarrow \mathrm{Aut}(X) \cong GL(3, 2)$. Let $\mathcal{U} < \mathcal{T} \in Syl_2(N)$, so that $|\mathcal{T}| = 2^9$. Then $\bar{\mathcal{T}} = O_2(\bar{N})\langle\bar{x}\rangle$ and $C_{\bar{N}}(X) = \bar{E}$ or $C_{\bar{N}}(X) = O_2(\bar{N})$.

Suppose that $C_{\bar{N}}(X) = \bar{E}$. Then $C_{\mathcal{T}}(X) = E$ char $\mathcal{T}$ since X char $\mathcal{T}$ by Lemma 6.13. Then $\mathcal{T} \in Syl_2(G)$ and $|G|_2 = 2^9$. Hence $C_{\bar{N}}(\bar{X}) = O_2(\bar{N})$, $C_N(X) = O_{2',2}(N)$, $\langle u\rangle \leq Z(N)$, and $\langle\bar{u}\rangle = Z(\bar{N})$. Set $\tilde{N} = \bar{N}/\langle\bar{u}\rangle$. Then $C_{O_2(\tilde{N})}(\tilde{t}) = \tilde{A}$ and $C_{O_2(\tilde{N})}(\tilde{\kappa}) = \langle\tilde{t}\rangle$. Applying Lemmas 2.6, 2.7, and 2.8 and setting $R = O(N)[O_{2',2}(N), \kappa]\langle u\rangle$, we conclude that $O_2(\tilde{N}) = \tilde{R}\langle\tilde{t}\rangle$ where $\tilde{\mathcal{V}} \leq \tilde{\mathcal{R}} \lhd \tilde{\mathcal{N}}$ and $\tilde{\mathcal{R}} = Z_8 \times Z_8$ or $\tilde{R}$ is isomorphic to a Sylow 2-subgroup of $L_3(4)$. Note that $C_{\bar{R}}(\bar{\kappa}) = \langle\bar{u}\rangle$ and that $\bar{R}/\langle\bar{u}\rangle \cong \tilde{R}$. Also set $\mathcal{Q} = \mathcal{T} \cap O_{2',2}(N)$, and $\mathcal{R} = R \cap \mathcal{T}$. Then $\mathcal{Q}$ char $\mathcal{T}$ since X char $\mathcal{T}$, $\mathcal{Q} = C_{\mathcal{T}}(X) = \mathcal{R}\langle t\rangle$ and $X = Z(\mathcal{Q})$.

Suppose that $\tilde{R}$ is of type $L_3(4)$. Then t acts freely on $\mathcal{R}/X$ since $\mathcal{R}/X \cong E_{16}$ and $\mathcal{Q}/X = \mathcal{R}\langle t\rangle/X$. Then $\mathcal{R}$ char $\mathcal{T}$ and $\langle I(\mathcal{Q} - \mathcal{R})\rangle = \langle t(\langle u\rangle \times \mathcal{V})\rangle = E$ char $\mathcal{T}$ and $|\mathcal{T}| = |G|_2 = 2^9$ which is impossible.

Suppose that $\tilde{R} \cong Z_8 \times Z_8$. If $\bar{R}$ is abelian, then $\bar{R} = \langle\bar{u}\rangle \times [\bar{R}, \bar{\kappa}]$ where $[\bar{R}, \bar{\kappa}] \cong \tilde{R}$ and $\bar{x}\bar{t}$ centralizes an element of $[\bar{R}, \bar{\kappa}]$ of order 8. Since this is impossible, we have $\bar{R}' = \langle\bar{u}\rangle$ and $\mho^1(\bar{R}) = \Phi(\bar{R}) = Z(\bar{R}) = \langle\bar{u}\rangle \times \bar{\mathcal{V}}$. Also $\bar{t}$ inverts $\bar{R}/\bar{X}$ and hence $J_0(\mathcal{Q}/X) = \mathcal{R}/X$ and $\mathcal{R}$ char $\mathcal{Q}$ char $\mathcal{T}$. If $\bar{t}$ does not invert $\bar{R}/\langle\bar{u}\rangle$, then $\Omega_1(\mathcal{Q}) = E$ and $T \in Syl_2(G)$ which is impossible. Thus $\bar{t}$ inverts $\bar{R}/\langle\bar{u}\rangle$.

Now $\mathcal{R}' = \langle u\rangle$, X, $\mathcal{R}$, and $\mathcal{Q}$ are all characteristic subgroups of $\mathcal{T}$, $\Phi(\mathcal{R}) = \langle u\rangle \times \mathcal{V}$, $\mathcal{Q} = \mathcal{R}\langle t\rangle$, and $E = \Phi(\mathcal{R})\langle t\rangle$. Also $N = O(N)N_N(\mathcal{Q})$, X char $\langle u\rangle \times \mathcal{V}$ char $\mathcal{R}$ char $\mathcal{Q}$ and $|\mathrm{Aut}(\mathcal{Q})|_{2'} = 3$. Since $I(E - \Phi(\mathcal{R})) = t^{\mathcal{Q}} \cup (tu)^{\mathcal{Q}}$, it follows that $N = O(N)\mathcal{Q}(N_N(\mathcal{Q}) \cap H)$ and hence there is a 3-element $\gamma \in N_N(\mathcal{Q}) \cap H$ such that $\gamma^x = \gamma^{-1}$, $\gamma^3 \in O(N)$, $C_{\mathcal{Q}}(\gamma) = \langle t, u\rangle$, and $[\mathcal{Q}, \gamma] = \mathcal{R}$. Setting $J = N_G(\mathcal{Q})$ and $\bar{J} = J/O(J)$, we have $\langle\mathcal{T}, \gamma\rangle \leq J \leq N_G(X) \cap C_G(u)$, $C_G(\mathcal{Q}) = O(J) \times X$, and $\bar{J} = O_2(\bar{J})\langle\bar{\gamma}, \bar{x}\rangle$. Since $C_{O_2(\bar{J})}(\bar{\gamma}) = \langle\bar{t}, \bar{u}\rangle$, it follows that $O_2(\bar{J})$ contains a maximal subgroup $\bar{P}$ containing $\bar{\mathcal{R}}$ such that $O_2(\bar{J}) = \bar{P}\langle\bar{t}\rangle$ and $\bar{P} \lhd \bar{J}$. Also since $Z_8 \times Z_8 \cong \bar{\mathcal{R}}/\langle\bar{u}\rangle \leq \bar{P}/\langle\bar{u}\rangle$, we conclude that $\bar{P}/\langle\bar{u}\rangle \cong Z_{2^n} \times Z_{2^n}$ for some integer $n \geq 4$. Thus $\bar{P}' = \langle\bar{u}\rangle$ and $\Phi(\bar{P})$ is abelian.

Since $\bar{\mathscr{R}} \leq \Phi(\bar{P})$, this is impossible. This contradiction completes the proof of Lemma 6.4.

7. The case $|\bar{M}/\bar{A}| = 48$

In view of our results to this point, it suffices to prove:

LEMMA 7.1. *If* $|\bar{M}/\bar{A}| = 48$, *then* $|O^2(G)|_2 \leq 2^{10}$.

Since the remainder of our paper is devoted to proving this lemma, we shall assume that $|\bar{M}/\bar{A}| = 48$ for the rest of the paper.

LEMMA 7.2. (i) $t^G \cap A = t^M \cap A = tX$.
(ii) $A \nsim B$ *in* G *and* $N_G(S) \leq M$.
(iii) $\bar{M}/\bar{A} \cong Z_2 \times \Sigma_4$.

Proof. Since $|(\bar{M}/\bar{A})\colon (C_{\bar{M}/\bar{A}}(\bar{t}))| = 8$, (i) is clear. Suppose that $A \sim B$ in G. Then $t^G \cap B = t^{N_G(B)}$ and hence $A \sim B$ in H, which is false. Thus (ii) holds. Clearly $\bar{M}/\bar{A} \hookrightarrow GL(4, 2)$, $C_{\bar{M}/\bar{A}}(\bar{t}) \cong \Sigma_3$, and $O_3(\bar{M}/\bar{A}) = 1$ since $C_{\bar{A}}(\bar{\rho}) = \langle \bar{t}, \bar{u}\rangle \ntriangleleft \bar{M}$. As $GL(4, 2)$ has no subgroup isomorphic to $GL(2, 3)$, we have (iii).

Let $S < S_1 \leq \mathscr{U} \in Syl_2(M)$, $W = O_{2', 2}(M)$, $\mathscr{W} = \mathscr{U} \cap W$, and $F = \langle y, z\rangle$. Clearly we have:

(7.1.) $\mathscr{W}$ is a maximal subgroup of $\mathscr{U}$, $\mathscr{W}/A \cong E_8$, $\mathscr{U} = \mathscr{W}\langle x\rangle$, $|\mathscr{U}| = 2^8$, $|\mathscr{W}| = 2^7$, $C_{\mathscr{W}}(t) = A$, $t^{\mathscr{W}} = t^G \cap A = tX$, and $\mathscr{W}/A$ acts regularly on tX.

Since $M = O(M)N_M(\mathscr{W})$ and $t^G \cap A = t^{\mathscr{W}}$ we also have:

(7.2) $M = O(M)\mathscr{W}(N_M(\mathscr{W}) \cap H)$ where $x \in N_M(\mathscr{W}) \cap H$. Also there is a 3-element $\kappa \in N_M(\mathscr{W}) \cap H$ such that $\kappa^x = \kappa^{-1}$, $\kappa^3 \in O(M)$, $[A, \kappa] = F$, and $C_A(\kappa) = \langle t, u\rangle$.

Set $\mathscr{Y} = C_{\mathscr{W}}(\kappa)$. Then we have:

(7.3) $|\mathscr{Y}| = 8$, $C_A(\kappa) = \langle t, u\rangle \lhd \mathscr{Y}$, $t^{\mathscr{Y}} = \{t, tu\}$, $\mathscr{Y} \cong D_8$, $\mathscr{Y}' = Z(\mathscr{Y}) = \langle u\rangle$, $\langle \mathscr{U}, \kappa\rangle \leq N_G(A\mathscr{Y})$, and $A\mathscr{Y} \leq N_{\mathscr{M}}(S) = S_1$.

Since $X = \langle A - (t^G \cap A)\rangle$, we have:

(7.4) $X \lhd M$.

Set $\mathscr{V} = [\mathscr{W}, \kappa]$. Then:

(7.5) $F \leq \mathscr{V}$, $\mathscr{W} = \mathscr{V}\mathscr{Y}$, $\kappa^3 \in C_G(\mathscr{W})$, $\mathscr{V}A/A \cong E_4$, and $\langle \mathscr{U}, \kappa\rangle \leq N_G(\mathscr{V})$.

Since $\mathscr{Y} = C_{\mathscr{W}}(\kappa)$ acts on $[A, \kappa] = F$, we have:

(7.6) $[F, \mathscr{Y}] = 1$.

Thus:

(7.7) $\mathscr{Y}$ has the following orbits on $t^G \cap A$: $\{t, tu\}$, $\{tz, tuz\}$, $\{ty, tuy\}$, and $\{tyz, tuyz\}$, $\mathscr{V}A/A$ acts regularly on these four orbits and $O_2(Z(M)) = \langle u \rangle$.

LEMMA 7.3. (i) $\mathscr{W}' = \Phi(\mathscr{W}) = \mho^1(\mathscr{W}) = X$.
(ii) $F = \langle y, z \rangle \leq \mathscr{V} \cap A \leq X$.
(iii) $\langle [\mathscr{Y}, x], [\mathscr{Y}, xt] \rangle = \langle u \rangle$ *and* x *or* xt *centralizes* $\mathscr{Y}$.

Proof. Clearly $X \leq \mathscr{W}' \leq A$. Since $\mathscr{W}/A \cong E_8$ and no element of tX is a square, it follows that $\mathscr{W}/X \cong E_{16}$ and (i) holds. Thus $t \notin \mathscr{V}$ and (ii) follows. Finally $x \in N_G(\mathscr{Y})$ and $[x, \langle t, u \rangle] = 1$. Thus x or xt centralizes $\mathscr{Y}$ and (iii) holds.

LEMMA 7.4. *$\mathscr{V}$ satisfies one of the following five conditions:*

(i) $\mathscr{V} \cong E_{16}$ *and* $C_{\mathscr{V}}(t) = F$.
(ii) $\mathscr{V} \cong Z_4 \times Z_4$, $F = \Omega_1(\mathscr{V})$, *and* t *inverts* $\mathscr{V}$.
(iii) *There is a* $\langle \kappa, x \rangle$*-invariant subgroup* $\mathscr{Q}$ *of* $\mathscr{V}$ *such that* $\mathscr{V} = F \times \mathscr{Q}$, $\mathscr{Q} \cong Q_8$, $\mathscr{Q}' = \langle u \rangle$, *and* $\mathscr{Q}\langle \kappa, x \rangle / \langle \kappa^3 \rangle \cong GL(2, 3)$.
(iv) $\mathscr{V}' = \langle u \rangle < X = Z(\mathscr{V}) = \Phi(\mathscr{V}) = \mho^1(\mathscr{V}) = \Omega_1(\mathscr{V})$, $\exp(\mathscr{V}) = 4$, $\mathscr{V}/\mathscr{V}' \cong Z_4 \times Z_4$, $\mathscr{V}/F \cong Q_8$, t *inverts* $\mathscr{V}/\mathscr{V}'$, *and* $(\mathscr{V}\langle \kappa, x \rangle)/(\langle \kappa^3 \rangle \times F) \cong GL(2, 3)$. *Also if* $\alpha \in \mathscr{V} - Z(\mathscr{V})$, *then* $|\alpha| = 4$, $C_{\mathscr{V}}(\alpha) = \langle \alpha, Z(\mathscr{V}) \rangle$, *and* $\alpha^2 \notin \langle u \rangle \cup F$.
(v) $\mathscr{V}' = Z(\mathscr{V}) = \langle u \rangle$, *$\mathscr{V}$ contains subgroups* Q_1 *and* Q_2 *with* $Q_1 \cong Q_2$ *quaternion of order* 8 *such that* $\mathscr{V} = Q_1 * Q_2$, $\mathscr{V}$ char $\mathscr{V}A = \mathscr{V}\langle t \rangle$, *and* $Q_1^t = Q_2$.

Proof. Suppose that $\langle u \rangle < Z(\mathscr{V}A)$. Then, as $\mathscr{V}A$ is $\langle \kappa \rangle$-invariant, $t \notin Z(\mathscr{V}A)$ and $C_{\mathscr{V}}(t) = A$, it follows that $Z(\mathscr{V}A) = X$. Thus $F \lhd \mathscr{V}A$, $C_{\mathscr{V}A/F}(\kappa) = \langle tF, uF \rangle$, $|\mathscr{V}A/F| = 2^4$ and $[\mathscr{V}A/F, \kappa] = \mathscr{V}/F$. Since no element of $tF \cup tuF$ is a square, we have $\mathscr{V}/F \cong E_4$ or $\mathscr{V}/F \cong Q_8$ with $(\mathscr{V}/F)' = \langle uF \rangle$. If $\mathscr{V}/F \cong E_4$, then $|\mathscr{V}| = 2^4$ and clearly (i) or (ii) hold. In the other case, $X = Z(\mathscr{V})$ and $C_{\mathscr{V}}(\kappa) = \langle u \rangle$. Then Lemma 2.10 yields (iii) or (iv).

Finally suppose that $Z(\mathscr{V}A) = \langle u \rangle$. Then $(\bar{\mathscr{V}}\bar{A}\langle \bar{\kappa}, \bar{x} \rangle)/\bar{A} \cong \Sigma_4$ and [7, VI, Lemma 2.6] implies that $\mathscr{V}A$ is of type $\mathscr{A}_8$. Also $C_{VA}(\kappa) = \langle t, u \rangle$, $|\mathscr{V}A| = 2^6$, and $\mathscr{V}A$ contains a characteristic maximal subgroup $\mathscr{Q}$ such that $\mathscr{Q}$ contains subgroups Q_1 and Q_2 with $Q_1 \cong Q_2 \cong Q_8$ and $\mathscr{Q} = Q_1 * Q_2$. Moreover if $v \in \mathscr{V}A - \mathscr{Q}$, then $Q_1^v = Q_2$. Clearly $\mathscr{V} = [\mathscr{V}A, \kappa] = \mathscr{Q}$ and (v) holds. This completes the proof of Lemma 7.4.

Our analysis of each of these five possibilities in Lemma 7.4 is presented in one of the remaining five sections of the paper.

8. The case of Lemma 7.4(i)

In this section, we shall prove:

LEMMA 8.1. *If $\mathscr{V}$ satisfies* (i) *of Lemma* 7.4, *then* $|O^2(G)|_2 \le 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V} \cong E_{16}$, $C_{\mathscr{V}}(t) = F$, and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

Clearly $\mathscr{Y} \cap \mathscr{V} = 1 = [u, \mathscr{V}]$, $C_{\mathscr{V}}(x) \ne C_{\mathscr{V}}(t)$, and $\mathscr{Y}$ normalizes $C_{\mathscr{V}}(x)$ since $[x, \mathscr{Y}] \le \langle u\rangle$. It follows that $C_{\mathscr{Y}}(\mathscr{V}) = \mathscr{P}$ is a maximal subgroup of $\mathscr{Y}$. Clearly $u \in \mathscr{P}$, $[\mathscr{P}, \mathscr{V}] = 1$, and $\langle \mathscr{U}, \kappa\rangle \le N_G(\mathscr{V}) \cap N_G(\mathscr{P})$. Setting $\mathscr{Q} = \mathscr{P} \times \mathscr{V}$, we have $\mathscr{W} = \mathscr{Q}\langle t\rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t\rangle$, $Z(\mathscr{U}) = \langle u, z\rangle$, and $[\mathscr{P}, t] = \langle u\rangle$. Also $|C_{\mathscr{V}}(xt)| = 4$ and xt does not centralize $\mathscr{Y}$ since $tx \sim t$ in G. Hence $[\mathscr{Y}, x] = 1$ and $[\mathscr{P}, xt] = \langle u\rangle$.

LEMMA 8.2. $\mathscr{P} \cong Z_4$.

