# REPRESENTATIONS OF ARCHIMEDEAN FUNCTION RINGS

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## 1. Introduction

In [3], Birkhoff and Pierce proved that each complete (resp. $\sigma$-complete) $f$-algebra $A$ with a ring unit is isomorphic to an $f$-algebra of extended functions on the Boolean space of the complete (resp. $\sigma$-complete) Boolean algebra of components of the ring unit of $A$. Another theorem of this type has been obtained by Henriksen and Johnson. In [4], those authors showed that each $\Phi$-algebra $A$ (i.e., an archimedean $f$-algebra with a ring unit) is isomorphic to an $f$-algebra of extended functions on the compact Hausdorff space of all maximal $l$-ideals in $A$. The latter result has been generalized by Johnson [5], who showed that each archimedean $f$-algebra (resp. $f$-ring) which contains no nonzero nilpotents is isomorphic to an $f$-algebra (resp. $f$-ring) of extended functions on a certain locally compact Hausdorff space.

It is our purpose here to show that an archimedean $f$-algebra (resp. $f$-ring) containing no nonzero nilpotents may be represented as an $f$-algebra (resp. $f$-ring) of extended functions on a variety of topological spaces, where the spaces in question are collections of $l$-prime ideals equipped with the hull-kernel topology. This is the content of Theorem 4.2, the main result of the present article. That theorem will be used to give new proofs of the various representation theorems quoted earlier.

Several of the preliminary results appearing here are essentially due to Amemiya. We mention especially Lemma 3.3, which is a slight generalization of the corollary to Theorem 4.5 of [1], and Lemma 4.1, which is Theorem 18.1 of [1]. Amemiya's results are phrased in terms of spectral functions, so in order to keep the present article reasonably self-contained, we take the liberty of restating them in terms of $l$-prime modules. (The relationship between spectral functions and $l$-prime modules was observed in [6].)

## 2. Preliminaries

The present section and the next are devoted to a brief discussion of that part of the theory of lattice-ordered groups ($l$-groups) and lattice-ordered rings ($l$-rings) which is to be applied in subsequent portions of this article. For a more complete discussion of these systems, the reader is referred to [2] and [3].

All $l$-groups $G$ considered here are assumed to be commutative. If $x$ is an element of $G$, then let $x^+ = x \vee 0, x^- = (-x) \vee 0$, and $|x| = x^+ + x^-$.

A subgroup $I$ of $G$ is called an *l-module* if $|x| \leqq |y|$ and $y \in I$ imply that $x \in I$. (This differs from the terminology in [6]; we make the change since

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we wish to reserve the term "ideal" for a ring concept.) An $l$-module $I$ is called an *$l$-prime module* if $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$. It is easy to see that the $l$-module $I$ is an $l$-prime module if, and only if, the quotient $l$-group $G/I$ is totally ordered.

Let $\mathcal{V}$ denote a collection of $l$-prime modules in $G$. For any subset $\mathcal{S}$ of $\mathcal{V}$, the *kernel* of $\mathcal{S}$, denoted by $k(\mathcal{S})$, is the set of elements in $G$ that are common to all of the $l$-modules in $\mathcal{S}$. For any subset $A$ of $G$, the *hull* of $A$, denoted by $h(A)$, is the set of all $P$ in $\mathcal{V}$ such that $P \supseteq A$. It was shown in [6] that the correspondence $\mathcal{S} \to \mathcal{S}^-$, where $\mathcal{S}^- = h(k(\mathcal{S}))$, is a closure operator which makes $\mathcal{V}$ into a topological space. The topology so defined on $\mathcal{V}$ is called the *Stone* or *hull-kernel topology*. If $x$ is an element of $G$, then let $\mathcal{V}_x = \{P \in \mathcal{V} : x \notin P\}$. The collection $\{\mathcal{V}_x : x \in G\}$ is a basis for the open sets for the hull-kernel topology on $\mathcal{V}$. If $k(\mathcal{V}) = \{0\}$, then $\mathcal{V}$ is said to be *dense*.

An $l$-prime module is called *minimal* if there is no $l$-prime module properly contained in it. Proofs of the assertions in the following result can be found in [6]. (A topological space is called *zero-dimensional* if the family of all closed-and-open subsets is a basis.)