Proof. Assume that $\mathscr{P} \cong E_4$. Then $\mathscr{Q} \cong E_{64}$, $|C_{\mathscr{Q}}(x)| = 2^4$ and $|C_{\mathscr{Q}}(t)| = |C_{\mathscr{Q}}(xt)| = 2^3$. Hence $\mathscr{Q} = J_e(\mathscr{U})$ char $\mathscr{U}$ and $\mathscr{U} \notin Syl_2(N_G(\mathscr{Q}))$. Let $\mathscr{U} \le \mathscr{T} \notin Syl_2(N_G(\mathscr{Q}))$. Then $\mathscr{U} < \mathscr{T}$, $t^G \cap \mathscr{Q} = \emptyset$, $I(t\mathscr{Q}) = t^{\mathscr{Q}}$ and hence $N_{\mathscr{T}}(\mathscr{Q}\langle t\rangle) = \mathscr{Q}S = \mathscr{U}$. This implies that $C_{\mathscr{T}}(\mathscr{Q}) = \mathscr{Q}$. Setting $\bar{\mathscr{T}} = \mathscr{T}/\mathscr{Q}$, we have $\bar{\mathscr{T}} \hookrightarrow \mathrm{Aut}\,(\mathscr{Q}) \cong GL(6, 2)$ and $C_{\bar{\mathscr{T}}}(\bar t) = \langle \bar t, \bar x\rangle$. Thus $\bar{\mathscr{T}}$ is dihedral or semidihedral and $|\bar{\mathscr{T}}| \le 2^4$ since the 2-exponent of $GL(6, 2)$ is 8. Hence $2^3 \le |\bar{\mathscr{T}}| \le 2^{10}$ and $\mathscr{T} \notin Syl_2(G)$. Also $Z(\bar{\mathscr{T}}) = \langle \bar x\rangle$ and $t\mathscr{Q} \sim xt\mathscr{Q}$ in $\mathscr{T}$ since $|C_{\mathscr{Q}}(\bar t)| = |C_{\mathscr{Q}}(\bar x\bar t)| \ne |C_{\mathscr{Q}}(\bar x)|$. As $\mathscr{T} \notin Syl_2(G)$, there is a 2-element $\omega \in N_G(\mathscr{T}) - \mathscr{T}$ such that $\omega^2 \in \mathscr{T}$ and $\mathscr{Q}_1 = \mathscr{Q}^{\omega} \lhd \mathscr{T}$ and $\mathscr{Q}_1 \ne \mathscr{Q}$. Hence $\bar x \in \bar{\mathscr{Q}}_1$ and $|\bar{\mathscr{Q}}_1| \le 4$. Since $|C_{\mathscr{Q}}(x)| = 2^4$, we have $\mathscr{Q} \cap \mathscr{Q}_1 = C_{\mathscr{Q}}(x)$ and $\langle C_{\mathscr{Q}}(x), x\rangle \le \mathscr{Q}_1$. Thus $|\bar{\mathscr{Q}}_1| = 4$ and $\bar{\mathscr{T}} \cong D_8$ since $\bar{\mathscr{Q}}_1 \lhd \bar{\mathscr{T}}$. Let $\alpha \in \mathscr{Q}_1 - \langle C_{\mathscr{Q}}(x), x\rangle$. Then

$$\mathscr{Q}_1 = \langle C_{\mathscr{Q}}(x), x, \alpha\rangle,\ \alpha\colon t\mathscr{Q} \leftrightarrow tx\mathscr{Q} \quad\text{and}\quad t\colon \alpha\mathscr{Q} \leftrightarrow \alpha x\mathscr{Q}.$$

On the other hand, $I(\mathscr{T}) = \mathscr{Q}^{\#} \cup I(x\mathscr{Q}) \cup I(\alpha\mathscr{Q}) \cup I(\alpha x\mathscr{Q}) \cup I(t\mathscr{Q}) \cup I(xt\mathscr{Q})$ and $\langle x, \mathscr{Q} \cap \mathscr{Q}_1, \alpha\rangle \le C_G(\mathscr{Q} \cap \mathscr{Q}_1)$ where $\mathscr{Q} \cap \mathscr{Q}_1 \cong E_{16}$. Thus $t^G \cap \mathscr{T} = I(t\mathscr{Q}) \cup I(xt\mathscr{Q}) = t^{\mathscr{T}}$ and hence $S < C_{\langle \mathscr{T}, \omega\rangle}(t)$, which is impossible. This concludes the proof of Lemma 8.2.

Let $\mathscr{P} = \langle \omega\rangle$ where $\omega^2 = u$. Clearly $\omega^t = \omega^{xt} = \omega^{-1}$, $\Omega_1(\mathscr{Q}) = \langle u\rangle \times \mathscr{V}$, and $\mho^1(\mathscr{Q}) = \langle u\rangle$. Set $E = \Omega_1(\mathscr{Q})$ and $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$ and $D = C_N(E)$. Thus $C \le D \trianglelefteq N \le C_G(u)$, D/C is a 2-group, $O(N) = O(C) = O(D)$, $\langle \mathscr{U}, \kappa\rangle \le N$, and $C_{\mathscr{Q}}(\kappa) = \mathscr{P}$. Setting $\bar N = N/O(N)$, we prove:

LEMMA 8.3. (i) $\bar C = C_{\bar C}(\bar\kappa) \times \bar{\mathscr{V}}$ *where* $\bar{\mathscr{V}} = [\bar C, \bar\kappa]$, $C_{\bar C}(\bar\kappa)$ *is a cyclic* 2-*group, and* $\bar{\mathscr{P}} = \Omega_2(C_{\bar C}(\bar\kappa))$.

(ii) $\bar S$ *normalizes* $C_{\bar C}(\kappa)$ *and* $C_{\bar C}(\bar\kappa)\langle \bar t\rangle$ *is dihedral or semidihedral.*

(iii) $\bar{\mathscr{Q}} \le \bar C \le \bar D \le O_2(\bar N)$.

Proof. Set $\bar Y = \bar C\langle \bar t\rangle$. Clearly $\bar{\mathscr{V}} \lhd \bar Y$, $\bar Y$ is $\langle \bar\kappa, \bar x\rangle$ invariant, $I(\bar t\bar{\mathscr{V}}) = \bar t^{\bar{\mathscr{V}}}$, $\bar S \in Syl_2(C_{\bar N}(\bar t))$, and $\bar A = \bar S \cap \bar Y \in Syl_2(C_{\bar Y}(\bar t))$. Thus $E_4 \cong \bar A\bar{\mathscr{V}}/\bar{\mathscr{V}} \in$

$Syl_2(C_{\bar{Y}/\bar{\mathscr{V}}}(\bar{t}\bar{\mathscr{V}}))$ and $\bar{Y}/\bar{\mathscr{V}}$ has dihedral or semidihedral Sylow 2-subgroups. Since $Z_4 \cong \bar{\mathscr{Q}}/\bar{\mathscr{V}} \trianglelefteq \bar{Y}$, [1, I, Proposition 1] and [8, Theorem 1] imply that $\bar{Y}/\bar{\mathscr{V}}$ has a normal 2-complement. As $\bar{\mathscr{V}} \leq Z(\bar{C})$, we conclude that $\bar{Y}$ is a 2-group and hence (iii) holds. Also $\bar{C}/\bar{\mathscr{V}}$ is a maximal subgroup of $\bar{Y}/\bar{\mathscr{V}}$ and $\bar{\mathscr{Q}}/\bar{\mathscr{V}} \leq Z(\bar{C})$. Thus $\bar{C}/\bar{\mathscr{V}}$ is cyclic, $\bar{C}$ is abelian, and (i) holds. Since $\bar{S} = \bar{A}\langle \bar{x}, \bar{t}\rangle$, (ii) also holds.

LEMMA 8.4. (i) $\bar{D} = C_{\bar{D}}(\bar{\kappa}) \times \bar{\mathscr{V}}$ *where* $\bar{\mathscr{V}} = [\bar{D}, \bar{\kappa}]$.

(ii) $\bar{S}$ *normalizes* $C_{\bar{D}}(\bar{\kappa})$ *and* $C_{\bar{D}}(\bar{\kappa})\langle \bar{t}\rangle$ *is dihedral or semidihedral;*

(iii) *either* $C_{\bar{D}}(\bar{\kappa}) = C_{\bar{C}}(\bar{\kappa})$ *(and* $C = D$*) or* $C_{\bar{D}}(\bar{\kappa})$ *is dihedral or generalized quaternion and* $C_{\bar{C}}(\bar{\kappa})$ *is the unique cyclic maximal subgroup of* $C_{\bar{D}}(\bar{\kappa})$ *when* $(C_{\bar{D}}\bar{\kappa})$ *is not isomorphic to* Q_8.

(iv) $t^G \cap D = \emptyset$.

(v) $\bar{\mathscr{Q}} = \bar{\mathscr{P}} \times \bar{\mathscr{V}}$ char $\bar{D}$ *if* $C_{\bar{D}}(\bar{\kappa})$ *is not isomorphic to* Q_8.

(vi) $C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\kappa}, \bar{x}\rangle$.

Proof. Set $\bar{Y} = \bar{D}\langle \bar{t}\rangle$. Clearly $\bar{\mathscr{V}} \lhd \bar{Y}$ and $\bar{A} \in Syl_2(C_{\bar{Y}}(\bar{t}))$. As in the proof of the preceding lemma, $\bar{Y}/\bar{\mathscr{V}}$ is dihedral or semidihedral and $\bar{\kappa}$ acts trivially on $\bar{Y}/\bar{\mathscr{V}}$. Thus (i)–(iii) and (v) hold. Since every involution of $\bar{D}$ centralizes $\bar{E}$, (iv) also holds. Also $C_N(t)$ normalizes $C_E(t) = X$ and hence $C_N(t) = (O(C_H(A)) \cap N)A\langle \kappa, t\rangle$. Since $O(C_H(A)) \cap N \leq C$ by [6, Theorem 5.3.4], (vi) also holds.

From the nature of the remainder of the proof of Lemma 8.1 and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$.

Set $\mathscr{R} = C_D(\kappa)$. Then $D = \mathscr{R} \times \mathscr{V}$, $\mathscr{R}\langle t\rangle$ is dihedral or semidihedral, $Z(\mathscr{R}\langle t\rangle) = \langle u\rangle$ and $\mathscr{R} = C_{\mathscr{R}\langle t\rangle}(E)$ is cyclic, dihedral, or generalized quaternion. Also $E = \langle u\rangle \times \mathscr{V} \leq Z(D)$ and $t^G \cap D = \emptyset$. Let γ be a generator of the cyclic maximal subgroup of $\mathscr{R}\langle t\rangle$. Then $\mathscr{P} \leq \langle \gamma\rangle$, $\gamma \in C$ if and only if $C = D$ and $\langle \gamma^2\rangle = C_C(\kappa)$ if and only if $C \neq D$. Also $I(tD) = I(t\mathscr{R}) \times F$ and hence $I(tD) = t^D$ if $\mathscr{R}\langle t\rangle$ is semidihedral and $I(tD) = t^D \cup (t\gamma)^D$ if $\mathscr{R}\langle t\rangle$ is dihedral. However, if $\mathscr{R}\langle t\rangle$ is dihedral and $C \neq D$, then $\mathscr{R}$ is dihedral and $t^G \cap tD = t^D$ since $t^G \cap D = \emptyset$.

Also $|C_{\mathscr{V}}(xt)| = 4$ and $|C_{\mathscr{R}\langle xt\rangle}(xt)| = \langle u, xt\rangle$. Hence $\mathscr{R}\langle xt\rangle$ is dihedral or semidihedral with $\mathscr{P} \lhd \mathscr{R}\langle xt\rangle$ and if $\tau \in I(xtD)$, then

$$C_{D\langle xt\rangle}(\tau) = \langle \tau, u\rangle \times C_{\mathscr{V}}(x) \cong E_{16}.$$

On the other hand, $\mathscr{P} \leq \mathscr{R} \cap C_N(x)$. Thus, by enumerating the possibilities for $\mathscr{R}\langle x\rangle$ and applying Lemmas 2.2 and 2.3, if necessary, if follows that if $\tau \in I(\mathscr{R}x)$, then either $|C_{\mathscr{R}\langle x\rangle}(\tau)| > 2^3$ or $C_{\mathscr{R}\langle x\rangle}$ is abelian of order 8. Hence $t^G \cap (D\langle x\rangle) = \emptyset$.

Setting $\hat{E} = E/\langle u\rangle$, we have $\hat{E} \cong E_{16}$, N acts on $\hat{E}$ and $D \leq C_N(\hat{E}) \trianglelefteq N$.

LEMMA 8.5. $C_N(\hat{E}) = D$ *and* $N/D \hookrightarrow$ Aut $(\hat{E}) \cong GL(4, 2)$.

Proof. Clearly $C_N(\hat{E}) \le O_2(N)$. Suppose that $D \ne C_N(\hat{E})$. Clearly

$$X = C_N(\hat{E}) \cap C_N(t) = C_D(t).$$

Let $D < Y \le C_D(\hat{E})$ be such that $Y/D = C_{C_N(\hat{E})/D}(t)$. Then Y is $\langle \kappa \rangle$-invariant and Y normalizes $D\langle t \rangle$. Thus $|Y/D| = 2$ and $[Y, \kappa] = \mathscr{V} \lhd Y$. Hence $Y \le C_N(E) = D$ which is false and we are done.

Choose $v_1 \in \mathscr{V}^{\#}$ such that $C_{\mathscr{V}}(x) = \langle z, v_1 \rangle$. Then $v_1^t = v_1 z$ and $C_{\mathscr{V}}(xt) = \langle z, v_1 y \rangle$. Since x normalizes $\mathscr{R}\langle t \rangle$, we have $\mathscr{R}\langle x, t \rangle = \langle \gamma, x, t \rangle$ and x normalizes $\langle \gamma \rangle$. Thus $(D\langle x, t \rangle)' = \langle \gamma^2 \rangle \times \langle z, y, v_1 \rangle$ and

$$C_{D\langle x, t \rangle}(\Omega_1((D\langle x, t \rangle)') = D \text{ char } D\langle x, t \rangle.$$

Thus E char $D\langle x, t \rangle$. Suppose that $\mathscr{Q} = \mathscr{P} \times \mathscr{V}$ is not characteristic in $D\langle x, t \rangle$. Then $\gamma^2 = u$ and $C = D = \mathscr{Q}$, which is a contradiction. Thus $\mathscr{Q}$ char $D\langle x, t \rangle$. Let $D\langle x, t \rangle \le \mathscr{T} \in Syl_2(N)$.

LEMMA 8.6. $\mathscr{T} \ne D\langle x\ t \rangle$.

Proof. Suppose that $\mathscr{T} = D\langle x, t \rangle \in Syl_2(N)$. Then $\mathscr{T} \in Syl_2(G)$, $|\mathscr{T}| \ge 2^{11}$, $Z(\mathscr{T}) = \langle u, z \rangle$, $|\gamma| \ge 2^5$, $\mathscr{T}' = \langle \gamma^2 \rangle \times \langle z, y, v_1 \rangle$, and $\Omega_1(\mho^1(\mathscr{T}')) = \langle u \rangle$. However, since $x \sim u$ in G, there is an element $g \in G$ such that $x^g = u$ and $C_{\mathscr{T}}(x)^g \le \mathscr{T}$. Since $\langle \gamma^2, x, t \rangle \le C_{\mathscr{T}}(x)$, we have $\langle \gamma^4 \rangle^g \le (C_{\mathscr{T}}(x)^g)'$. Thus $\langle u \rangle^g \le \Omega_1(\mho^1((C_{\mathscr{T}}(x)^g)')) \le \Omega_1(\mho^1(\mathscr{T}')) = \langle u \rangle$ and we have a contradiction.

LEMMA 8.7. (i) $C_{N/D}(tD) = \langle tD \rangle \times \langle \kappa D, xD \rangle = C_N(t)D/D$.
(ii) $O_2(N) = D$.
(iii) $\mathscr{T}/D \cong D_8$, $Z(\mathscr{T}/D) = \langle xD \rangle$ *and* $tD \sim xtD$ *in* $\mathscr{T}$.

Proof. Assume that $C_{N/D}(tD) \ne C_N(t)D/D$. Then $\mathscr{R}\langle t \rangle$ is dihedral, $C = D$, $\mathscr{R} = \langle \gamma \rangle$, and $|C_{N/D}(tD): ((C_N(t)D)/D)| = 2$. As $N/D \hookrightarrow GL(4, 2) \cong \mathscr{A}_8$, it follows from the structures of the centralizers of involutions in $\mathscr{A}_8$ that $C_{N/D}(tD)$ has Sylow 2-subgroups of type D_8. Hence $xD \sim xtD$ in N/D which is false since $t^G \cap D\langle x \rangle = \emptyset$. Thus (i) holds. Now (ii)–(iii) are immediate.

LEMMA 8.8. $\mathscr{T} \in Syl_2(G)$.

Proof. Assume that there is a 2-element $\tau \in N_G(\mathscr{T}) - \mathscr{T}$ such that $\tau^2 \in \mathscr{T}$. Let $\mathscr{Q}_1 = \mathscr{Q}^\tau$ and $E_1 = E^\tau = \Omega_1(\mathscr{Q}_1)$. Then $\mathscr{Q} \ne \mathscr{Q}_1 \lhd \mathscr{T}$ and $E_1 \lhd \mathscr{T}$. Since $t^G \cap E = \emptyset$, we have $C_{E_1}(t) = \langle u, y, z \rangle$ or $C_{E_1}(t) = \langle u, x, z \rangle$. Suppose that $C_{E_1}(t) = \langle u, y, z \rangle$. Then $[E_1, A] \le A$ and $E_1 \in C_{\mathscr{U}}(X) = \mathscr{Q}\langle t \rangle$ and $|E \cap E_1| \ge 2^4$. This implies that $E_1 = E$ and $\mathscr{Q}_1 \le C_{\mathscr{T}}(E) = D = \mathscr{R} \times \mathscr{V}$. Hence $\mathscr{Q}_1 = (\mathscr{Q}_1 \cap \mathscr{R}) \times \mathscr{V}$ where $\mathscr{Q}_1 \cap \mathscr{R} \cong Z_4$ and $\mathscr{Q}_1 \cap \mathscr{R}$ is t-invariant. This forces $\mathscr{Q}_1 \cap \mathscr{R} = \mathscr{P}$ and $\mathscr{Q}_1 = \mathscr{Q}$ which is false. Thus $C_{E_1}(t) = \langle u, x, z \rangle$. Since $\mathscr{T}/D \cong D_8$, we have $|E_1 \cap D| \ge 2^3$. We also have $\langle u, z \rangle \le E_1 \cap D \le C_D(x) \le \mathscr{R} \times \langle z, v_1 \rangle$, $[E_1, t] \le \langle u, x, z \rangle$, and $E_1 \le N_{\mathscr{T}}(B)$. Thus $E_1 \cap D \le N_R(B) \times \langle z, v_1 \rangle$. However $[N_{\mathscr{R}}(B), t] \le \mathscr{R} \cap B = \langle u \rangle$ and hence

$N_{\mathscr{R}}(B) = \mathscr{P}$ and $E_1 \cap D = \langle u, z, v_1 \rangle$. Thus $xD \in E_1D/D \cong E_4$ and $\mathscr{Q}_1 \cap (tD) = \emptyset = \mathscr{Q}_1 \cap (xtD)$.

Since $\tau: \mathscr{Q} \leftrightarrow \mathscr{Q}_1$, τ leaves $I(\mathscr{T} - \mathscr{Q}\mathscr{Q}_1) = I(tD \cup xtD)$ invariant. Since $tD \sim xtD$ in $\mathscr{T}$, it follows that $C = D$, $\mathscr{R} = \langle y \rangle$, and $y^t = y^{-1}$. Moreover we may assume that $t^\tau = ty$. Also

$$\langle u, z, v_1 \rangle \leq \mathscr{Q}_1 \cap \mathscr{Q} \leq C_{\mathscr{Q}}(x) \leq \mathscr{R} \times \langle z, v_1 \rangle$$

and $\mathscr{Q}_1$ is abelian. Thus $\mathscr{Q}_1 D = E_1 D$ and hence $|\mathscr{Q}_1 \cap \mathscr{Q}| = 2^4$ and $\mathscr{Q}_1 \cap \mathscr{Q} = \mathscr{P} \times \langle z, v_1 \rangle$. Since τ normalizes $\mathscr{Q}_1 \cap \mathscr{Q}$, $\tau \in C_G(u)$ and hence $\langle t, u \rangle^\tau = \langle ty, u \rangle$. But $\langle y, z, \kappa \rangle \leq C_G(\langle t, u \rangle) \cap C_G(ty, u)$ since $[\mathscr{R}, \kappa] = 1$. Now (4.12) implies that

$$\tau: \mathscr{T} \cap O^2(C_G(t, u)) = \langle y, z \rangle \to \mathscr{T} \cap O^2(C_G(t^y, u)) = \langle u, z \rangle.$$

But $y \in \mathscr{Q}$ and $y \notin \mathscr{Q}_1 = \mathscr{Q}^\tau$, which is a contradiction and the lemma follows.

We shall now conclude the proof of Lemma 8.1. Let Y be the maximal subgroup of $\mathscr{T}$ such that $D < Y$ and $Y/D \cong Z_4$. Clearly $\Omega_1(Y/D) = \langle xD \rangle$ and hence $t^G \cap Y = \emptyset$. Hence $t \notin O^2(G)$ by [17, Lemma 5.38]. Since $|O^2(G)|_2 \geq 2^{11}$, we have $|\mathscr{T}| \geq 2^{12}$ and $|y| \geq 2^5$. Since x normalizes $\langle y \rangle$ and centralizes $\mathscr{P} = \Omega_2(\langle y \rangle)$, we have $\langle y^2, x, t \rangle \leq C_{\mathscr{T}}(x)$. Since t inverts y^2, we have $\langle y^4 \rangle \leq C_{\mathscr{T}}(x)' \leq \mathscr{T}' \leq D\langle x \rangle$. Hence $\langle y^8 \rangle \leq D = \mathscr{R} \times \mathscr{V}$ and $y^{16} \in \mho^1(\mathscr{R}) = \langle y^2 \rangle$. Since $|y| \geq 2^5$, and $u \in Z(\mathscr{T})$, we obtain a contradiction as in Lemma 8.6. Thus Lemma 8.1 is established.

9. The case of Lemma 7.4(ii)

In this section, we shall prove:

LEMMA 9.1. *If $\mathscr{V}$ satisfies* (ii) *of Lemma* 7.4, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V} \cong Z_4 \times Z_4$, $F = \Omega_1(\mathscr{V})$, t inverts $\mathscr{V}$, and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

Clearly $\langle \kappa, x \rangle$ normalizes $\mathscr{V}$ and $[\kappa^3, \mathscr{V}] = 1 = \mathscr{Y} \cap \mathscr{V}$. Let $v_1 \in \mathscr{V}$ be such that $v_1^2 = y$ and set $v_2 = v_1^x$ and $v = v_1 v_2$. Then $v_2^2 = yz$, $v^2 = z$, and $C_{\mathscr{V}}(x) = \langle v \rangle$. Then $\mathscr{P} = C_{\mathscr{Y}}(\mathscr{V})$ is a maximal subgroup of $\mathscr{Y}$. Clearly $u \in \mathscr{P}$, $[\mathscr{P}, \mathscr{V}] = 1$, and $\langle \mathscr{U}, \kappa \rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P})$. Setting $\mathscr{Q} = \mathscr{P} \times \mathscr{V}$, we have $\mathscr{W} = \mathscr{Q}\langle t \rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t \rangle$, $Z(\mathscr{U}) = \langle u, z \rangle$, and $[\mathscr{P}, t] = \langle u \rangle$. Also $C_{\mathscr{V}}(xt) = \langle vy \rangle$ and $\langle u \rangle \times C_{\mathscr{V}}(xt) \leq C_{\mathscr{Q}}(xt)$. Thus

$$(\langle u \rangle \times \langle vy \rangle)(\langle x, t \rangle) \leq C_{\mathscr{U}}(xt)$$

and hence $(\langle u \rangle \times \langle vy \rangle)(\langle x, t \rangle) = C_{\mathscr{U}}(xt) \in Syl_2(C_G(xt))$, $[x, \mathscr{P}] = 1$, and $[xt, \mathscr{P}] = \langle u \rangle$.

Let $\tau \in I(x\mathscr{Q})$. Then $C_{\mathscr{Q}}(\tau) = \mathscr{P} \times \langle v \rangle$, $|C_{\mathscr{Q}\langle x \rangle}(\tau)| = 2^5$ and $C_{\mathscr{Q}\langle x \rangle}(\tau)$ is abelian. Thus $t^G \cap (\mathscr{Q}\langle x \rangle) = \emptyset$.

Note also that $\mathscr{U}' = \langle u \rangle \times \langle v \rangle \times \langle y \rangle$, $C_{\mathscr{U}}(\mathscr{U}') = \mathscr{Q}$, $\Omega_1(\mathscr{U}') = X$, $C_{\mathscr{U}}(X) = \mathscr{Q}\langle t \rangle$, and $\mho^1(\mathscr{U}') = \langle z \rangle$.

LEMMA 9.2. $\mathscr{P} \cong Z_4$.

Proof. Assume that $\mathscr{P} = \langle u, \omega \rangle$ where $\omega^2 = 1$. Let $\mathscr{U}$ be of index 2 in the 2-subgroup $\mathscr{T}$ of G. Clearly $\mathscr{Q} \lhd \mathscr{T}$ and $\mathscr{Q}\langle t \rangle \lhd \mathscr{T}$. Since $t^G \cap (x\mathscr{Q}) = \emptyset$, we have $\mathscr{Q}\langle xt \rangle \lhd \mathscr{T}$. Then, since $I(xt\mathscr{Q}) = (xt)^{\mathscr{Q}}$, we have $|C_{\mathscr{T}}(xt)| = 2^6$ which is a contradiction since $t \sim xt$ in G. This completes the proof of the lemma.

Let $\mathscr{P} = \langle \omega \rangle$ where $\omega^2 = u$. Thus t inverts $\mathscr{Q} = \mathscr{P} \times \mathscr{V}$, $\Omega_1(\mathscr{Q}) = X$, $\omega^{xt} = \omega u$, $I(t\mathscr{Q}) = t\mathscr{Q}$, $I(x\mathscr{Q}) = x(\langle u \rangle \times \langle vy \rangle)$, and $I(xt\mathscr{Q}) = xt(\mathscr{P} \times \langle v \rangle)$.

LEMMA 9.3. *Let $\mathscr{U} \leq \mathscr{R}$ where $\mathscr{R}$ is a 2-subgroup. Then $X \lhd \mathscr{R}$ and X is the unique normal element of $\mathscr{E}_8(\mathscr{R})$.*

Proof. Since X char $\mathscr{U}$, it suffices, by induction on $|\mathscr{R}|$, to assume that $X \lhd \mathscr{R}$ and to show that X is unique. Thus let $X \neq Y \lhd \mathscr{R}$ where $Y \in \mathscr{E}_8(\mathscr{R})$. Since $2^7 < |\mathscr{R}|$, $t^G \cap Y = \emptyset$. Thus $|C_Y(t)| \geq 4$ and $C_Y(t) \leq \langle u, x, z \rangle$ or $C_Y(t) \leq \langle u, y, z \rangle$. Note that if $\tau_1 \in I(t\mathscr{Q})$, $\tau_2 \in I(x\mathscr{Q})$, and $\tau_3 \in I(xt\mathscr{Q})$, then $\langle \tau_1, X \rangle \leq \langle \tau^2 \rangle$, $\langle \tau_2, vy \rangle \leq \langle \tau_2^{\mathscr{Q}} \rangle$, and $\langle \tau_3, v \rangle \leq \langle \tau_3^{\mathscr{Q}} \rangle$. Hence $C_Y(t) \leq X$ and $Y \leq N_{\mathscr{R}}(A) = \mathscr{U}$. Then $Y \leq \mathscr{Q}$, $Y = X$ and we are done.

Clearly $\mathscr{Q} = J_0(\mathscr{W})$ char $\mathscr{W} = \mathscr{Q}\langle t \rangle$, $\langle \mathscr{U}, \kappa \rangle \leq N_G(\mathscr{W}) \leq N_G(\mathscr{Q})$, $C_G(\mathscr{W}) = O(C_G(\mathscr{W})) \times X$, and $\kappa^3 \in O(C_G(\mathscr{W}))$.

LEMMA 9.4. (i) *$\mathscr{Q} \leq N_G(\mathscr{W}) \cap C_G(\mathscr{Q}) \leq N_G(\mathscr{W})$ and $O(N_G(\mathscr{W}))$ is a normal 2-complement of $N_G(\mathscr{W}) \cap C_G(\mathscr{Q})$.*

(ii) *Either $\mathscr{Q} \in Syl_2(N_G(\mathscr{W}) \cap C_G(\mathscr{Q}))$ and $t^{N_G(\mathscr{W}) \cap C_G(\mathscr{Q})} = tX = t^{\mathscr{Q}}$ or $\mathscr{Q}$ is a maximal subgroup of a Sylow 2-subgroup of $N_G(\mathscr{W}) \cap C_G(\mathscr{Q})$ and $t^{N_G(\mathscr{W}) \cap C_G(\mathscr{Q})} = t(\mathscr{P} \times F)$.*

Proof. Let $N = N_G(\mathscr{W})$, $\bar{N} = N/O(N)$, and $J = C_N(\mathscr{Q})$. Clearly $\mathscr{Q} \leq Z(J)$, $J \lhd N \leq N_G(\mathscr{Q})$, $O(N) = O(J) = O(C_G(\mathscr{W}))$, and $\bar{J} = C_{\bar{N}}(\bar{\mathscr{Q}})$. Let $\tau \in J$. Then $t^\tau \in \tau\mathscr{Q}$ and hence $t^{\tau^2} \in tX = t^{\mathscr{Q}}$. Thus $\tau^2 \in C_N(\mathscr{W})\mathscr{Q} = O(N) \times \mathscr{Q}$. Hence $\bar{J}/\bar{\mathscr{Q}}$ is an elementary abelian 2-group and (i) holds. Then

$$\bar{J}/\bar{\mathscr{Q}} = ((C_{\bar{J}}(\bar{\kappa})\bar{\mathscr{Q}})/\bar{\mathscr{Q}}) \times (([\bar{J}, \bar{\kappa}]\bar{\mathscr{Q}})/\bar{\mathscr{Q}}).$$

Note that $C_{\bar{J}}(\bar{t}) = C_{\bar{N}}(\bar{\mathscr{W}}) = \bar{X}$ and $\bar{t}^{\bar{\mathscr{Q}}} = \bar{t}\bar{X}$.

Let $\bar{\mathscr{Z}} = [\bar{J}, \bar{\kappa}]\bar{\mathscr{Q}}$. Then $\bar{\mathscr{Z}}$ is $(\langle \bar{t} \rangle \times \langle \bar{\kappa}, \bar{x} \rangle)$-invariant. Suppose that $|\bar{\mathscr{Z}}/\bar{\mathscr{Q}}| \geq 8$. Then there is an element $\bar{\tau} \in \bar{\mathscr{Z}} - \bar{\mathscr{Q}}$ such that $\bar{t}^{\bar{\tau}} = \bar{t}\bar{\omega}$. Hence $[\bar{\tau}, \bar{\kappa}] \in C_{\bar{J}}(\bar{t}) = \bar{X} \leq \bar{\mathscr{Q}}$. Since $C_{\bar{\mathscr{Z}}/\bar{\mathscr{Q}}}(\bar{\kappa}) = 1$, this is impossible.

Suppose that $|\bar{\mathscr{Z}}/\bar{\mathscr{Q}}| = 4$. Then $|\bar{\mathscr{Z}}| = 2^8$, $|\bar{\mathscr{Z}}/\bar{\mathscr{V}}| = 2^4$, $(\langle \bar{t} \rangle \times \langle \bar{\kappa}, \bar{x} \rangle)$ normalizes $\bar{\mathscr{Z}}/\bar{\mathscr{V}}$ and $\bar{\mathscr{Q}}/\bar{\mathscr{V}} = C_{\bar{\mathscr{Z}}/\bar{\mathscr{V}}}(\bar{\kappa}) \leq Z(\bar{\mathscr{Z}}/\bar{\mathscr{V}})$. Set $\bar{\mathscr{X}} = [\bar{\mathscr{Z}}, \bar{\kappa}]$. Since $\bar{\mathscr{V}} \leq \bar{\mathscr{X}}$, we conclude that either $|\bar{\mathscr{X}}| = 2^6$ or $\bar{X}/\bar{\mathscr{V}} \cong Q_8$. Assume that $|\bar{\mathscr{X}}| = 2^6$. Then since $\bar{\mathscr{V}} \leq Z(\bar{\mathscr{X}})$ and $C_{\bar{\mathscr{X}}}(\bar{\kappa}) = 1$, [7, IV, Lemma 2.5] implies that either $\bar{\mathscr{X}} \cong Z_8 \times Z_8$ or $\bar{\mathscr{X}} = \bar{\mathscr{V}} \times \bar{\mathscr{V}}_1$ where $\bar{\mathscr{V}}_1$ is a $\langle \bar{\kappa} \rangle$-invariant 4-group. In either case, we have $|C_{\bar{J}\langle \bar{x}, \bar{t} \rangle}(\bar{x}\bar{t})| \geq 2^6$ and we have a contradiction. Thus $\bar{\mathscr{X}}/\bar{\mathscr{V}} \cong Q_8$ and $C_{\bar{\mathscr{X}}}(\bar{\kappa}) = \langle \bar{u} \rangle = \bar{\mathscr{X}}'$. If $\bar{\mathscr{X}}/\langle \bar{u} \rangle \cong Z_8 \times Z_8$, then there is an element $\bar{\lambda} \in \bar{\mathscr{X}} - \bar{\mathscr{Q}}$ such that $\bar{\lambda}^{\bar{x}\bar{t}} = \bar{\lambda}\bar{u}$. Hence $(\bar{\lambda}\bar{\omega})^{\bar{x}\bar{t}} = \bar{\lambda}\bar{\omega}$ and $|C_{J\langle\bar{x},\bar{t}\rangle}(\bar{x}\bar{t})| \geq 2^6$ which is impossible. Hence $\bar{\mathscr{X}}/\langle\bar{u}\rangle \cong Z_4 \times Z_4 \times E_4$ and there is an element $\bar{\lambda} \in \bar{\mathscr{X}} - \bar{\mathscr{Q}}$ such that $\bar{\lambda}^{\bar{x}\bar{t}} = \bar{\lambda}\bar{u}$ and we obtain a contradiction in the same way. Hence $\bar{\mathscr{T}} = \bar{\mathscr{Q}}$. Finally, let $\bar{\tau} \in C_J(\bar{\kappa})$. Then $\bar{t}^{\tau} \in \bar{t}C_{\bar{\mathscr{Q}}}(\bar{\kappa}) = \bar{t}\bar{\mathscr{P}}$ and (ii) holds.

For the remainder of this section, let $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar{N} = N/O(N)$. Clearly $\mathscr{Q}(\langle t\rangle \times \langle\kappa, x\rangle) \leq N$. Let $Y = C\langle t\rangle$ and let $\mathscr{U} = \mathscr{Q}\langle x, t\rangle \leq \mathscr{T} \in Syl_2(N)$. Clearly $Y \lhd N$ as t inverts $\mathscr{Q}$ and $\kappa^3 \in C$. Also let $O(N) \leq \mathscr{R} \leq C$ be such that $\bar{\mathscr{R}} = C_{\bar{C}}(\bar{\kappa})$.

LEMMA 9.5. (i) $\bar{C} = \bar{\mathscr{R}} \times \bar{\mathscr{V}}$, $\bar{\mathscr{V}} = [\bar{C}, \bar{\kappa}]$, $\bar{\mathscr{R}}$ *is a cyclic* 2-*group, and* $\bar{\mathscr{P}} = \Omega_2(\bar{\mathscr{R}})$.

(ii) $\bar{S}$ *normalizes* $\bar{\mathscr{R}}$ *and* $\bar{\mathscr{R}}\langle\bar{t}\rangle$ *is dihedral or semidihedral.*

(iii) $C_{\bar{N}}(\bar{t}) = \bar{A}\langle\bar{\kappa}, \bar{x}\rangle$.

Proof. Clearly $\mathscr{S} = \mathscr{T} \cap Y \in Syl_2(Y)$, $\mathscr{S} = (\mathscr{S} \cap C)\langle t\rangle$, $C_{\mathscr{S}}(t) = A$, and $\mathscr{V} \lhd Y$. Set $\tilde{Y} = Y/\mathscr{V}$. Then $C_{\tilde{\mathscr{S}}}(\tilde{t}) = \langle\tilde{u}, \tilde{t}\rangle$ by Lemma 9.4(ii). Hence $\tilde{\mathscr{S}}$ is dihedral or semidihedral. Also $\mathscr{S} \cap C \in Syl_2(C)$, $\mathscr{Q} = \mathscr{P} \times \mathscr{V} \leq Z(C)$ and hence $\tilde{\mathscr{P}} \leq Z(\tilde{C})$. Thus $(S \cap C)^{\sim}$ is cyclic and (i)–(ii) hold. Finally

$$C_N(t) = (O(C_N(t)) \cap O(N_H(A))A\langle\kappa, x\rangle$$

as $C_N(t) \leq N_H(A) = O(N_H(A))A\langle\kappa, x\rangle$. Since $O(C_N(t)) \cap O(N_H(A)) \leq C_G(\mathscr{Q})$ by [6, Theorem 5.3.4], (iii) also holds.

From the nature of the remainder of the proof of Lemma 9.1 and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$. Then $C = \mathscr{R} \times \mathscr{V}$ and $C\langle x, t\rangle \leq \mathscr{T} \in Syl_2(N)$.

LEMMA 9.6. (i) $\mathscr{Q} = \Omega_2(C)$ char $C\langle x, t\rangle$.

(ii) $\mathscr{T} \neq C\langle x, t\rangle$.

Proof. Clearly $t^G \cap C = \emptyset = t^G \cap Cx$. Also X char $C\langle x, t\rangle$ by Lemma 9.3. Thus $C\langle t\rangle = C_{C\langle x,t\rangle}(X)$ char $C\langle x, t\rangle$. Since $C = J_0(C\langle t\rangle)$ and $\Omega_2(C) = \mathscr{Q}$, (i) holds.

Suppose that $\mathscr{T} = C\langle x, t\rangle$. Then $\mathscr{T} \in Syl_2(G)$ and $t^G \cap (C\langle x\rangle) = \emptyset$ implies that $|\mathscr{T}| \geq 2^{12}$ by [17, Lemma 5.38]. Hence $|\mathscr{R}| \geq 2^6$. Letting $\mathscr{R} = \langle\gamma\rangle$, we have $\gamma^2 \in C_{\mathscr{T}}(x)$. Since $Z(\mathscr{T}) = \langle u, z\rangle$, there is an element $g \in G$ such that $C_{\mathscr{T}}(x)^g \leq \mathscr{T}$ and $x^g = u$. Thus $(\alpha^2)^g \in \mathscr{T}$ and hence $1 \neq (\alpha^{16}) \in \mho^3(\mathscr{T}) \leq \mathscr{R}$. Then $u^g = u$, a contradiction; hence (ii) holds.

LEMMA 9.7. (i) $|N|_{2'} = 3$ *and* $Y = C\langle t\rangle \leq O_2(N)$.