LEMMA 2.1. *If $G$ is an $l$-group, then*

(a) *an $l$-prime module $P$ in $G$ is minimal if, and only if, for each $x \in P$ there is an element $y \notin P$ such that $|x| \wedge |y| = 0$;*

(b) *the set $\mathfrak{M}$ of all minimal $l$-prime modules is dense; and*

(c) *the set $\mathfrak{M}$, when equipped with the hull-kernel topology, is a zero-dimensional Hausdorff space; in fact, for each $x$ in $G$, the set $\mathfrak{M}_x$ is closed (and open) in $\mathfrak{M}$.*

Let $A$ be a lattice-ordered ring. A subset of $A$ which is both an $l$-module and a ring ideal will be called an *$l$-ideal*. An *$l$-prime ideal* is an $l$-ideal $I$ which is also an $l$-prime module, i.e., $x \wedge y \in I$ implies that $x \in I$ or $y \in I$. A lattice-ordered ring is called a *function ring* or an *$f$-ring* if $a \wedge b = 0$ and $c \geqq 0$ imply that $ca \wedge b = ac \wedge b = 0$.

It is known (see [1] or [3]) that the multiplicative operation in an archimedean $f$-ring is commutative.[1] (An $l$-group is called *archimedean* if $na \leqq b$ for all nonnegative integers $n$ implies that $a \leqq 0$.) Since the main results of the present article are stated for archimedean $f$-rings, we henceforth assume that all lattice-ordered rings considered here are commutative.

LEMMA 2.2. *The following statements are equivalent in a lattice-ordered ring $A$.*

(i) *$A$ has a dense set of $l$-prime ideals.*

(ii) *$A$ is an $f$-ring.*

(iii) *Every minimal $l$-prime module in $A$ is a ring ideal.*

[1] The semi-normal rings of [1] are the same as $f$-algebras. This can be seen as follows. When the appropriate results in [6] are used to rephrase Amemiya's definition of semi-normal ring in terms of minimal $l$-prime modules, we get condition (iii) of Lemma 2.2. The assertion follows from the equivalence of (ii) and (iii) of that lemma, which is obviously valid without the assumption of commutativity.

*Proof.* (i) *implies* (ii). Let $\mathcal{V}$ be a dense set of $l$-prime ideals. If $a \wedge b = 0$, then $a \in P$ or $b \in P$, where $P$ is any element of $\mathcal{V}$. Thus, if $c \geqq 0$, then $ca \in P$ or $b \in P$, and so $ca \wedge b \in P$. Since $\mathcal{V}$ is dense, we must have $ca \wedge b = 0$.

(ii) *implies* (iii). Let $P$ be a minimal $l$-prime module, and suppose that $a \in P$, $x \in A$. By Lemma 2.1(a), there is an element $b \notin P$ such that $|a| \wedge |b| = 0$. Thus, $(|x|\cdot|a|) \wedge |b| = 0$, and since $P$ is an $l$-prime module, we must have $|x|\cdot|a| \in P$. But $|uv| \leqq |u|\ |v|$ in any $l$-ring, and so $xa \in P$.

(iii) *implies* (i). This is an immediate consequence of Lemma 2.1(b).

The proof of the next result is similar to the proof of Lemma 2 in [8], so will be omitted.

LEMMA 2.3. *If $a$ is a positive element of the $f$-ring $A$, and if $I$ is an $l$-ideal such that $a \notin I$, then there is an $l$-ideal $P$ of $A$ which contains $I$, and which is maximal with respect to the property of not containing $a$; moreover, $P$ is an $l$-prime ideal.*

A collection $S$ of nonzero positive elements of the $l$-group $G$ is called a *meet orthogonal set* if $s \wedge t = 0$ for each pair of distinct elements of $S$. By Zorn's lemma, each $l$-group contains a maximal meet orthogonal set. It is clear that $S$ is a maximal meet orthogonal set if, and only if, $|x| \wedge s = 0$ for each $s \in S$ implies that $x = 0$. A collection $S$ of nonzero elements of the $f$-ring $A$ is called an *orthogonal set* if $st = 0$ for each pair of distinct elements of $S$. Again, Zorn's lemma insures the existence of maximal orthogonal sets. If $A$ has no nonzero nilpotents, then the orthogonal set $S$ is a maximal orthogonal set if, and only if, $xs = 0$ for each $s$ in $S$ implies that $x = 0$. In an $f$-ring, squares are positive, and $a \wedge b = 0$ implies that $ab = 0$. These remarks can be used to prove the following result.

LEMMA 2.4. *Let $A$ be an $f$-ring which contains no nonzero nilpotents, and let $S$ be a maximal orthogonal subset of $A$. Then the set $\{s^2 : s \in S\}$ is both maximal orthogonal and maximal meet orthogonal.*

## 3. Lattice-ordered groups

If $I$ is an $l$-module in the $l$-group $G$, then we shall also use the symbol $I$ to denote the natural homomorphism of $G$ upon the quotient $l$-group $G/I$. By $G_+$, we mean the set of all positive elements in $G$. Let $a$ be a nonzero positive element of $G$, and let $P$ be an $l$-prime module of $G$ such that $a \notin P$. For $x \in G_+$, let $(x/a, P)$ be the infimum of all rational numbers $m/n$, $n > 0$, for which $nP(x) \leqq mP(a)$. (The infimum of the empty set is understood to be $+\infty$.) If $x$ is an arbitrary element of $G$, then put $(x/a, P) = (x^+/a, P) - (x^-/a, P)$. Since $x^+ \wedge x^- = 0$, either $x^+ \in P$ or $x^- \in P$, and so $(x/a, P)$ is well defined. It is easy to see that

$$\begin{aligned}(x/a, P) &= \inf\{m/n, n > 0 : (ma - nx)^- \in P\}\\ &= \inf\{m/n, n > 0 : (ma - nx)^+ \notin P\}.\end{aligned}$$

Techniques identical to those used to prove the first part of Theorem 5.1 in [6] will yield a proof of statement (a) of the following result. Assertion (b) follows from a routine calculation, using the fact that $G/P$ is a totally ordered group when $P$ is an $l$-prime module in $G$. If $G$ is a vector lattice, and if $I$ is an $l$-module of the underlying $l$-group, then $I$ is closed under multiplication by real numbers, i.e., $I$ is a linear subspace of $G$. This observation together with (a) and (b) yields statement (c) below.

LEMMA 3.1. *Let $\mathcal{V}$ be a dense collection of $l$-prime modules in the $l$-group $G$, let $T$ be a meet orthogonal subset of $G$, and let the set $\mathfrak{X} = \bigcup \{\mathcal{V}_t : t \in T\}$ be equipped with the hull-kernel topology. For $P \in \mathfrak{X}$, and $x \in G$, put $\bar{x}(P) = (x/t, P)$, where $t$ is the unique element of $T$ for which $t \notin P$. Then,*

(a) *$\bar{x}$ is a continuous mapping from $\mathfrak{X}$ to the two-point compactification of the real line;*

(b) *if $\bar{x}(P)$ and $\bar{y}(P)$ are finite, then*

$$\overline{x \wedge y}(P) = \bar{x}(P) \wedge \bar{y}(P),$$

$$\overline{x \vee y}(P) = \bar{x}(P) \vee \bar{y}(P),$$

$$\overline{x + y}(P) = \bar{x}(P) + \bar{y}(P); \quad \textit{and}$$

(c) *if $G$ is a vector lattice, and if $\alpha$ is a real number, then in addition to* (a) *and* (b), *we have*

$$\overline{\alpha x}(P) = \alpha \bar{x}(P),$$

*provided $\bar{x}(P)$ is finite.*

If $B$ is a nonempty subset of the $l$-group $G$, then $B^{\perp}$ denotes the set of all $x$ in $G$ such that $|x| \wedge |b| = 0$ for each $b$ in $B$. If $B$ consists of a single element $b$, we write $b^{\perp}$ instead of $\{b\}^{\perp}$.

The next result can be found in [6]. It is stated here for ease of reference.