(ii) $N = O_2(N)\langle\kappa, x\rangle$ *and* $\mathscr{T} = O_2(N)\langle x\rangle$.

Proof. Clearly $Y = C\langle t\rangle \leq O_2(N)$ and $N/C \hookrightarrow \mathrm{Aut}(\mathscr{Q})$. Thus $|N|_{2'} = 3$ if $C = \mathscr{R} \times \mathscr{V} \neq \mathscr{Q} = \mathscr{P} \times \mathscr{V}$. On the other hand, suppose that $C = \mathscr{Q}$. Then every element of tC is an involution and $t^{\mathscr{T}} \neq t^{\mathscr{Q}} = tX$. Suppose that

$|N|_{2'} \neq 3$. Then $|N|_{2'} = 3 \cdot 7$ and $N/O_2(N) \hookrightarrow GL(3, 2)$. Since $\langle \kappa, x \rangle \cong \Sigma_3$, it follows that $N/O_2(N) \cong GL(3, 2)$. Also $C_N(t) = A\langle \kappa, x \rangle$, $|C_N(t)| = 3 \cdot 2^5$, N permutes the eight Y-conjugacy classes in tC, and $t^Y = tX$. Since $N_N(A) = \mathscr{Q}(\langle t \rangle \times \langle \kappa, x \rangle)$ is the stabilizer of t^Y, we have a contradiction and (i) holds. Clearly (ii) is immediate and we are done.

LEMMA 9.8. *$\mathscr{R}\langle t \rangle$ is dihedral.*

Proof. Assume that $\mathscr{R}\langle t \rangle$ is semidihedral. Then $tC = t(\mathscr{R} \times \mathscr{V})$ decomposes into four conjugacy classes under $Y = C\langle t \rangle$. Thus $N/Y \cong \Sigma_4$. Note that tC is not a square in N/C by Lemma 2.5. Thus $O_2(N)/C \cong E_8$ where $C_{O_2(N)/C}(\kappa) = \langle tC \rangle$. Since X char $\mathscr{T}$ by Lemma 9.3, we have $C_{\mathscr{T}}(X) = C\langle t \rangle$ or $O_2(N)$. Suppose that $C_{\mathscr{T}}(X) = C\langle t \rangle$. Then C char $\mathscr{T}$, $\mathscr{Q} = \Omega_2(C)$ char $\mathscr{T}$, and $\mathscr{T} \in Syl_2(G)$. Suppose that $C_{\mathscr{T}}(X) = O_2(N)$. Then $\mathscr{Q} \leq O_2(N)' \leq C$ and hence $\mathscr{Q} = \Omega_2(C_{\mathscr{T}}(X)')$ char $\mathscr{T}$. Thus $\mathscr{Q}$ char $\mathscr{T}$ and $\mathscr{T} \in Syl_2(G)$ in either case. Moreover $N_G(\mathscr{T}) = N_N(\mathscr{T}) = \mathscr{T}$.

Clearly $\langle u \rangle \leq Z(\mathscr{T}) \leq \langle u, z \rangle$ and all involutions of $X = \langle u, y, z \rangle$ are G-conjugate into $\langle u, z \rangle$. Suppose that $\omega \in u^G \cap (\langle u, z \rangle - \langle u \rangle)$. Then $Z(\mathscr{T}) = \langle u \rangle$, $O_2(N)$ does not centralize X and $\langle u, z \rangle$ is the unique normal 4-subgroup of $\mathscr{T}$ lying in X. Let $g \in G$ be such that $\omega^g = u$ and $C_{\mathscr{T}}(\omega)^g \leq \mathscr{T}$. Since $C\langle x, t \rangle \leq C_{\mathscr{T}}(\omega)$ max $\mathscr{T}$, $C_{\mathscr{T}}(\omega)^g \leq C_{\mathscr{T}}(u^g)$, and $u^g \notin Z(\mathscr{T})$, we have $C_{\mathscr{T}}(\omega)^g = C_{\mathscr{T}}(u^g)$ max $\mathscr{T}$ and $\langle u, u^g \rangle \leq Z(C_{\mathscr{T}}(\omega)^g)$.

Suppose $\mathscr{S}$ is an arbitrary maximal subgroup of $\mathscr{T}$ such that $|\Omega_1(Z(\mathscr{S}))| \geq 4$. Since $Z(\mathscr{T}) = \langle u \rangle$, we have $\langle u \rangle \leq \Omega_1(Z(\mathscr{S}))$ and $|\Omega_1(Z(\mathscr{S}))| = 4$. Since $\Omega_1(Z(\mathscr{S})) \trianglelefteq \mathscr{T}$, we have $\Omega_1(Z(\mathscr{S})) \leq N_{\mathscr{T}}(A) = \mathscr{Q}\langle x, t \rangle$. Hence $\Omega_1(Z(\mathscr{S})) \leq X$, $\Omega_1(Z(\mathscr{S})) = \langle u, z \rangle$, and $\mathscr{S} = C_{\mathscr{T}}(\omega)$.

This implies that $g \in N_G(C_{\mathscr{T}}(\omega))$. But $C_{\mathscr{T}}(\omega)/C \cong E_8$ and $X \leq (C\langle x, t \rangle)' \leq C_{\mathscr{T}}(\omega)'$. Thus $X = \Omega_1(C_{\mathscr{T}}(\omega)')$ and g normalizes $C_{\mathscr{T}}(\omega) \cap C_G(X) = C\langle t \rangle$. Hence $g \in N_G(\mathscr{Q}) = N$ which is impossible since $\langle u \rangle \leq Z(N)$. We have shown that $u^G \cap X = \{u\}$.

Let $\mathscr{R} = \langle \gamma \rangle$ and let $g \in G$ be such that $x^g = u$ and $C_{\mathscr{T}}(x)^g \leq \mathscr{T}$. Noting that $\gamma^2 \in C_{\mathscr{T}}(x)$, $|\gamma^2| \geq 4$, and $u^g \notin C$, we conclude that $|\gamma| = 2^3$ and $|\mathscr{T}| = 2^{11}$. Setting $w = u^g$, we also have $w = u^g \in \mho^1(\mathscr{T}/C) = (\mathscr{T}/C)'$ since $\mathscr{T}/C \cong Z_2 \times D_8$. Letting $\mathscr{S} = C[O_2(N), \kappa]\langle x \rangle$, we have $\mathscr{S}$ max $\mathscr{T}$ and $\mathscr{S}/C \cong D_8$. But $t^G \cap (C\langle x \rangle) = \emptyset$ and $t \in G'$ since $|\mathscr{T}| = 2^{11}$. Then [13, Corollary 2.1.2] implies that $t^G \cap Cw \neq \emptyset$. Let $s \in t^G \cap Cw$. Then $|C_{\mathscr{T}}(s)| \leq 2^5$ and hence $|s^{\mathscr{T}}| \geq 2^6$. But $w = u^g \in I(Cw)$, $C\langle w \rangle \lhd \mathscr{T}$, and $|Cw| = 2^7$. Then $s^{\mathscr{T}} \subseteq Cw$ and hence $D = \{c \in C \mid c^s = c^{-1}\}$ is a subgroup of C with $|D| > 2^6$. Thus s inverts C and hence $ts \in C_G(\mathscr{Q}) = C$ which is false and the proof of the lemma is complete.

In view of Lemma 9.8, we conclude that t inverts $C = \mathscr{R} \times \mathscr{V}$. Set $\mathscr{X} = O_2(N)$.

LEMMA 9.9. (i) $\mathscr{R}\langle t\rangle = C_{\mathscr{X}}(\kappa)$.
(ii) $\mathscr{X}/C \cong E_8$.
(iii) $\mathscr{Q}$ char $\mathscr{T}$ *and* $\mathscr{T} \in Syl_2(G)$.

Proof. Suppose that $\mathscr{R}\langle t\rangle \neq C_{\mathscr{X}}(\kappa)$. Then, as $C_{\mathscr{X}}(\kappa, t) = \langle t, u\rangle$, we conclude that $C_{\mathscr{X}}(\kappa)$ is dihedral or semidihedral. Also, since $C_Y(\kappa) = \mathscr{R}\langle t\rangle \lhd C_{\mathscr{X}}(\kappa)$, we conclude that $\mathscr{R}\langle t\rangle$ is a maximal subgroup of $C_{\mathscr{X}}(\kappa)$. Clearly $\mathscr{R} = \Phi(C_{\mathscr{X}}(\kappa))$, x normalizes $C_{\mathscr{X}}(\kappa)$, and x leaves invariant the three maximal subgroups of $C_{\mathscr{X}}(\kappa)$. Let α generate the maximal cyclic subgroup of $C_{\mathscr{X}}(\kappa)$. Then $C_{\mathscr{X}}(\kappa) = \langle\alpha, t\rangle$, $\mathscr{R} = \mho^1(\langle\alpha\rangle)$, $[\alpha, x] \in \mathscr{R}$, α normalizes $[C, \kappa] = \mathscr{V}$, and hence α normalizes $C_{\mathscr{V}}(x) = \langle v\rangle$. Thus, as α does not centralize $\mathscr{Q}$, α inverts $\mathscr{V}$. Thus $t\alpha \in C_N(\mathscr{V})$ and $t \sim t\alpha$ in G. Set $\mathscr{S} = \langle C, x, t, \alpha\rangle$, so that $\mathscr{S}/C \cong E_8$ and $C\langle x, t\rangle$ is a maximal subgroup of $\mathscr{S}$. Since xt inverts $\mathscr{P} = \Omega_2(\mathscr{R})$, we conclude that $\mathscr{R}\langle xt\rangle$ is dihedral or semidihedral. If $\mathscr{R}\langle xt\rangle$ is semidihedral, then $I(xtC) = (xt)^C$. But $C\langle xt\rangle \lhd \mathscr{S}$ and $C\langle t\rangle = Y \lhd \mathscr{S}$. Hence $C\langle x\rangle \lhd \mathscr{S}$, $C\langle xt\rangle \lhd \mathscr{S}$, and $|C_{\mathscr{S}}(xt)| = 2|C_{C\langle x,t\rangle}(xt)| = 2^6$ which is impossible. Thus $\mathscr{R}\langle xt\rangle$ is dihedral and $[\mathscr{R}, x] = 1$. Moreover $\mathscr{S}/C$ contains seven involutions with $C_C(tC) = X$, $C_C(\alpha C) = \mathscr{R} \times F$, $C_C(xC) = \mathscr{R} \times \langle v\rangle$, $C_C(t\alpha C) = \langle u\rangle \times \mathscr{V}$, $C_C(xtC) = \langle u\rangle \times \langle vy\rangle$, $C_C(x\alpha C) = \mathscr{R} \times \langle vy\rangle$, and $C_C(x\alpha tC) = \langle u\rangle \times \langle v\rangle$. Suppose that $\mathscr{S} \neq \mathscr{T}$ and let $\gamma \in N_{\mathscr{X}}(\mathscr{S}) - \mathscr{S}$. Then γ normalizes C, Ct, $C\alpha$, and $C\alpha t$. Note that $\langle\alpha, x\rangle$ is abelian or modular so that $I(C\alpha x) = \emptyset$. Hence $(Cx)^\gamma \neq C\alpha x$ and γ acts trivially on $\mathscr{S}/C$. Since $I(xtC) = (xt)^{\langle C,\alpha\rangle}$ and $|C_{\mathscr{U}}(xt)| = 2^5$, this is a contradiction and we conclude that $\mathscr{S} = \mathscr{T}$. Then $\mathscr{T}' = \mathscr{R} \times \langle v\rangle \times \langle y\rangle$, $C_{\mathscr{T}}(\mathscr{T}') = C$, $\mathscr{Q}$ char $\mathscr{T}$, and $\mathscr{T} \in Syl_2(G)$. Hence $|\alpha| \geq 2^5$ since $|\mathscr{T}| \geq 2^{11}$. Clearly $u \in Z(\mathscr{T})$ and there is an element $g \in G$ such that $x^g = u$ and $C_{\mathscr{T}}(x)^g \leq \mathscr{T}$. Then, since $\alpha^2 \in C_{\mathscr{T}}(x)$, we have $(\alpha^4)^g \in C = \mathscr{R} \times \mathscr{V}$. Hence $1 \neq (\alpha^{16}) \in \mathscr{R}$ and $u^g = u$, which is a contradiction. We conclude that (i) holds.

Now $C_{\mathscr{X}}(t) = A$, $\mathscr{X}/Y$ acts regularly on the orbits of Y on $t^{\mathscr{X}}$, κ acts nontrivially and fixed point freely on $\mathscr{X}/Y$, and tC decomposes into eight Y-conjugacy classes. Thus (ii) holds by Lemma 2.5. Also X char $\mathscr{T}$ and $C\langle t\rangle \leq C_{\mathscr{T}}(X) \leq \mathscr{X}$. If $C_{\mathscr{T}}(X) = C\langle t\rangle$, then $\mathscr{Q}$ char $\mathscr{T}$, $\mathscr{T} \in Syl_2(G)$ and (iii) holds. Suppose that (iii) does not hold. Then $C_{\mathscr{T}}(X) = \mathscr{X}$. Hence $[\mathscr{X}, t] \leq \mathscr{X}'$ and $\mathscr{X}' = \mho^1(\mathscr{R}) \times \mathscr{V}$; this forces $\mathscr{X}' = \langle u\rangle \times \mathscr{V}$ and $C_{\mathscr{X}}(\mathscr{X}') > C = \mathscr{R} \times \mathscr{V}$ where $C_{\mathscr{X}}(\mathscr{X}')/C \cong E_4$. It follows that $C_{\mathscr{X}}(\mathscr{X}') = C[\mathscr{X}, \kappa]$ is a maximal subgroup of $\mathscr{X}$. Setting $\mathscr{J} = [\mathscr{X}, \kappa]\mathscr{X}'$, we have $\mathscr{J} \leq C_{\mathscr{X}}(\mathscr{X}')$, $\mathscr{X}' \leq Z(\mathscr{J})$, $\mathscr{J} \lhd N$, and $|\mathscr{J}| = 2^7$. Also $\langle t\rangle \times \langle\kappa, x\rangle$ acts on $\tilde{\mathscr{J}} = \mathscr{J}/\langle u\rangle$, $C_{\tilde{\mathscr{J}}}(\kappa) = 1$, and $\tilde{\mathscr{V}} \cong \tilde{\mathscr{X}}' \leq Z(\tilde{\mathscr{J}})$. Since $|\tilde{\mathscr{J}}| = 2^6$, we conclude that $\tilde{\mathscr{J}}$ is abelian by [7, IV, Lemma 2.5]. Since $\langle\kappa, xt\rangle$ acts on $[\mathscr{X}, \kappa]$ and $\mathscr{V} \leq Z([\mathscr{X}, \kappa])$, it follows that $[\mathscr{X}, \kappa]$ is not abelian. Thus $\mathscr{J} = [\mathscr{X}, \kappa]$ and $\mathscr{J}' = \langle u\rangle$. Since $N_{\mathscr{X}}(A) = C\langle t\rangle$, we have $N_{\mathscr{J}}(A) = \mathscr{X}'$. Hence $\Omega_1(\tilde{\mathscr{J}}) = \tilde{X}$ as $C_{\tilde{\mathscr{J}}}(t) = \tilde{X}$. Thus $\tilde{\mathscr{J}} \cong Z_8 \times Z_8$ and there is an element $\beta \in \mathscr{J} - \mathscr{X}'$ with $\beta^{xt} = \beta u$. Then $\beta\omega \in C_{\mathscr{X}}(xt) \leq Y = \mathscr{Q}\langle t\rangle$ which is impossible and we are done.

We can now conclude the proof of Lemma 9.1. Since $|\mathscr{T}| \geq 2^{11}$, we have $|\mathscr{R}| \geq 2^3$ and $u \in Z(\mathscr{T})$. Also $N_G(\mathscr{T}) = N_N(\mathscr{T}) = \mathscr{T}$. Then the final portion of the proof of Lemma 9.8 applies to force a contradiction and the proof of Lemma 9.1 is complete.

10. The case of Lemma 7.4(iii)

In this section, we shall prove:

LEMMA 10.1. *If* $\mathscr{V}$ *satisfies* (iii) *of Lemma* 7.4, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus throughout this section, we assume that $\mathscr{V}$ contains a $\langle \kappa, x\rangle$-invariant subgroup $\mathscr{Q}$ such that $\mathscr{V} = \mathscr{Q} \times F$, $\mathscr{Q} \cong Q_8$, $\mathscr{Q}' = \langle u\rangle$, and $(\mathscr{Q}\langle \kappa, x\rangle)/\langle \kappa^3\rangle \cong GL(2, 3)$. We shall also assume that $|O^2(G)|_2 \geq 2^{11}$ and we shall proceed to obtain a contradiction.

Clearly $\mathscr{Y} \cap \mathscr{V} = \langle u\rangle$ and $\mathscr{Y}$ acts on $C^*_{\mathscr{V}}(x) = \{v \in \mathscr{V} \mid v^x = v \text{ or } v^x = v^{-1}\} = \langle z\rangle \times \langle q\rangle$ where $q \in \mathscr{Q}$ is such that $q^x = q^{-1} = qu$. Also $t^{\mathscr{Q}} = tF$ or $t^{\mathscr{Q}} = tuF$ and hence $q^t \in \{qz, quz\}$. Since $\mathscr{Q} = \langle q, q^\kappa, q^{\kappa^2}\rangle$, it follows that no element of $\mathscr{Y}$ can invert q. Then $C_{\mathscr{Y}}(\mathscr{V}) = \mathscr{P}$ is a maximal subgroup of $\mathscr{Y}$ and $\langle \mathscr{U}, \kappa\rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P})$. Also $\langle \mathscr{V}, \mathscr{P}, \kappa, x\rangle \leq N_G(\mathscr{Q})$ and $I(t\mathscr{V}) = t^{\mathscr{V}} \cup (tu)^{\mathscr{V}}$. Set $E = P\mathscr{V} = P * \mathscr{V}$. Then $\mathscr{W} = E\langle t\rangle$, $E \lhd \mathscr{U} = E\langle x, t\rangle$, $Z(\mathscr{U}) = \langle u, z\rangle$, $Z(E) = \mathscr{P} \times F$, and $[\mathscr{P}, t] = \langle u\rangle$.

LEMMA 10.2. $\mathscr{P} \cong Z_4$.

Proof. Assume that $\mathscr{P} = \langle u, \omega\rangle$ where $\omega^2 = 1$. Then $E = \langle \omega, y, z\rangle \times \mathscr{Q}$, $I(tE) = tX = t^E$, and $\langle I(tE)\rangle = A$. Note that $\langle x, q\rangle \cong D_8$ and $\langle x, q\rangle = \langle x, xq\rangle \leq \langle I(xE)\rangle$ and hence $\langle I(xE)\rangle$ is not abelian. A similar argument implies that $\langle I(xtE)\rangle$ is not abelian. But $\mathscr{U}' = F \times \langle q\rangle$ and hence $C_{\mathscr{U}}(\Omega_1(\mathscr{U}') = C_{\mathscr{U}}(X) = E\langle t\rangle$ char $\mathscr{U}$. Since $t^G \cap E = \emptyset$ and $t^E = tX = I(tE)$, we conclude that $A \lhd N_G(\mathscr{U})$. Hence $|\mathscr{U}| = |G|_2 = 2^8$, which is false and the proof is complete.