LEMMA 3.2. *If $\mathcal{V}$ is a dense collection of $l$-prime modules in the $l$-group $G$, then $b^{\perp} = k(\mathcal{V}_b)$ for each element $b$ of $G$.*

LEMMA 3.3. *Let $\mathcal{V}$ be a dense collection of $l$-prime modules in the archimedean $l$-group $G$, and let $a, b$ be nonzero elements of $G_+$ such that $\mathcal{V}_b \subseteq \mathcal{V}_a$. If $x$ is an element of $G_+$ such that $(x/a, P) = 0$ for each $P \in \mathcal{V}_b$, then $x \in b^{\perp}$.*

*Proof.* If $(x/a, P) = \inf \{m/n : n > 0, nP(x) \leqq mP(a)\} = 0$ for each $P \in \mathcal{V}_b$, then $nP(x) \leqq P(a)$ for each $n \in N$, and each $P \in \mathcal{V}_b$, where $N$ denotes the set of positive integers. Thus, $(a - nx)^- \in P$ for each $n \in N$, and each $P \in \mathcal{V}_b$. By Lemma 3.2, $(a - nx)^- \in b^{\perp}$, i.e., $(a - nx)^- \wedge b = 0$ for each $n \in N$. Therefore, $b \in ((nx - a)^+)^{\perp}$, and since Lemma 3.2 implies that $c^{\perp}$ is, in particular, a subgroup of $G$ for arbitrary $c$ in $G$, we conclude that $(nx - a)^+ \wedge nb = 0$ for each $n \in N$. Hence,

$$(nx - a) \wedge nb \leqq (nx - a)^+ \wedge nb = 0.$$

Now

$$nx \wedge (nb + a) - a = (nx - a) \wedge nb \leqq 0,$$

and so

$$n(x \wedge b) \leqq nx \wedge nb \leqq nx \wedge (nb + a) \leqq a.$$

We have shown that $n(x \wedge b) \leqq a$ for each $n \in N$, and since $G$ is archimedean, we conclude that $x \wedge b \leqq 0$, i.e., $x \wedge b = 0$. This completes the proof of the lemma.

## 4. The representation theorem

LEMMA 4.1. *If $P$ is an $l$-prime ideal of the $l$-ring $A$, and if $a$, $b$ are positive elements of $A$ such that $ab \notin P$, then $(xy/ab, P) = (x/a, P)(y/b, P)$ for all $x$, $y$ in $A$, provided the right-hand side of this equation makes sense.*

*Proof.* We may assume that $x$ and $y$ are elements of $A_+$. If $m/n$ and $m_1/n_1$ are positive rationals such that $0 \leqq nP(x) \leqq mP(a)$, and $0 \leqq n_1 P(y) \leqq m_1 P(b)$, then $0 \leqq nn_1 P(xy) \leqq mm_1 P(ab)$. Now

$$\inf \{mm_1/nn_1 : nP(x) \leqq mP(a),\ n_1 P(y) \leqq m_1 P(b)\} = (x/a, P)(y/b, P),$$

and so $(xy/ab, P) \leqq (x/a, P)(y/b, P)$.

Suppose that there is a positive rational $m/n$ for which

$$(xy/ab, P) < m/n < (x/a, P)(y/b, P).$$

Then either $(x/a, P) > m/n$, or $(y/b, P) > 1$, i.e., either $nP(x) > mP(a)$, or $P(y) > P(b)$, and thus $nP(xy) > mP(ab)$. But also, $nP(xy) \leqq mP(ab)$, which is impossible. This completes the proof of the lemma.

By an *extended* (*real-valued*) *function* on the topological space $X$, we mean a continuous mapping of $X$ into the two-point compactification of the real line $R$ which is real-valued on an everywhere dense subset of $X$. Let $D(X)$ denote the set of all extended functions on $X$. If $f$, $g$ are in $D(X)$, and if $\alpha$ is in $R$, then the functions $\alpha f$, $f \wedge g$, and $f \vee g$, which are defined pointwise, are in $D(X)$. Let $R(f)$ denote the set of points in $X$ at which $f$ is real-valued. If there is a function $h$ in $D(X)$ such that $h(x) = f(x) + g(x)$ for each $x$ in $R(f) \cap R(g)$, then $h$ is called the *sum* of $f$ and $g$, and is denoted by $f + g$. If there is a function $k$ in $D(X)$ such that $k(x) = f(x)g(x)$ for each $x$ in $R(f) \cap R(g)$, then $k$ is called the *product* of $f$ and $g$, and is denoted by $fg$. Since $R(f) \cap R(g)$ is dense in $X$, the sum and product are uniquely defined, provided they exist. A nonempty subset of $D(X)$ which is a ring with respect to the above operations of sum and product is called a *ring of extended functions* on $X$. Algebras, $f$-rings, and $f$-algebras of extended functions are defined analogously.

We now come to the main representation theorem. (An *isomorphism* of an $l$-ring (resp. $l$-algebra) is a ring (resp. algebra) isomorphism which preserves the lattice operations.)

THEOREM 4.2. *Let $A$ be an archimedean $f$-ring (resp. $f$-algebra) which contains no nonzero nilpotents, let $\mathcal{V}$ be a dense collection of $l$-prime ideals in $A$, and let $T$ be a subset of $A$ whose elements are squares of a maximal orthogonal subset of $A_+$. If the set $\mathfrak{X} = \bigcup \{\mathcal{V}_t : t \in T\}$ is equipped with the hull-kernel topology, then $A$ is isomorphic to an $f$-ring (resp. $f$-algebra) of extended functions on $\mathfrak{X}$.*

*Proof.* First assume that $A$ is an $f$-ring.

To see that a set $T$ with the prescribed properties exists, take any maximal orthogonal subset $S_1$, and let $S = S_1^2 = \{s^2 : s \in S_1\}$. By Lemma 2.4, $S$ is, in particular, a maximal orthogonal set of positive elements. Now put $T = S^2$. Another application of Lemma 2.4 shows that $T$ is a subset of $A_+$ which is both maximal orthogonal and maximal meet orthogonal.

Since $T$ is a meet orthogonal set, the sets $\mathcal{V}_t$, $t \in T$, are pairwise disjoint; since $\mathcal{V}$ is dense, none of these sets is empty. If $P \in \mathfrak{X}$, then let $t$ be the unique element of $T$ for which $t \notin P$. For an element $x \in A_+$, put $x^*(P) = (tx/t, P)$. If $x$ is an arbitrary element of $A$, then put $x^*(P) = (x^+)^*(P) - (x^-)^*(P)$. Since $x^+ \wedge x^- = 0$, we have $tx^+ \wedge tx^- = 0$. Thus, $tx^+ \in P$ or $tx^- \in P$, and so $x^*(P)$ is well defined.

We now show that $x^*$ is an extended function on $\mathfrak{X}$. For each $y$ in $A$, let $\bar{y}(P) = (y/t, P)$. Then $x^*(P) = \overline{tx}(P)$, and it follows from (a) of Lemma 3.1 that $x^*$ is a continuous mapping of $\mathfrak{X}$ into the two-point compactification of the real line.

To show that $R(x^*)$ is dense in $\mathfrak{X}$, we may assume that $x$ is an element of $A_+$. Let $\mathfrak{X}_b$ be a nonempty basic open set in $\mathfrak{X}$. Thus, $b \neq 0$, and we may take $b \in A_+$. Suppose that $x^*(P) = +\infty$ for each $P$ in $\mathfrak{X}_b$. Now $\mathfrak{X}_b = \mathcal{V}_b \cap \mathfrak{X} = \bigcup \{\mathcal{V}_{b \wedge t} : t \in T\}$. Since $(tx/t, P) = +\infty$ for each $P$ in $\mathcal{V}_{b\wedge t}$, it follows that $(t/tx, P) = 0$ for each such $P$. Clearly, $tx \notin P$ for $P \in \mathcal{V}_{b\wedge t}$, and so by Lemma 3.3, $t$ is in $(b \wedge t)^\perp$, i.e., $b \wedge t = 0$. Thus, $b \in T^\perp$, and since $T$ is a maximal meet orthogonal set, $b = 0$, a contradiction.

We have shown that the set $A^* = \{x^* : x \in A\}$ is a subset of $D(\mathfrak{X})$. It follows easily from (b) of Lemma 3.1 that

$$(x + y)^* = x^* + y^*, \qquad (x \wedge y)^* = x^* \wedge y^*, \quad \text{and} \quad (x \vee y)^* = x^* \vee y^*$$

for all $x$, $y$ in $A$.