Let $\mathscr{P} = \langle \omega\rangle$ where $\omega^2 = \mathscr{U}$ and $\omega^t = \omega^{-1} = \omega u$. Thus $I(tE) = tX \cup (t\omega)X$, $E = F \times \mathscr{P} * \mathscr{Q}$, $|E| = 2^6$, and if $j \in I(E) - Z(E)$, then $C_E(j)$ is abelian of order 2^5. Thus $t^G \cap E = \emptyset$. Also $C_E(t) = C_E(t\omega) = X = \Omega_1(Z(E))$ and $Z(E) = F \times \mathscr{P}$ and $X = \Omega_1(Z(E))$. If x inverts ω, then $I(xE) = x(\langle z\rangle \times \langle \omega, q\rangle)$ and if x centralizes ω, then $I(xE) = x(\langle z\rangle \times \langle q\rangle)$. A similar result holds for xt. Also $\mathscr{U}' = F \times \langle q\rangle$, $\Omega_1(\mathscr{U}') = X$, and $C_{\mathscr{U}}(X) = E\langle t\rangle$. Thus $\Omega_1(E) = E$ char $\mathscr{U}$ since $C_E(t) = C_E(t\omega) = X$.

Set $N = N_G(E)$, $\bar{N} = N/O(N)$, and $C = C_G(E)$. Thus $\langle \mathscr{U}, \kappa\rangle \leq N$, $\kappa^3 \in C$, and $Z(E) = F \times \mathscr{P} \leq Z(C)$. Also let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$ and set $Y = C\langle t\rangle$. Note that $E' = \langle u\rangle \leq Z(N)$ and $X = \Omega_1(Z(E)) \lhd N$. Let $O(N) \leq \mathscr{R} \leq C$ be such that $\bar{\mathscr{R}} = C_{\bar{C}}(\bar{\kappa})$.

LEMMA 10.3. (i) $\bar{C} = \bar{\mathscr{R}} \times \bar{F}$ *where* $[\bar{C}, \bar{\kappa}] = \bar{F}$, $\bar{\mathscr{R}} = C_{\bar{C}}(\bar{\kappa})$ *is a cyclic 2-group*, $\bar{\mathscr{P}} = \Omega_2(\bar{\mathscr{R}})$, *and* $\bar{X} = \Omega_1(\bar{C})$.

(ii) $\bar{S}$ *normalizes* $\bar{\mathscr{R}}$, $C_{\bar{\mathscr{R}}}(\bar{t}) = \langle \bar{u} \rangle$, *and* $\bar{\mathscr{R}}\langle \bar{t} \rangle$ *is dihedral or semidihedral.*

(iii) $C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\kappa}, \bar{x} \rangle$.

Proof. Let $\mathscr{T}_0 = \mathscr{T} \cap C$. Then $\mathscr{T}_0 \lhd \mathscr{T}$, $\mathscr{T}_0 \in Syl_2(C)$, and $\mathscr{T}_0\langle t \rangle \in Syl_2(Y)$. Clearly $F \lhd Y$, $C_E(t) = X$, and $\bar{F} \lhd \bar{Y}$. Set $\tilde{Y} = \bar{Y}/\bar{F}$. Then

$$C_{\tilde{\mathscr{T}}_0}(\tilde{t}) \leq (N_{\mathscr{T}_0}(A))^{\sim} \leq (\mathscr{U} \cap \mathscr{T}_0)^{\sim} = (\mathscr{U} \cap C)^{\sim} = \tilde{\mathscr{P}}$$

and hence $C_{\tilde{\mathscr{T}}_0}(\tilde{t}) = \langle \tilde{u} \rangle$. Thus $C_{\tilde{\mathscr{T}}_0\langle \tilde{t} \rangle}(\tilde{t}) = \langle \tilde{t}, \tilde{u} \rangle$ and $\tilde{\mathscr{T}}_0\langle \tilde{t} \rangle$ is dihedral or semidihedral. Since $\tilde{\mathscr{P}} \leq Z(\tilde{Y})$, it follows that $\mathscr{T}_0$ is cyclic. As $F \leq Z(C)$, (i) holds. Since $\bar{S} = \bar{X}\langle \bar{x}, \bar{t} \rangle$, (ii) holds. Also $C_N(t) = H \cap N_N(A) = (N \cap O(N_H(A)))A\langle \kappa, x \rangle$. Since $N \cap O(N_H(A))$ centralizes $C_E(t) = X$, we have $C_N(t) = (O(N) \cap H)A\langle \kappa, x \rangle$ by [6, Lemma 5.3.4] and we are done.

From the nature of the remainder of the proof of Lemma 10.1 and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$. Then $C = \mathscr{R} \times F$, $EC = F \times (\mathscr{R} * \mathscr{Q}) \lhd N$; $EC \leq C_N(X) \lhd N$, and $t^G \cap EC = \emptyset$. Since $X \lhd N$, we also have:

(10.1) $EC\langle x \rangle \not\sim EC\langle t \rangle \not\sim EC\langle xt \rangle$ in N.

LEMMA 10.4. $\mathscr{R}\langle t \rangle$ *is dihedral.*

Proof. Assume that $\mathscr{R}\langle t \rangle$ is semidihedral. Then $I(tEC) = t^{EC}$, $|\mathscr{R}| \geq 8$, $\mathscr{T} = EC\langle x, t \rangle$ by (10.1), and $Z(\mathscr{T}) = \langle u, z \rangle$. Also $\mathscr{T}' = F \times \mho^1(\mathscr{R}) * \langle q \rangle$ where $\mathscr{P} \leq \mho^1(\mathscr{R})$, $C_{\mathscr{T}}(\mathscr{T}') = F \times \mathscr{R} * \langle q \rangle$, $\mho^1(C_{\mathscr{T}}(\mathscr{T}')) = \mho^1(\mathscr{R})$, and $C_{\mathscr{T}}(\mho^1(\mathscr{R})) = EC\langle x \rangle$ or $EC\langle xt \rangle$. Let $|\mathscr{R}| = 2^a$ for some integer $a \geq 3$ and let $\mathscr{S} = C_{\mathscr{T}}(\mho^1(\mathscr{R})) = C_{\mathscr{T}}(\mho^1(C_{\mathscr{T}}(\mathscr{T}')))$. Note that if $j \in I(EC)$, then $|C_{\mathscr{S}}(j)| \geq 2^{a+3}$ and if $j \in I(\mathscr{S} - EC)$, then $|C_{\mathscr{S}}(j)| = 2^{a+2}$. Thus $\Omega_1(EC) = E \lhd N_G(\mathscr{T})$ and hence $\mathscr{T} \in Syl_2(G)$. Then $a \geq 5$ and $|\mho^1(\mathscr{R})| \geq 2^4$. But then x centralizes $\mho^1(\mathscr{R})$ and there is an element $g \in G$ such that $x^g = u$ and $C_{\mathscr{T}}(x)^g \leq \mathscr{T}$. Then

$$(\mho^2(\mathscr{R}))^g \leq \mathscr{T}' = F \times \mho^1(\mathscr{R}) * \langle q \rangle$$

and $u^g = u$ which is a contradiction and the lemma follows.

Let $\mathscr{R} = \langle \gamma \rangle$. Then $I(tEC) = t^{EC} \cup (t\gamma)^{EC}$. Since $S \leq EC\langle x, t \rangle$ and $t^G \cap EC = \emptyset$, we conclude that $|N_N(EC\langle t \rangle): EC(\langle t \rangle \times \langle \kappa, x \rangle)| \leq 2$.

LEMMA 10.5. $N_N(EC\langle t \rangle) \neq EC(\langle t \rangle \times \langle \kappa, x \rangle)$.

Proof. Assume that $N_N(EC\langle t \rangle) = EC(\langle t \rangle \times \langle \kappa, x \rangle)$. Then $\mathscr{T} = EC\langle x, t \rangle$ by (10.1). Suppose that $\mathscr{R} = \mathscr{P}$. Then $\mathscr{T}' = F \times \langle q \rangle$, $\Omega_1(\mathscr{T}') = X$, $C_{\mathscr{T}}(X) = E\langle t \rangle$, and $\Omega_1(E) = E \lhd N_G(\mathscr{T})$ since $t^G \cap E = \emptyset$. Then $|\mathscr{T}| = |G|_2 = 2^8$ which is false. Hence $|\mathscr{R}| \geq 2^3$, $\mathscr{T}' = F \times \mho^1(\mathscr{R}) * \langle q \rangle$, $C_{\mathscr{T}}(\mathscr{T}') = F \times$

$\mathscr{R} * \langle q \rangle$, $\mho^1(C_{\mathscr{T}}(\mathscr{T}')) = \mho^1(\mathscr{R})$, and $C_{\mathscr{T}}(\mho^1(\mathscr{R})) = EC\langle x \rangle$ or $EC\langle xt \rangle$. But then we obtain a contradiction as in the proof of Lemma 10.4 and we are done.

Set $J = N_N(EC\langle t \rangle)$. Then $J = O_2(J)\langle \kappa, x \rangle$ and $[O_2(J), \kappa] = [EC, \kappa] = F \times \mathscr{Q} = \mathscr{V} \lhd J$. Hence $F \lhd J$, $[O_2(J), X] = 1$ and $O_2(J) = \mathscr{V} C_{O_2(J)}(\kappa)$. Thus $\mathscr{R}\langle t \rangle = C_{EC\langle t \rangle}(\kappa)$ is a maximal subgroup of $C_{O_2(J)}(\kappa)$. Also $C_{O_2(J)}(t, \kappa) = \langle t, u \rangle$ and $C_{O_2(J)}(\kappa)$ is $\langle x \rangle$-invariant and dihedral or semidihedral. Then Lemmas 2.3 and 2.4 imply that $\langle C_{O_2(J)}(\kappa), x \rangle' \leq \mathscr{R} \leq C_N(\mathscr{V})$. Hence $C_{O_2(J)}(\kappa, \mathscr{V}) = \mathscr{R}_1$ is a maximal subgroup of $C_{O_2(J)}(\kappa)$, $C_{O_2(J)}(\kappa) = \mathscr{R}_1\langle t \rangle$, $\mathscr{R}_1$ is dihedral or generalized quaternion, and $\mathscr{R}$ is the cyclic maximal subgroup of $\mathscr{R}_1$. Also $\mathscr{R}_1 EC = F \times \mathscr{R}_1 * \mathscr{Q}$ and $\mathscr{S} = (F * \mathscr{R}_1 * \mathscr{Q})\langle x, t \rangle \in Syl_2(J)$. Then $\mathscr{S}' = F \times \mathscr{R} * \langle q \rangle$, $\Omega_1(\mathscr{S}') = F \times \langle u, \omega q \rangle$, $Z(\mathscr{S}) = \langle u, z \rangle$, $y^{\mathscr{S}} = y\langle z \rangle$, $(\omega q)^{\langle \mathscr{Q}, t \rangle} = \omega q\langle u, z \rangle$, and $(\omega q y)^{\langle \mathscr{Q}, t \rangle} = \omega q y\langle u, z \rangle$. Hence $X = F \times \langle u \rangle$ char $\mathscr{S}$ and $C_{\mathscr{S}}(X) = (F \times (\mathscr{R}_1 * \mathscr{Q}))\langle t \rangle$ char $\mathscr{S}$. But $t^G \cap (F \times (\mathscr{R}_1 * \mathscr{Q})) = \emptyset$ and $I(t(F \times \mathscr{R}_1 * \mathscr{Q})) = t^{(F \times R_1 * \mathscr{Q})}$ as is easily seen. Thus $\mathscr{S} = \mathscr{T} \in Syl_2(G)$, $|\mathscr{R}| \geq 2^4$, and $[x, \mathscr{R}] = 1$ since $\langle x \rangle$ normalizes the cyclic maximal subgroup of $R_1\langle t \rangle$. Letting $g \in G$ be such that $x^g = u$ and $C_{\mathscr{T}}(x)^g \leq \mathscr{T}$, we conclude that $(\mho^1(\mathscr{R}))^g \leq \mathscr{T}' = F \times \mathscr{R} * \langle q \rangle$ and hence $u^g = u$. This contradiction completes the proof of Lemma 10.1.

11. The case of Lemma 7.4(iv)

In this section we shall prove:

LEMMA 11.1. *If $\mathscr{V}$ satisfies* (iv) *of Lemma* 7.4, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V}$ satisfies (iv) of Lemma 7.4 and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

Thus, if $q \in \mathscr{V} - X$, then $q^t = q^{-1}u = q^3 u$ since t inverts $\mathscr{V}/\langle u \rangle$ and

$$\mathscr{V} = qX \cup q^{\kappa}X \cup q^{\kappa^2}X$$

cannot be inverted by t. Since $[\mathscr{Y}, x] \leq \langle u \rangle = Z(\mathscr{Y}) \leq C_{\mathscr{W}}(\mathscr{V})$, Lemma 2.11(iii) implies that $\mathscr{P} = C_{\mathscr{Y}}(\mathscr{V})$ is a maximal subgroup of $\mathscr{Y}$. Clearly $u \in \mathscr{P}$, $\langle \mathscr{U}, \kappa \rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P})$ and $I(t\mathscr{V}) = t^{\mathscr{V}} \cup (tu)^{\mathscr{V}}$. Set $\mathscr{Q} = \mathscr{P}\mathscr{V} = \mathscr{P} * \mathscr{V}$. Then $\mathscr{W} = \mathscr{Q}\langle t \rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t \rangle$, $Z(\mathscr{U}) = \langle u, z \rangle$, and $[\mathscr{P}, t] = \langle u \rangle$. Note also that $C_{\mathscr{V}/\langle u \rangle}(xt) \cong Z_4$ and hence there is an element $v \in \mathscr{V} - X$ such that $v^2 = uz$ and $v^{xt} \in v\langle u \rangle$. If $v^{xt} = v$, then

$$\mathscr{S} = \langle u, z, v, xt, t \rangle = C_{\mathscr{U}}(xt) \in Syl_2(C_G(xt)).$$

But then $\mathscr{S}' = \mho^1(\mathscr{S}) \cong Z_2$, $v^2 = uz$, and $[v, t] = v^{-2}u = z$, which is impossible. Thus $v^{xt} = vu$.

LEMMA 11.2. $\mathscr{P} \cong E_4$.

Proof. Assume that $\mathscr{P} = \langle \omega \rangle$ where $\omega^2 = u$. Clearly $\omega^t = \omega^{-1} = \omega u$ and $\Omega_1(\mathscr{Q}) = X$. Suppose that xt inverts $\mathscr{P}$. Then $C_{\mathscr{U}}(xt) = \langle u, z, \omega v, x, t \rangle \in$

$Syl_2(C_G(xt))$. Since $(\omega v)^2 = z$ and $[\omega v, t] = uz$, this is impossible. Thus $[\mathscr{P}, xt] = 1$ and $\omega^x = \omega^{-1}$.

Set $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar{N} = N/O(N)$. Thus $Z(\mathscr{Q}) = \mathscr{P} \times F \leq Z(C)$, $\langle \mathscr{U}, \kappa \rangle \leq N$, $\kappa^3 \in C$, $Z(\mathscr{Q}) = \mathscr{P} \times F = \mathscr{Q} \cap C \lhd N$, and $X \leq Z(C\langle t \rangle)$. Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$. Then $\mathscr{T}_1 = C \cap \mathscr{T} \lhd \mathscr{T}$, $\mathscr{P} \times F \leq \mathscr{T}_1 \in Syl_2(C)$, and $\mathscr{T}_1\langle t \rangle \in Syl_2(C\langle t \rangle)$.

Suppose that $\tau_1 \in \mathscr{T}_1$ is such that $t^{\tau_1} \in tF$. Then

$$\tau_1 \in N_{\mathscr{T}_1}(A) = \mathscr{U} \cap \mathscr{T}_1 = \mathscr{P} \times F$$

and hence $\tau_1 \in C_{\mathscr{T}_1}(t) = X$. Then since $F \leq Z(\mathscr{T}_1\langle t \rangle)$, we conclude that $C_{\mathscr{T}_1\langle t \rangle/F}(tF) = A/F$ and that $\mathscr{T}_1\langle t \rangle/F$ is dihedral or semidihedral. Since $\bar{F} \leq Z(\bar{C})$ and $\mathscr{P} \cong (\bar{\mathscr{P}} \times \bar{F})/\bar{F} \leq Z(\bar{C}/\bar{F})$, it follows that $\bar{C}/\bar{F}$ is a cyclic 2-group and $\bar{C} = C_{\bar{C}}(\bar{\kappa}) \times \bar{F}$ where $C_{\bar{C}}(\bar{\kappa})$ is cyclic $\bar{\mathscr{P}} = \Omega_2(C_C(\bar{\kappa}))$ and $\bar{F} = [\bar{C}, \bar{\kappa}]$.

From the nature of the remainder of the proof of Lemma 11.2 and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$.

Set $\mathscr{R} = C_C(\kappa)$. Then $C = \mathscr{R} \times F$, $\mathscr{R}$ is a cyclic 2-group, $\mathscr{P} = \Omega_2(\mathscr{R})$, $[C, \kappa] = F$, $\kappa^3 = 1$, and $C\langle x, t \rangle \leq \mathscr{T}$. Since $[xt, \mathscr{P}] = 1$, it follows that $[xt, \mho^1(\mathscr{R})] = 1$ and hence $|\mathscr{R}| \leq 2^3$. Suppose that $|\mathscr{R}| = 2^3$. Then $\mathscr{R}\langle xt \rangle$ is a modular 2-group and letting $\mathscr{R} = \langle \gamma \rangle$, we conclude that $\gamma v \in C_{\mathscr{T}}(xt)$. Since $|\gamma v| = 2^3$, this is impossible. Thus $\mathscr{R} = \mathscr{P}$.

Since $N/C \hookrightarrow \mathrm{Aut}(\mathscr{Q})$ where $\mathscr{Q} = \mathscr{P} * \mathscr{V}$, it is easy to see that $|N|_{2'} = 3$. Thus $N = O_2(N)\langle \kappa, x \rangle$ and $\mathscr{T} = O_2(N)\langle x \rangle$.

Let $\mathscr{X} = O_2(N)$. Then $C\langle t \rangle \leq \mathscr{X}$, $\mathscr{V} \leq \mathscr{X}$, $C\mathscr{V} = \mathscr{P} * \mathscr{V} = \mathscr{Q} \leq \mathscr{X}$ and $C_{\mathscr{X}}(t) = A$. Clearly $t^G \cap \mathscr{Q} = \emptyset$.

Suppose that $\mathscr{X} = \mathscr{Q}\langle t \rangle$. Then $\mathscr{T} = \mathscr{U}$, $\Omega_1(\mathscr{T}') = X$ char $\mathscr{T}$, $C_{\mathscr{T}}(X) = \mathscr{Q}\langle t \rangle$ char $\mathscr{T}$, $\Omega_1(\mathscr{Q}\langle t \rangle) = \langle \omega, t \rangle \times F$ char $\mathscr{T}$, $\mathscr{P} \times F$ char $\mathscr{T}$, $\mathscr{Q} = C_{\mathscr{T}}(\mathscr{P} \times F)$ char $\mathscr{T}$, and $\mathscr{T} \in Syl_2(G)$ which is false. Thus $\mathscr{W} = \mathscr{Q}\langle t \rangle \neq \mathscr{X}$.

Let $\mathscr{X}_1 = N_{\mathscr{X}}(\mathscr{W})$. Since $I(t\mathscr{Q}) = t^{\mathscr{Q}} \cup (t\omega)^{\mathscr{Q}}$, it follows that $|\mathscr{X}_1/\mathscr{W}| = 2$ and $\mathscr{X}_1$ is $\langle \kappa, x \rangle$ invariant. Thus $[\mathscr{X}, \kappa] = \mathscr{V} \lhd \mathscr{X}_1$ and $\mathscr{Y} = \mathscr{P}\langle t \rangle$ is of index 2 in $\mathscr{Y}_1 = C_{\mathscr{X}_1}(\kappa)$. Also $C_{\mathscr{Y}_1}(t) = \langle t, u \rangle$, $\mathscr{Y}_1$ is dihedral or semidihedral, $\mathscr{Y}_1$ is $\langle x \rangle$ invariant, $\mathscr{Y}_1' = \mathscr{P}$, and $\mathscr{Y}_1 \cap \mathscr{V} = \langle u \rangle$. Since x normalizes the unique cyclic maximal subgroup of $\mathscr{Y}_1$, $\langle \kappa, x \rangle$ acts trivially on $\mathscr{Y}_1/\mathscr{Y}_1'$ and hence $\mathscr{R}_1 = C_{\mathscr{Y}_1}(\mathscr{V})$ is a maximal subgroup of $\mathscr{Y}_1$ by Lemma 2.1(iii). Thus $\mathscr{R}_1$ is dihedral or generalized quaternion of order 8, $\mathscr{R}_1\mathscr{V} = \mathscr{R}_1 * \mathscr{V}$, $\mathscr{R}_1 \cap \mathscr{V} = \langle u \rangle$, and $\mathscr{Y}_1 = (\mathscr{R}_1 * \mathscr{V})\langle t \rangle$. Clearly $t^G \cap (\mathscr{R}_1 * \mathscr{V}) \neq \emptyset$ and $I(t(\mathscr{R}_1 * \mathscr{V})) = t^{\mathscr{X}_1}$. Thus $\mathscr{X} = \mathscr{X}_1$ and $\mathscr{T} = (\mathscr{R}_1 * \mathscr{V})\langle x, t \rangle$. Hence $X = \Omega_1(\mathscr{T}')$ and $C_{\mathscr{T}}(X) = \mathscr{X}_1$ char $\mathscr{T}$. Since $S \leq \mathscr{T}$, it follows that $\mathscr{T} \in Syl_2(G)$ and $|G|_2 = |\mathscr{T}| = 2^9$. This contradiction yields Lemma 11.2.

Hence $\mathscr{P} = \langle u, \omega \rangle$ for some involution ω, $\mathscr{Q} = \langle \omega \rangle \times \mathscr{V}$, $\mathscr{Q}' = \langle u \rangle$, $\mho^1(\mathscr{Q}) = X = \Phi(\mathscr{Q})$, $\mathscr{W} = \mathscr{Q}\langle t \rangle$, and $\mathscr{U} = \mathscr{Q}\langle x, t \rangle$. Setting $E = \Omega_1(\mathscr{Q}) = \langle \omega \rangle \times X$, we have $E = Z(\mathscr{Q}) \cong E_{16}$ and hence $t^G \cap \mathscr{Q} = \emptyset$. Since $\Omega_1(\mathscr{U}') = X$ char $\mathscr{U}$, we have $C_{\mathscr{U}}(X) = \mathscr{W}$ char $\mathscr{U}$. Also, if $\tau \in I(\mathscr{W} - \mathscr{Q})$, then $C_{\mathscr{W}}(\tau) = \langle \tau, X \rangle$ and hence E char $\mathscr{W}$ and $\mathscr{Q} = C_{\mathscr{W}}(E)$ char $\mathscr{W}$.

Clearly $\omega^{xt} \in \omega\langle u \rangle$. Suppose that $\omega^{xt} = \omega u$. Then

$$C_{\mathscr{U}}(xt) = \langle \omega v, t \rangle \times \langle z, xt \rangle \in Syl_2(C_G(xt)) \text{ and } I(xt\mathscr{Q}) = xt\langle vy, z \rangle = (xt)^{\mathscr{Q}}.$$

Also $|[E, \mathscr{Q}x]| = |[E, x]| = 2 \neq |[E, \mathscr{Q}xt]| = |[E, xt]| = 4$ since $[x, \omega] = 1$. As $\mathscr{U} \notin Syl_2(G)$, this is impossible. Thus $\omega^{xt} = \omega$,

$$C_{\mathscr{U}}(xt) = \mathscr{Y} \times \langle z, xt \rangle \in Syl_2(C_G(xt)),$$

$u \sim z$ in G and $I(xt\mathscr{Q}) = xt(\langle \omega \rangle \times \langle vy, z \rangle) = (xt)^{\mathscr{Q}} \cup (xt\omega)^{\mathscr{Q}}$. Also $\omega^x = \omega u$, $I(x\mathscr{Q}) = x\langle v, z \rangle = x^{\mathscr{Q}}$ and $[E, x] = \langle u, z \rangle$.

Set $N = N_G(\mathscr{W})$, $C = C_G(\mathscr{W})$, and $\bar{N} = N/O(N)$. Clearly $\langle \mathscr{U}, \kappa \rangle \leq N \leq N_G(\mathscr{Q})$ and $\mathscr{U} \cap C = X$. Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$. Then $\mathscr{U} \neq \mathscr{T}$ since $\mathscr{U} \notin Syl_2(G)$ and $N_{\mathscr{T}}(A) = \mathscr{U}$. Hence $X = C \cap \mathscr{T} \in Syl_2(C)$, $C = O(N) \times X$, and $\bar{C} = \bar{X}$. Also $N/C \hookrightarrow \operatorname{Aut}(\mathscr{W})$ and hence $|N/C|_{2'} = 3$, $\langle \bar{\kappa} \rangle \in Syl_3(\bar{N})$, $\bar{N} = O_2(\bar{N}) \cdot \langle \bar{\kappa}, \bar{x} \rangle$, and $\bar{\mathscr{U}} < \bar{\mathscr{T}} = O_2(\bar{N})\langle \bar{x} \rangle$.

From the nature of the remainder of the proof of Lemma 11.1 and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$. Set $\mathscr{X} = O_2(N)$. The $C_{\mathscr{X}}(t) = A$, $C_{\mathscr{X}}(\kappa, t) = \langle t, u \rangle$, and hence $C_{\mathscr{X}}(\kappa) = \mathscr{Y} = \langle t, \omega \rangle$ since $I(t\langle u, \omega \rangle) = t\langle u \rangle$. Also, since $\mathscr{Q} \lhd N$, if $q \in \mathscr{V} - X$, then the four $\mathscr{W}$-conjugacy classes of involutions in $t\mathscr{Q}$ are represented by t, $t\omega q$, $t\omega q^{\kappa}$, and $t\omega q^{\kappa^2}$. Thus $\mathscr{X}/\mathscr{W} \cong E_4$, $|\mathscr{X}| = 2^9$, κ acts nontrivially on $\mathscr{X}/\mathscr{W}$, and $\mathscr{X}$ is transitive on $I(t\mathscr{Q})$. Thus $\Omega_1(\mathscr{Q}) = E$ is strongly closed in $\mathscr{W}$ with respect to G.

LEMMA 11.3. *Let $\mathscr{U} < \mathscr{R}$ where $\mathscr{R}$ is a 2-group. Then $E \lhd \mathscr{R}$ and E is the unique normal element of $\mathscr{E}_{16}(\mathscr{R})$.*

Proof. Since E char $\mathscr{U}$, it suffices, by induction, to assume that $E \lhd \mathscr{R}$ and to prove that E is unique. Thus let $E \neq Y \lhd \mathscr{R}$ with $Y \in \mathscr{E}_{16}(\mathscr{R})$. Since $|\mathscr{U}| = 2^8 < |\mathscr{R}|$, we have $t^G \cap Y = \emptyset$. Also $4 \leq |C_Y(t)|$ and if $\tau \in x\langle u, z \rangle$, there is an element $q \in \mathscr{V} - X$ such that $|[q, \tau]| = 4$. Hence $C_Y(t) \leq X$. Suppose that $Y \leq \mathscr{W} = \mathscr{Q}\langle t \rangle$. Then $Y \leq \mathscr{Q}$ and hence $Y = E$. Thus $Y \not\leq \mathscr{W}$. Since $[Y, t] \leq C_Y(t) \leq X$ and $\mathscr{Q}$ is transitive on tX, it follows that $Y \leq \mathscr{Q}C_{\mathscr{R}}(t) = \mathscr{Q}S = \mathscr{U}$. Thus there is an involution $\tau \in Y \cap (\mathscr{Q}x \cup \mathscr{Q}xt)$. Since for any such τ there is an element $q \in \mathscr{V} - X$ such that $|[q, \tau]| = 4$, we have a contradiction and the lemma follows.

Let $\tau \in N_{\mathscr{X}}(\mathscr{U}) - \mathscr{W}$. Then τ normalizes $\{x\mathscr{Q}, xt\mathscr{Q}\}$ and $\tau^2 \in \mathscr{W}$. Since $|[E, x\mathscr{Q}]| \neq |[E, xt\mathscr{Q}]|$, we conclude that $\langle [x, \tau], [xt, \tau] \rangle \leq \mathscr{Q}$ and hence a Sylow 2-subgroup of $N/\mathscr{Q}$ is not semidihedral. Thus

$$N/\mathscr{Q} \cong Z_2 \times \Sigma_4 \quad \text{where } \langle t\mathscr{Q} \rangle = C_{\mathscr{X}/\mathscr{Q}}(\kappa),\ \mathscr{V} < [\mathscr{X}, \kappa], \text{ and } t \notin [\mathscr{X}, \kappa].$$

Note that $\mathscr{T} = \mathscr{X}\langle x\rangle$, $|\mathscr{T}| = 2^{10}$, $\mathscr{T} \notin Syl_2(G)$, E char $\mathscr{T}$, $\mathscr{Q} \leq C_{\mathscr{T}}(E) = C_N(E) = C_{\mathscr{X}}(E)$ char $\mathscr{T}$, and $C_{\mathscr{X}}(E) \lhd N$ since $E = Z(\mathscr{Q})$.

Suppose that $\mathscr{Q} = C_{\mathscr{T}}(E)$. Set $J = N_G(E)$ and $\bar{J} = J/O(J)$. Clearly $\langle \mathscr{T}, \kappa\rangle \leq J$. Let $\mathscr{T} \leq \mathscr{S} \in Syl_2(J)$. Then $\mathscr{T} \neq \mathscr{S}$ and $\mathscr{S} \in Syl_2(G)$ since E char $\mathscr{S}$ by Lemma 11.3. Also

$\mathscr{Q} = C_G(E) \cap \mathscr{T} \leq C_G(E) \cap \mathscr{S} \trianglelefteq \mathscr{S}$ and $C_G(E) \cap \mathscr{S} = C_{\mathscr{S}}(E) \in Syl_2(C_G(E))$. Suppose that $\mathscr{Q} \neq C_{\mathscr{S}}(E)$. Then there is an element $\tau \in C_{\mathscr{S}}(E) - \mathscr{Q}$ such that $t^\tau \in t\mathscr{Q}$. But $\mathscr{X}$ is transitive on $I(t\mathscr{Q})$ and hence $\tau \in \mathscr{T}$ which is false. Thus $\mathscr{Q} = C_{\mathscr{S}}(E) \in Syl_2(C_G(E))$. Also $\mathscr{Q}/E \cong E_4$ and any element of odd order in $N_J(\mathscr{Q}) \cap C_J(E)$ centralizes $\mathscr{Q}$. Thus $C_G(E) = O(J)\mathscr{Q}$ and $\bar{J} = N_{\bar{J}}(\bar{\mathscr{Q}}) = \overline{N_J(\mathscr{Q})}$. Also $C_{\bar{J}}(\bar{\mathscr{Q}}) = \bar{E}$ and hence $\bar{J}/\bar{E} \hookrightarrow \operatorname{Aut}(\bar{\mathscr{Q}})$. Thus $\bar{J} = O_2(\bar{J})\langle\bar{\kappa}, \bar{x}\rangle$ and $O_2(\bar{J}) = C_{\bar{J}}(\bar{\mathscr{Q}}/\bar{E})$ and hence $O_2(\bar{J})$ acts trivially on $\bar{X} = \mho^1(\bar{\mathscr{Q}})$. Since $C_{\bar{J}}(\bar{E}) = \bar{\mathscr{Q}}$, we have $|O_2(\bar{J})\colon \bar{\mathscr{Q}}| \mid 2^3$ and hence $|G|_2 = |J|_2 = |\mathscr{S}| = |\mathscr{T}|$ which is a contradiction.

Thus $\mathscr{Q} \neq C_{\mathscr{T}}(E) = C_{\mathscr{X}}(E)$. Setting $\mathscr{X}_1 = C_{\mathscr{X}}(E)$, we have $\mathscr{X} = \mathscr{X}_1\langle t\rangle$, $t \notin \mathscr{X}_1$,

$$\mathscr{X}_1 = C_{\mathscr{T}}(E) \text{ char } \mathscr{T} = \mathscr{X}_1\langle x, t\rangle.$$

Also $\mathscr{X}_1 = \mathscr{Q}[\mathscr{X}, \kappa] \lhd N$, $|\mathscr{X}_1| = 2^8$, and $\mathscr{P} = C_{\mathscr{X}_1}(\kappa) = \langle u, \omega\rangle \lhd N$. Set $\tilde{N} = N/\mathscr{P}$. Then $|\tilde{\mathscr{X}}_1| = 2^6$, $C_{\tilde{\mathscr{X}}_1}(\tilde{\kappa}) = 1$, and $Z_4 \times Z_4 \cong \tilde{\mathscr{Q}} \lhd \tilde{N}$. Clearly $\langle u\rangle = \mathscr{Q}' \leq \mathscr{X}_1'$. Suppose that $\mathscr{X}_1' = \langle u\rangle$. Then $\mathscr{V} \leq [\mathscr{X}, \kappa] < \mathscr{X}_1$, $[\mathscr{X}, \kappa] \lhd N$, and $\mathscr{Q} \cap [\mathscr{X}, \kappa] = \mathscr{V}$. Since $I(t\mathscr{V}) = t^{\mathscr{V}}$ and $C_{[\mathscr{X}, \kappa]}(t) = X$, this is impossible. Thus $\mathscr{X}_1' \neq \langle u\rangle$ and $\tilde{\mathscr{X}}_1$ is not isomorphic to $Z_8 \times Z_8$.

Suppose that $\tilde{\mathscr{X}}_1 \cong Z_4 \times Z_4 \times E_4$. Let $\mathscr{K}$ denote the inverse image in $\mathscr{X}_1$ of $\Omega_1(\tilde{\mathscr{X}}_1)$. Then $\mathscr{K} \lhd N$, $\mathscr{K} \cap \mathscr{Q} = E$ and t fixes an element $\mathscr{Y}/E$ which is impossible. Thus $\tilde{\mathscr{X}}_1$ is isomorphic to a Sylow 2-subgroup of $L_3(4)$ by Lemma 2.9. Thus $\langle u\rangle < \mathscr{X}_1' \leq \Phi(\mathscr{X}_1) \leq E = Z(\mathscr{X}_1)$. If $\Phi(\mathscr{X}_1) = X$, then $\mathscr{V} \leq [\mathscr{X}, \kappa] < \mathscr{X}_1$, $[\mathscr{X}, \kappa] \lhd N$, $[\mathscr{X}, \kappa] \cap \mathscr{Q} = \mathscr{V}$ and we obtain a contradiction as above. Thus, utilizing κ, we have $E = \Phi(\mathscr{X}_1) = \mathscr{X}_1' = Z(\mathscr{X}_1)$, $\exp(\mathscr{X}_1) = 4$, $C_{\mathscr{X}_1/E}(t) = \mathscr{Q}/E$, and $I(t\mathscr{X}_1) = t^{\mathscr{X}_1}$. Note also that $t^G \cap \mathscr{X}_1 = \emptyset$. Also $\langle \kappa, x\rangle$ acts faithfully on $\mathscr{X}_1/\mathscr{Q}$; hence $(\mathscr{X}_1\langle x\rangle)\,\mathscr{Q} \cong D_8$ and $I(x\mathscr{X}_1) = x^{\mathscr{X}_1}$ since $I(x\mathscr{Q}) = x^{\mathscr{Q}}$. It follows that $t^G \cap (\mathscr{X}_1\langle x\rangle) = \emptyset$.

Set $J = N_G(E)$, $\bar{J} = J/O(J)$, and let $\mathscr{T} \leq \mathscr{S} \in Syl_2(J)$. Clearly $\langle \mathscr{T}, \kappa\rangle \leq J$, $\mathscr{S} \in Syl_2(J)$, $\mathscr{T} \neq \mathscr{S}$, and $\mathscr{S} \in Syl_2(G)$. On the other hand,

$$\mathscr{X}_1 = C_G(E) \cap \mathscr{T} \leq \mathscr{S} \cap C_G(E) \trianglelefteq \mathscr{S}.$$

Since $I(t\mathscr{X}_1) = t^{\mathscr{X}_1}$, we conclude that $\mathscr{X}_1 = \mathscr{S} \cap C_G(E) \in Syl_2(C_G(E))$. By the same token, $C_{\mathscr{S}/\mathscr{X}_1}(t\mathscr{X}_1) = \mathscr{T}/\mathscr{X}_1 \cong E_4$ and hence $\mathscr{S}/\mathscr{X}_1$ is dihedral or semidihedral. But $\mathscr{S}/\mathscr{X}_1 \hookrightarrow \operatorname{Aut}(E)$ and hence $\mathscr{S}/\mathscr{X}_1 \cong D_8$. Since $|[E, \mathscr{X}_1x]| \neq |[E, \mathscr{X}_1t]| = |[E, \mathscr{X}_1xt]|$, we conclude that $\langle x\mathscr{X}_1\rangle = (\mathscr{S}/\mathscr{X}_1)'$. Since $|\mathscr{S}| = 2^{11} = |G_2|$ and $t^G \cap (\mathscr{X}_1\langle x\rangle) = \emptyset$, [17, Lemma 5.38] implies that $|O^2(G)|_2 \leq 2^{10}$. This contradiction completes the proof of Lemma 11.1.

12. The case of Lemma 7.4(v)

In this section we shall conclude the proof of Theorem 2 by proving:

LEMMA 12.1. *If* $\mathscr{V}$ *satisfies* (v) *of Lemma* 7.4, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V}$ satisfies (v) of Lemma 7.4 and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

Thus $Z(\mathscr{V}A) = Z(\mathscr{U}) = Z(\mathscr{V}) = \langle u \rangle$, $\mathscr{V}A = \mathscr{V}\langle t \rangle$ is of type $\mathscr{A}_8$, $\mathscr{V}\langle t, x \rangle \cong D_8 \wr Z_2$ by utilizing the proof of [7, VI, Lemma 2.7(iii)], $[\mathscr{V}A, \kappa] = \mathscr{V} = [\mathscr{W}, \kappa] = Q_1 * Q_2$ where Q_1 and Q_2 are quaternion of order 8 and $Q_1^t = Q_2$. Also $\mathscr{E}_{16}(\mathscr{V}A) = \{A\}$ and every element of $\mathscr{V}A - \mathscr{V}$ interchanges Q_1 and Q_2. Now $\mathscr{Y} = C_{\mathscr{W}}(\kappa)$ acts on $\mathscr{V} = [\mathscr{W}, \kappa]$ and hence $\mathscr{Y}$ contains a maximal subgroup $\mathscr{P}$ normalizing both Q_1 and Q_2. Since $\mathscr{P} \leq C_{\mathscr{W}}(\kappa)$, we have $[\mathscr{P}, Q_1] = [\mathscr{P}, Q_2] = 1$. Then $\langle \mathscr{U}, \kappa \rangle \leq N_G(\mathscr{P}) \cap N_G(\mathscr{V})$ and $\mathscr{P} \cap \mathscr{V} = \langle u \rangle$. Set $\mathscr{Q} = \mathscr{P}\mathscr{V} = \mathscr{P} * \mathscr{V}$. Then $\mathscr{W} = \mathscr{Q}\langle t \rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t \rangle$, $Z(\mathscr{U}) = \langle u \rangle$, $Z(\mathscr{Q}) = \mathscr{P}$, and $\mathscr{Q}' = \Phi(\mathscr{Q}) = \langle u \rangle$.

LEMMA 12.2. $\mathscr{P} \cong Z_4$.

Proof. Assume that $\mathscr{P} = \langle u, \omega \rangle$ where $\omega^2 = 1$. Then $\mathscr{Q} = \langle \omega \rangle \times \mathscr{V}$, $I(t\mathscr{Q}) = tX = t^{\mathscr{Q}}$, and $\mathscr{U} = \mathscr{Q}\langle x, t \rangle \notin Syl_2(G)$. Note also that $Z(\overline{M}) = \langle \bar{u} \rangle$ and set $\tilde{M} = \overline{M}/\langle \bar{u} \rangle$. Then it is easy to see that $\mathscr{E}_{32}(\tilde{\mathscr{U}}) = \{\tilde{\mathscr{Q}}\}$. Since $Z(\mathscr{U}) = \langle u \rangle$ it follows that $\mathscr{E}_{32}(\mathscr{U}/\langle u \rangle) = \{\mathscr{Q}/\langle u \rangle\}$ and hence $\mathscr{Q}$ char $\mathscr{U}$ and $\mathscr{P} = Z(\mathscr{Q})$ char $\mathscr{U}$.

Set $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\overline{N} = N/O(N)$. Thus $\langle \mathscr{U}, \kappa \rangle \leq N$. Also let $\mathscr{U} = \mathscr{Q}\langle x, t \rangle \leq \mathscr{T} \in Syl_2(N)$. Clearly $\mathscr{U} \neq \mathscr{T}$, $\mathscr{U} \cap C = \mathscr{P}$, and $I(t\mathscr{P}) = t\langle u \rangle = t^{\mathscr{P}}$. Hence $C = O(N) \times \mathscr{P}$, $\overline{C} = \overline{\mathscr{P}}$, and $\overline{N}/\overline{\mathscr{P}} \hookrightarrow \mathrm{Aut}(\mathscr{Q})$. Also $C_{\mathscr{Q}}(t) = X$ and hence $C_N(t) = C_N(t) \cap N_H(A)$. Thus $C_N(t) = (O(N) \cap C_N(t))A(\kappa, x)$ and $C_{\overline{N}}(\bar{t}) = \bar{A}\langle \bar{\kappa}, \bar{x} \rangle$. Moreover $I(t\bar{\mathscr{Q}}) = t^{\bar{\mathscr{Q}}}$ and hence $C_{\overline{\mathscr{T}}/\overline{\mathscr{Q}}}(\bar{t}\overline{\mathscr{Q}}) = \overline{\mathscr{U}}/\overline{\mathscr{Q}} \cong E_4$. Thus $\overline{\mathscr{T}}/\overline{\mathscr{Q}}$ is dihedral or semidihedral. But $\overline{\mathscr{U}}/\overline{\mathscr{Q}} \neq \overline{\mathscr{T}}/\overline{\mathscr{Q}}$. Hence $t\mathscr{Q} \sim xt\mathscr{Q}$ in $\mathscr{T}$, $Z(\mathscr{T}/\mathscr{Q}) = \langle x\mathscr{Q} \rangle$, and $O_2(\overline{N}) = \overline{\mathscr{Q}}$ since $\bar{x} \notin O_2(\overline{N})$.

On the other hand, $\mathscr{Q} = \langle \omega \rangle \times \mathscr{V}$, $\mathscr{Q}' = \langle u \rangle$, $Z(\mathscr{Q}) = \langle \omega, u \rangle$, $\mathscr{V} = Q_1 * Q_2$ and $\overline{N}/\overline{C} \hookrightarrow \mathrm{Aut}(\overline{\mathscr{Q}})$. Hence every Sylow p-subgroup of $\overline{N}/\overline{C}$ with p odd acts on $\overline{\mathscr{Q}}/\langle \bar{\omega} \rangle \cong \overline{\mathscr{V}}$. Thus $|\overline{N}|_{2'} = 3^2$,

$$O(\overline{N}/\overline{C}) \cong Z_3 \times Z_3, \quad \mathscr{V} = [\mathscr{Q}, O_{2',2,2'}(N)] \lhd N,$$

and

$$C_N(\mathscr{V}) = O(N) \times \mathscr{P}.$$

Thus $\overline{N}/\overline{\mathscr{P}} \hookrightarrow \mathrm{Aut}(\overline{\mathscr{V}}) \cong \Sigma_4 \wr Z_2$ and hence $\mathscr{T}/\mathscr{U} \cong D_8$. Since $|\mathscr{T}| = 2^9$, $\mathscr{T}$ is a maximal subgroup of some 2-subgroup $\mathscr{S}$ of G. Clearly $Z(\mathscr{S}) = Z(\mathscr{T}) = \langle u \rangle$ and $\mathscr{Q} \ntrianglelefteq \mathscr{S}$. Let $\alpha \in \mathscr{S} - \mathscr{T}$ and set $\tilde{\mathscr{S}} = \mathscr{S}/\langle u \rangle$ and $\mathscr{Q}_1 = \mathscr{Q}^{\alpha}$. Then $\mathscr{Q} \neq \mathscr{Q}_1 \lhd \mathscr{T}$ and hence $\mathscr{Q}\langle x \rangle \leq \mathscr{Q}\mathscr{Q}_1$. Since $C_{\overline{\mathscr{V}}/\langle \bar{u} \rangle}(\bar{\kappa}) = 1$, we have $|C_{\tilde{\mathscr{V}}}(\tilde{x})| = 4$ and $|C_{\tilde{\mathscr{Q}}}(\tilde{x})| = 8$. Thus $|\tilde{\mathscr{Q}} \cap \tilde{\mathscr{Q}}_1| = 8$, $\mathscr{Q}_1\mathscr{Q}/\mathscr{Q} \cong E_4$, and

$$I(\mathscr{T}) = I(\mathscr{Q}\mathscr{Q}_1) \cup I(t\mathscr{Q}) \cup I(xt\mathscr{Q}).$$

Since α leaves $\mathscr{Q}\mathscr{Q}_1$ invariant and $t\mathscr{Q} \sim xt\mathscr{Q}$ in $\mathscr{T}$, we may assume that α leaves $I(t\mathscr{Q}) = t^{\mathscr{Q}}$ invariant. As $S = C_{\mathscr{S}}(t) \leq \mathscr{T}$, this is a contradiction and the proof of Lemma 12.2 is complete.

Let $\mathscr{P} = \langle \omega \rangle$ where $\omega^2 = u$. Then $I(t\mathscr{Q}) = t^{\mathscr{Q}} \cup (t\omega)^{\mathscr{Q}}$, $\mathscr{Q}$ char $\mathscr{U}$, and $\mathscr{P}$ char $\mathscr{U}$, as in the preceding lemma.

Suppose that xt normalizes Q_1 and Q_2. Then there is an element $q_1 \in Q_1 - \langle u \rangle$ such that $q_1^{xt} = q_1^{-1}$. Setting $q_2 = q_1^t$, we have $C_{\mathscr{V}}(xt) = C_{\mathscr{V}}(x, t) = \langle u, z \rangle = \langle u, q_1 q_2 \rangle$. If $\omega^{xt} = \omega$, then

$$C_{\mathscr{U}}(xt) = \mathscr{Y} \times \langle z, xt \rangle \in Syl_2(C_G(xt))$$

and $u \sim z$ in G. If $\omega^{xt} = \omega^{-1}$, then $C_{\mathscr{U}}(xt) = C_{\mathscr{Q}}(xt)\langle x, t \rangle \in Syl_2(C_G(xt))$ where $C_{\mathscr{Q}}(xt) = \langle u, z, \omega q_1 \rangle \cong E_8$.

Suppose that $Q_1^{xt} = Q_2$. Then $x \in N_G(Q_1) \cap N_G(Q_2)$ and there is an element $q_1 \in Q_1 - \langle u \rangle$ such that $q_1^x = q_1^{-1}$ and $C_{\mathscr{V}}(x) = C_{\mathscr{V}}(x, t) = \langle u, z \rangle = \langle u, q_1 q_2 \rangle$. Also $\langle u, z \rangle < C_{\mathscr{V}}(xt) \cong E_8$ and hence $\omega^{xt} = \omega^{-1}$ and $C_{\mathscr{U}}(xt) = C_{\mathscr{V}}(xt)\langle x, t \rangle \in Syl_2(C_G(xt))$.

Set $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar{N} = N/O(N)$. Clearly $\langle \mathscr{U}, \kappa \rangle \leq N$ and $\kappa^3 \in C$. Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$, so that $\mathscr{U} \neq \mathscr{T}$. As in the proof of Lemma 12.2, we have $C_N(t) = (O(C_N(t)) \cap C)A\langle \kappa, x \rangle$.

LEMMA 12.3. (i) *$C = O(N)(C \cap \mathscr{T})$ where $\bar{C} = \overline{C \cap \mathscr{T}}$ is cyclic, $C \cap \mathscr{T} \lhd \mathscr{T}$, $(C \cap \mathscr{T})\langle t \rangle$ is dihedral or semidihedral and $\mathscr{P} \leq (C \cap \mathscr{T}) \cap Z(C)$.*

(ii) *$(C \cap \mathscr{T})\mathscr{Q} = (C \cap \mathscr{T}) * \mathscr{V}$ and $t^G \cap ((C \cap \mathscr{T}) * \mathscr{V}) = \emptyset$.*

(iii) *$\bar{N}/(\bar{C}\bar{\mathscr{V}}) \hookrightarrow Z_2 \times \Sigma_6$.*

(iv) *$C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\kappa}, \bar{x} \rangle$.*

Proof. Clearly $C \cap \mathscr{T} \lhd \mathscr{T}$, $(C \cap \mathscr{T})\langle t \rangle \in Syl_2(C\langle t \rangle)$, and $C_{(C \cap \mathscr{T})}(t) = \langle u \rangle$. Since $\mathscr{P} \leq (C \cap \mathscr{T}) \cap Z(C)$, (i) and (iv) hold. Since

$$((C \cap \mathscr{T})\mathscr{V})' = ((C \cap \mathscr{T}) * \mathscr{V})' = \mathscr{V}' = \langle u \rangle,$$

$|((C \cap \mathscr{T}) * \mathscr{V})| \geq 2^6$ and $Z((C \cap \mathscr{T}) * \mathscr{V}) = C \cap \mathscr{T}$ is cyclic of order at least 4, (ii) holds. Finally, since $\bar{N}/\bar{C} \hookrightarrow$ Aut $(\mathscr{Q})$, we have (iii) by [10, Section 1] and we are done.

From the nature of the remainder of the proof of this lemma and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$. Then $C = C \cap \mathscr{T}$, $(C * \mathscr{V})\langle x, t \rangle = C\mathscr{U} \leq \mathscr{T}$, and $C\mathscr{Q} = C * \mathscr{V} \leq O_2(N)$. Let $C = \langle \gamma \rangle$ and $|C| = 2^a$ for some integer $a \geq 2$.

LEMMA 12.4. (i) *$C * \mathscr{V}$ char $C\mathscr{U} = (C * \mathscr{V})\langle x, t \rangle$ and $\mathscr{Q}$ char $C\mathscr{U}$.*
(ii) *$\mathscr{T} \neq C\mathscr{U}$.*

Proof. If $\mathcal{P} = C$, then $C\mathcal{U} = \mathcal{U}$, and $C * \mathcal{V} = \mathcal{Q}$ char $\mathcal{U}$. Suppose that $\mathcal{P} < C$. Clearly $Z(C\mathcal{U}) = \langle u \rangle$. Setting $(C\mathcal{U})^{\sim} = (C\mathcal{U})/\langle u \rangle$, we have

$$(C\mathcal{U})^{\sim} = (\tilde{C} \times \tilde{\mathcal{V}})\langle \tilde{x}, \tilde{t} \rangle.$$

Since $a \geq 3$, it is clear that $J_0((C\mathcal{U})^{\sim}) = \tilde{C} \times \tilde{\mathcal{V}}$ and hence $C * \mathcal{V}$ char $C\mathcal{U}$. Since $\Omega_2(C * \mathcal{V}) = \mathcal{Q}$, (i) holds. Suppose that $\mathcal{T} = C\mathcal{U}$. Then $\mathcal{T} \in Syl_2(G)$ and $|\gamma| = 2^a \geq 2^5$ and xt acts dihedrally or semidihedrally on $C = \langle \gamma \rangle$. Hence $\langle \gamma^2 \rangle \leq C_{\mathcal{T}}(x)$. Thus $\langle \gamma^4 \rangle \leq C_{\mathcal{T}}(x)' \leq \mathcal{T}' \leq C * \mathcal{V}$. Since $|\gamma^4| \geq 2^3$ and $Z(\mathcal{T}) = \langle u \rangle$, we have $x \not\sim u$ in G. This contradiction shows that (ii) holds.

LEMMA 12.5. (i) $|C_{N/(C*\mathcal{V})}(t): ((C * \mathcal{V})(\langle t \rangle \times \langle \kappa, x \rangle)/(C * \mathcal{V}))| = 2$.
(ii) $C\langle t \rangle$ *is dihedral.*

Proof. Assume that $N_N((C * \mathcal{V})\langle t \rangle) = (C * \mathcal{V})(\langle t \rangle \times \langle \kappa, x \rangle)$. If $O_2(N) \neq C * \mathcal{V}$, then $O_2(N) = (C * \mathcal{V})\langle t \rangle$ and hence $N = (C * \mathcal{V})(\langle t \rangle \times \langle \kappa, x \rangle)$ which is false by Lemma 12.4(ii). Thus $O_2(N) = C * \mathcal{V}$ and $\mathcal{T}/O_2(N)$ is dihedral or semidihedral. Then Lemma 12.3(iii) implies that $\mathcal{T}/O_2(N) \cong D_8$.

Suppose that $\mathcal{T} \in Syl_2(G)$. Then $|C| = |\gamma| \geq 2^4$ and $\mathcal{T}'(C * \mathcal{V})$ centralizes $C = \langle \gamma \rangle$. Hence $\mathcal{T}'(C * \mathcal{V}) = (C * \mathcal{V})\langle x \rangle$ and we obtain a contradiction as in the proof of Lemma 12.4(ii). Thus $\mathcal{T}$ is a maximal subgroup of some 2-subgroup $\mathcal{S}$ of G. Let $\alpha \in \mathcal{S} - \mathcal{T}$, set $Y = C * \mathcal{V}$, and $Y_1 = Y^{\alpha}$. Clearly $Y_1 \neq Y$ since $\mathcal{Q} = \Omega_2(Y)$ and hence $N_{\mathcal{S}}(Y) = N_{\mathcal{S}}(Y_1) = \mathcal{T}$. Also $Z(\mathcal{S}) = \langle u \rangle$. Setting $\tilde{\mathcal{S}} = \mathcal{S}/\langle u \rangle$, we have $\tilde{Y} = \tilde{C} \times \tilde{\mathcal{V}} \cong Z_{2^{a-1}} \times E_{16} \cong \tilde{Y}_1$. Also $C_{\tilde{Y}}(\tilde{x}) = C_{\tilde{C}}(\tilde{x}) \times C_{\tilde{\mathcal{V}}}(\tilde{x})$ where $C_{\tilde{\mathcal{V}}}(\tilde{x}) \cong E_4$; similarly for $C_{\tilde{Y}}(\tilde{x}\tilde{t})$. Since $\tilde{Y}_1$ is abelian, we have $\mathcal{T} \neq YY_1$. Thus $|Y_1Y/Y| = 4$ and $|\tilde{Y}_1 \cap \tilde{Y}| = 2^{a+1}$. Since $x \in Y_1Y$ or $xt \in Y_1Y$, $|C_{\tilde{Y}}(\tilde{x}\tilde{Y})| = 4|C_{\tilde{C}}(\tilde{x})|$ and $|C_{\tilde{Y}}(\tilde{x}\tilde{t}\tilde{Y})| = 4|C_{\tilde{C}}(\tilde{x}\tilde{t})|$, we have $\tilde{C} \leq \tilde{Y}_1$, $\tilde{Y}_1 \cap \tilde{Y} = \tilde{C} \times (\tilde{\mathcal{V}} \cap \tilde{Y}_1)$ and $\tilde{t} \notin \tilde{Y}_1$ since $t^G \cap Y = \emptyset$. Since α leaves YY_1 invariant, it follows that we may assume that $t^{\alpha} \in tC$. Thus, since $C_{\mathcal{S}}(t) = S \leq \mathcal{T}$, it follows that $C\langle t \rangle$ is dihedral and that we may assume that $t^{\alpha} = t\gamma$. Hence $C_G(t, u)^{\alpha} = C_G(t\gamma, u)$. However

$$\langle y, z, \kappa \rangle \leq O^2(C_G(t, u)) \cap O^2(C_G(t\gamma, u))$$

and

$$(\mathcal{T} \cap O^2(C_G(t, u)))^{\alpha} = \mathcal{T} \cap O^2(C_G(t\gamma, u)).$$

Thus $\langle y, z \rangle^{\alpha} = \langle y, z \rangle$ by (4.12). Since $[y, x] = z = [y, xt]$ and $xY \cap Y_1 \neq \emptyset$ or $xtY \cap Y_1 \neq \emptyset$, we have $y \notin Y_1$. This contradiction shows that

$$N_N(C * \mathcal{V})\langle t \rangle) \neq (C * \mathcal{V})(\langle t \rangle \times \langle \kappa, x \rangle).$$

But $I(t(C * \mathcal{V})) = t^{(C*\mathcal{V})}$ if $C\langle t \rangle$ is semihidedral and $I(t(C * \mathcal{V})) = t^{C*\mathcal{V}} \cup (t\gamma)^{C*\mathcal{V}}$ if $C\langle t \rangle$ is dihedral. Then $C_N(t) = A\langle \kappa, x \rangle$ implies Lemma 12.5.

Let $Y = N_N((C * \mathcal{V})\langle t \rangle)$. Then $(C * \mathcal{V})\langle t \rangle$ is of index 2 in $O_2(Y)$, $Y = O_2(Y)\langle \kappa, x \rangle$, $C\langle t \rangle$ is of index 2 in $C_{O_2(Y)}(\kappa)$, $[O_2(Y), \kappa] = \mathcal{V} \lhd Y$, $C_{O_2(Y)}(\kappa, t) = \langle t, u \rangle$ and $C_{O_2(Y)}(\kappa)$ is dihedral or semidihedral. Set

$$\mathcal{R} = C_{O_2(Y)}(\kappa) \cap N_Y(Q_1) \cap N_Y(Q_2).$$

Then $\mathscr{R}$ is of index 2 in $C_{O_2(Y)}(\kappa)$, $[\mathscr{R}, \mathscr{V}] = 1$, $t \notin \mathscr{R}$, C is a maximal subgroup of $\mathscr{R}$, and hence $\mathscr{R}$ is not abelian. Thus $\mathscr{R}$ is dihedral or generalized quaternion, $\mathscr{R}\mathscr{V} = \mathscr{R}\mathscr{Q} = \mathscr{R} * \mathscr{V}$, $\mathscr{R} \cap \mathscr{V} = \langle u \rangle$, $Y = (\mathscr{R} * \mathscr{V})(\langle t \rangle \times \langle \kappa, x \rangle)$, and $\mathscr{R}\langle t \rangle = C_{O_2(Y)}(\kappa)$.

Clearly $t^G \cap (\mathscr{R} * \mathscr{V}) = \emptyset$ by Lemma 2.12 and $I(t(\mathscr{R} * \mathscr{V})) = t^{(\mathscr{R}*\mathscr{V})}$. Also

$$(\mathscr{R} * \mathscr{V})/C \lhd N/C$$

by [10, Section 1] and hence $R * \mathscr{V} \leq O_2(N)$. Thus $O_2(N) = \mathscr{R} * \mathscr{V}$ or $O_2(N) = (\mathscr{R} * \mathscr{V})\langle t \rangle = O_2(Y)$.

LEMMA 12.6. *$O_2(N) = \mathscr{R} * \mathscr{V}$ and $\mathscr{T} \neq (\mathscr{R} * \mathscr{V})\langle x, t \rangle$.*

Proof. Assume that $O_2(N) = (\mathscr{R} * \mathscr{V})\langle t \rangle$. Then, since $t^G \cap (\mathscr{R} * \mathscr{V}) = \emptyset$, $\Omega_1(\mathscr{R} * \mathscr{V}) = \mathscr{R} * \mathscr{V}$ and $I(t(\mathscr{R} * \mathscr{V})) = t^{(\mathscr{R}*\mathscr{V})}$, we have $N = O_2(N)\langle \kappa, x \rangle = Y$ where $\mathscr{R} \lhd N$ and $\mathscr{V} \lhd Y$. Also $\mathscr{R}\langle t \rangle$ is dihedral or semidihedral, $|\mathscr{R}| \geq 8$, and $(\mathscr{R}\langle t \rangle)' = C$. Hence we may assume that $\mathscr{T} = (\mathscr{R} * \mathscr{V})\langle t, x \rangle$. Then $Z(\mathscr{T}) = \langle u \rangle$ and $\langle u \rangle \leq \mho^1(\mathscr{T}') = \mho^1(C)$. Setting $\tilde{\mathscr{T}} = \mathscr{T}/\mho^1(\mathscr{T}')$, we have $(\mathscr{R}\mathscr{V})^\sim = \tilde{\mathscr{R}} \times \tilde{\mathscr{V}}$ and $\mathscr{E}_{64}(\tilde{\mathscr{T}}) = \{(\mathscr{R}\mathscr{V})^\sim\}$. Thus $\mathscr{R} * \mathscr{V}$ char $\mathscr{T}$.

Suppose that $|\mathscr{R}| \geq 2^4$. Then $\mathscr{P} = \Omega_2((\mathscr{R} * \mathscr{V})')$ char $\mathscr{T}$, $C_{\mathscr{R}*\mathscr{V}}(\mathscr{P}) = C * \mathscr{V}$ char $\mathscr{T}$, $\mathscr{Q}$ char $\mathscr{T}$, and $\mathscr{T} \in Syl_2(G)$. Since the cyclic maximal subgroup of $\mathscr{R}\langle t \rangle$ is $\langle x, t \rangle$-invariant and $|C| \geq 8$, we have $[x, C] = 1$. Thus $\mho^1(C) \leq C_{\mathscr{T}}(x)' \leq C * \mathscr{V}$ where $\mho^1(C)$ is cyclic of order at least 4. Since $x \sim u$ in G, this is impossible. Thus $|\mathscr{R}| = 2^3$, $C = \mathscr{P}$, $|\mathscr{T}| = 2^9$, $\mathscr{R} * \mathscr{V}$ is extraspecial of order 2^7, and $\mathscr{R} * \mathscr{V}$ char $\mathscr{T}$.

Let $J = N_G(\mathscr{R} * \mathscr{V})$, $\bar{J} = J/O(J)$ and let $\mathscr{T} \leq \mathscr{S} \in Syl_2(J)$. Clearly $\mathscr{T} \neq \mathscr{S}$ and $Z(\mathscr{S}) = \langle u \rangle$. Set $\tilde{\mathscr{S}} = \mathscr{S}/(\mathscr{R} * \mathscr{V})$. Then $C_{\tilde{\mathscr{S}}}(\tilde{t}) = \langle \tilde{x}, \tilde{t} \rangle$ and $\tilde{\mathscr{S}}$ is dihedral or semidihedral with $Z(\tilde{\mathscr{S}}) = \langle \tilde{x} \rangle$ and $\tilde{t} \sim \tilde{t}\tilde{x}$ in $\tilde{\mathscr{S}}$ since $I(t(\mathscr{R} * \mathscr{V})) = t^{(\mathscr{R}*\mathscr{V})}$. Thus $\tilde{\mathscr{S}}$ acts faithfully on $\mathscr{R}\mathscr{V}/\langle u \rangle$ and hence $\exp(\tilde{\mathscr{S}}) \leq 8$.

Suppose that $\tilde{\mathscr{S}} \cong D_8$. Then $|\mathscr{S}| = 2^{10}$ and $\mathscr{S} \notin Syl_2(G)$. Thus there is a 2-element $\beta \in N_G(\mathscr{S}) - \mathscr{S}$ such that $\beta^2 \in \mathscr{S}$. Set $\mathscr{X} = \mathscr{R} * \mathscr{V}$ and $\mathscr{X}_1 = \mathscr{X}^\beta$. Then $\mathscr{X} \neq \mathscr{X}_1 \lhd \mathscr{S}$ and $1 \neq \tilde{\mathscr{X}}_1 \lhd \tilde{\mathscr{S}}$. Since $\tilde{x} \in \tilde{\mathscr{X}}_1$, it follows that $\tilde{\mathscr{X}}_1 \cong E_4$ and $\tilde{t} \notin \tilde{\mathscr{X}}_1$. Then β leaves $I(t\mathscr{X}) \cup I(xt\mathscr{X})$ invariant. Since $t\mathscr{X} \sim xt\mathscr{X}$ in $\mathscr{S}$ and $I(t\mathscr{X}) = t^{\mathscr{X}}$, this is impossible.

Assume that $|\tilde{\mathscr{S}}| = 16$. Then $\mathscr{S} \in Syl_2(G)$ since no element of $\mathscr{E}_4(\tilde{\mathscr{S}})$ is normal in $\tilde{\mathscr{S}}$. Hence $|\mathscr{S}| = 2^{11}$ and $O^2(G) = G$. Thus $t^G \cap x(\mathscr{R} * \mathscr{V}) \neq \emptyset$ by [17, Lemma 5.38]. Since $\langle \tilde{x} \rangle = \mho^2(\tilde{\mathscr{S}})$, it follows that x acts trivially on $\mathscr{R}/\langle u \rangle$. Since $|C_{\mathscr{V}/\langle u \rangle}(\tilde{x})| = 4$, it follows that every element of $I(x(\mathscr{R} * \mathscr{V}))$ is conjugate via $\mathscr{R} * \mathscr{V}$ into an element of $\mathscr{R}\langle x \rangle$.

Let $\tau \in I(\mathscr{R}x)$ and let $\mathscr{Z}$ be the inverse image in $\mathscr{R} * \mathscr{V}$ of $C_{(\mathscr{R}*\mathscr{V})/\langle u \rangle}(\tilde{x})$. Then $\mathscr{Z}/\langle u \rangle \cong E_{16}$, $\mathscr{Z} = \mathscr{R} * (\mathscr{V} \cap \mathscr{Z})$, $|\mathscr{Z}| = 2^5$, τ normalizes $\mathscr{Z}$, and $\mathscr{Z}$ normalizes $\langle \tau, u \rangle$. Hence $|C_{\mathscr{Z}\langle \tau \rangle}(\tau)| \geq 2^5$. Also Lemmas 2.2 and 2.3 imply that $\mathscr{R}\langle x \rangle = \mathscr{R} * Z(\mathscr{R}\langle x \rangle)$ where $u \in Z(\mathscr{R}\langle x \rangle)$ and $|Z(\mathscr{R}\langle x \rangle)| = 4$. Note that there is an involution $\mu \in t^G \cap (\mathscr{R}\langle x \rangle - \mathscr{R})$.

Suppose that $Z(\mathscr{R}\langle x \rangle)$ is cyclic. Then $\mathscr{R}\langle x \rangle \cong Z_4 * Q_8$ and $\mathscr{R}\langle x \rangle$ has four conjugacy classes of involutions. If $Z(\mathscr{R}\langle x \rangle) \cong E_4$, then $\mathscr{R}\langle x \rangle - \mathscr{R}$ contains two or four conjugacy classes of involutions. Since x and μ are not conjugate in $\mathscr{S}$, $(\mathscr{R} * \mathscr{V})\langle x\rangle \lhd \mathscr{S}$, and $|\mathscr{S}/((\mathscr{R} * \mathscr{V})\langle x\rangle)| = 8$, it follows that $|C_{\mathscr{S}}(\mu)| \geq 2^7$, which is impossible. This establishes Lemma 12.6.

Since $N/(C * \mathscr{V})$ has 2-exponent at most 4 and

$$N_N((\mathscr{R} * \mathscr{V})\langle t\rangle) = (\mathscr{R} * \mathscr{V})(\langle t\rangle \times \langle \kappa, x\rangle),$$

we have $\mathscr{T}/O_2(N) \cong D_8$. Also $tO_2(N) \sim xtO_2(N)$, $t \sim xt$ in $\mathscr{T}$, and $Z(\mathscr{T}/O_2(N)) = \langle xO_2(N)\rangle$.

Assume that $|\mathscr{R}| = 2^3$. Then $|\mathscr{T}| = 2^{10}$, $\mathscr{T} \notin Syl_2(G)$, and $O_2(N) = \mathscr{R} * \mathscr{V}$ is extraspecial or order 2^7. Let $\mathscr{T}$ be of index 2 in the 2-subgroup $\mathscr{T}_1$ of G. Then $Z(\mathscr{T}_1) = \langle u\rangle$ and $\mathscr{R} * \mathscr{V} \lhd \mathscr{T}_1$ since $I(tO_2(N) \cup xtO_2(N)) = t^{\mathscr{T}}$ by the usual argument. Letting

$$\mathscr{T}_1 \leq \mathscr{S} \in Syl_2(N_G(O_2(N))),$$

we have $z(\mathscr{S}) = \langle u\rangle$, $\mathscr{T} \neq \mathscr{S}$, and $C_{\mathscr{S}/O_2(N)}(t) \cong E_4$. Also $\mathscr{S}/O_2(N)$ acts faithfully on $O_2(N)/\langle u\rangle$ and hence $\mathscr{S}/O_2(N)$ has exponent 8, $\mathscr{S} = \mathscr{T}_1$, and $\mathscr{S}/O_2(N)$ is dihedral or semidihedral or order 16. Also $|\mathscr{S}| = 2^{11}$, $Z(\mathscr{S}/O_2(N)) = \langle xO_2(N)\rangle$ and hence $\mathscr{R} * \mathscr{V}$ char $\mathscr{S}$. Thus $\mathscr{S} \in Syl_2(G)$ and the argument at the end of Lemma 12.6 applies to yield a contradiction.

Consequently, letting $|\mathscr{R}| = 2^a$, we have $a \geq 4$. Then $\mathscr{P} \leq O_2(N)' = R'$ and $\mathscr{P} = \Omega_2(O_2(N)')$ char $O_2(N)$, $C_{O_2(N)}(\mathscr{P}) = C * \mathscr{V}$ char $O_2(N)$, and hence $\mathscr{Q}$ char $O_2(N)$. Also $\langle x, t\rangle$ normalizes the cyclic maximal subgroup of $\mathscr{R}\langle t\rangle$. Hence $[C, x] = 1$ and x stabilizes the chain $\mathscr{R} > \mathscr{R}' > 1$.

Clearly $O_2(N)\langle x\rangle \lhd \mathscr{T}$ and $t^G \cap O_2(N) = \emptyset$. Let $\tilde{\mathscr{T}} = \mathscr{T}/O_2(N)'$. Then $(O_2(N))^{\sim} = \tilde{\mathscr{R}} \times \tilde{\mathscr{V}}$ where $\tilde{\mathscr{R}} \cong E_4$ and $\tilde{\mathscr{V}} \cong E_{16}$. Clearly every involution of $xO_2(N)$ is conjugate via $O_2(N)$ to an involution of $\mathscr{R}x$. Also if $\tau \in I(xO_2(N))$ and $\mathscr{X}$ is the inverse image of $C_{(O_2(N))^{\sim}}(\tilde{x})$ in $O_2(N)$, then $\mathscr{R} \leq \mathscr{X}$, $\mathscr{X} = \mathscr{R} * (\mathscr{V} \cap \mathscr{X})$, τ normalizes $\mathscr{X}$, $|\mathscr{X}| = 2^{a+2}$, $|\mathscr{X}\langle\tau\rangle| = 2^{a+3}$, and $\mathscr{X}$ normalizes $\langle \tau, \mathscr{R}'\rangle$.

Since $[x, C] = 1$, it follows from Lemmas 2.2 and 2.3 that $|Z(\mathscr{R}\langle x\rangle)| = 4$. Suppose that $\mu \in t^G \cap \mathscr{R}x$. If $Z(\mathscr{R}\langle x\rangle)$ is cyclic, then $|C_{\mathscr{R}\langle x\rangle}(\mu)| = 2^3$, $|C_{\mathscr{X}}(\mu)| = 2^5$, and $t^G \cap (\mathscr{R}\langle x\rangle)$ consists of one or two $\mathscr{R}\langle x\rangle$ conjugacy classes. Hence $|C_{\mathscr{T}}(\mu)| \geq 2^6$ which is impossible.

If $Z(\mathscr{R}\langle x\rangle) = \langle \mu, \lambda\rangle$ where $\lambda^2 = 1$, then $\langle x, u\rangle = Z(\mathscr{R}\langle x\rangle)$, $\mathscr{R}\langle x\rangle = \mathscr{R} \times \langle x\rangle$, and $\mathscr{R}$ is dihedral since $t^G \cap \{x, xu\} = \emptyset$. Also, since $t^G \cap \mathscr{R} = \emptyset$, $t^G \cap \mathscr{R}\langle x\rangle$ consists of at most two $\mathscr{R}\langle x\rangle$ conjugacy classes and again we have a contradiction. Thus $t^G \cap (O_2(N)\langle x\rangle) = \emptyset$.

Suppose that $\mathscr{T} \in Syl_2(G)$. Then $t \notin O^2(G)$ by [17, Lemma 5.38], $|\mathscr{T}| \geq 2^{12}$, and $|C| \geq 2^4$. Since $C \leq C_{\mathscr{T}}(x)$, we conclude that $\mho^1(C) \leq C_{\mathscr{T}}(x)' \leq \mathscr{T}' \leq O_2(N)\langle x\rangle$ and we obtain a contradiction from the fact that $x \sim u$ in G. Hence $\mathscr{T} \notin Syl_2(G)$.

Let $\mathscr{T}$ be a maximal subgroup of the 2-subgroup $\mathscr{S}$ of $\mathscr{T}$ and let $\alpha \in \mathscr{S} - \mathscr{T}$. Then $\tau = t^\alpha \in \mathscr{T} - (O_2(N)\langle x, t\rangle)$, $S^\alpha \in Syl_2(C_G(\tau))$, and $\langle u\rangle \neq (S^\alpha)'$. Moreover, since $t^G \cap (O_2(N)\langle x\rangle) = \emptyset$, we conclude that $N/O_2(N)$ has a normal 2-complement. Also $3 \mid |O_{2,2'}(N)/O_2(N)|$ and $O_{2,2'}(N)/O_2(N) \hookrightarrow \mathscr{A}_6$. Thus

$$O_{2,2'}(N)/O_2(N) \cong Z_3 \times Z_3 \quad \text{and} \quad N/O_2(N) \cong \Sigma_3 \wr Z_2.$$

Choose $\mathscr{N} \in Syl_3(N)$ such that $\kappa \in \mathscr{N}$. Then $C_{O_2(N)}(\kappa) = \mathscr{R}$ and $[O_2(N), \kappa] = \mathscr{V}$ are $\mathscr{N}$-invariant. Then $O^2(O_{2,2'}(N)) = \mathscr{V}\mathscr{N} \lhd N$, $\mathscr{V} \lhd N$, $C_{O_2(N)}(\mathscr{V}\mathscr{N}) = \mathscr{R} \lhd N$, and $O_2(N)\mathscr{T}'$ normalizes Q_1 and Q_2. Thus x normalizes Q_1 and Q_2. Set $\tilde{N} = N/O_2(N)$. Then $\tilde{\mathscr{N}} = C_{\tilde{\mathscr{N}}}(\tilde{\tau}) \times [\tilde{\mathscr{N}}, \tilde{\tau}]$ where $|C_{\tilde{\mathscr{N}}}(\tilde{\tau})| = |[\tilde{\mathscr{N}}, \tilde{\tau}]| = 3$. Let $\delta, \omega \in \mathscr{N}$ be such that $C_{\tilde{\mathscr{N}}}(\tilde{\tau}) = \langle\tilde{\delta}\rangle$ and $[\tilde{\mathscr{N}}, \tilde{\tau}] = \langle\tilde{\omega}\rangle$. Thus τ normalizes $\mathscr{V}\langle\delta\rangle$ and $\mathscr{V}\langle\omega\rangle$. By choice of notation, we may assume that $C_{\mathscr{V}}(\delta) = Q_1$ and $C_{\mathscr{V}}(\omega) = Q_2$. Then τ normalizes $[\mathscr{V}, \mathscr{V}\langle\delta\rangle] = Q_2$ and Q_1. But $[\delta, \tau] \in \mathscr{V}$ and hence $[\delta, \tau] \in C_{\mathscr{V}}(Q_1) = Q_2$. Thus δ acts nontrivially on $Q_2\langle\tau\rangle$. Applying Lemma 2.2 and noting that S has no subgroup isomorphic to Q_2, it follows that $Q_2\langle\tau\rangle \cong Z_4 * Q_8$. Thus $C_{Q_2}(\tau) = \langle q_2\rangle$ where $q_2^2 = u$. Hence $C_{\mathscr{T}}(\tau)' = (S^\alpha)' = \langle u\rangle$ which is the final contradiction. Thus the proofs of Lemma 12.1 and Theorem 2 are complete.

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