Now let $P \in \mathfrak{X}$, and let $t$ be the unique element of $T$ for which $t \notin P$. Suppose that $t = s^2$, where $s \in S$. If $x^*(P)$ and $y^*(P)$ are finite, then by Lemma 4.1 we have

$$(xy)^*(P) = (s^2xy/s^2, P) = (sx/s, P)(sy/s, P).$$

Another application of Lemma 4.1 gives

$$(s^2z/s^2, P) = (s/s, P)(sz/s, P) = (sz/s, P)$$

for each $z$ in $A$, and so $(xy)^*(P) = x^*(P)y^*(P)$.

Thus, the set $A^*$ is an $l$-ring of extended functions on $\mathfrak{X}$, and the mapping $x \rightarrow x^*$ is a homomorphism of the $f$-ring $A$ upon $A^*$.

To show that the above mapping is an isomorphism, let $x^* = 0$. We may suppose that $x$ is an element of $A_+$. If $t \in T$, then $x^*(P) = (tx/t, P) = 0$ for each $P \in \mathcal{V}_t$, and so Lemma 3.3 insures that $tx \in t^{\perp}$. Therefore, $tx \wedge t = 0$ for each $t \in T$. But then $tx \wedge tx = 0$, or $tx = 0$ for each $t \in T$. Since $T$ is a maximal orthogonal set, we have $x = 0$. Since $A^*$ is the isomorphic image of an $f$-ring, it is also an $f$-ring.

This completes the proof of the theorem when $A$ is an $f$-ring. The assertion for $f$-algebras follows from the result for $f$-rings and (c) of Lemma 3.1.

## 5. Locally compact spaces

Throughout this section, we let $\mathcal{P}$ denote the collection of all $l$-prime ideals of the $f$-ring $A$. In virtue of Lemma 2.2, $\mathcal{P}$ is dense.

If $t$ is a nonzero positive element of $A$, then let $\mathcal{P}^t$ denote the set of all $l$-ideals in $A$ which are maximal with respect to the property of not containing $t$. By Lemma 2.3, such ideals exist, and moreover, each is prime, i.e., $\mathcal{P}^t \subseteq \mathcal{P}_t$.

LEMMA 5.1. *If $t$ is a nonzero positive element of the $f$-ring $A$, then the set $\mathcal{P}^t$ is a compact Hausdorff space when equipped with the hull-kernel topology.*

*Proof.* We first prove that the set $\mathcal{P}_t$ is compact. Consider a collection of relatively closed subsets of $\mathcal{P}_t$ whose intersection is empty. We may suppose that they have the form $h(b) \cap \mathcal{P}_t$, where $b$ ranges over some subset $B$ of $A_+$, and where $h(b) = \{P \in \mathcal{P}: b \in P\}$. Thus, $h(B) \cap \mathcal{P}_t = \emptyset$, or $h(B) \subseteq h(t)$. In virtue of Lemma 2.3, each $l$-ideal is the intersection of all $l$-prime ideals containing it, and so $t \in (B)$, the $l$-ideal generated by $B$. Hence, there exist nonnegative integers $n_1, \cdots, n_k$, elements $b_1, \cdots, b_k$ in $B$, and elements $a_1, \cdots, a_k$ in $A_+$ such that

$$t \leqq a_1 b_1 + \cdots + a_k b_k + n_1 b_1 + \cdots + n_k b_k .$$

Since $\mathcal{P}_{x+y} = \mathcal{P}_x \cup \mathcal{P}_y$, and $\mathcal{P}_{xy} \subseteq \mathcal{P}_x \cap \mathcal{P}_y$ for any positive elements $x$, $y$, it follows that $\mathcal{P}_t \subseteq \bigcup_1^k \mathcal{P}_{b_i}$, or $\bigcap_1^k (h(b_i) \cap \mathcal{P}_t) = \emptyset$. This proves that $\mathcal{P}_t$ is compact.

To show that $\mathcal{P}^t$ is compact, let $\mathcal{P}^t \subseteq \bigcup \{\mathcal{P}_b : b \in B\}$, where $B$ is some subset of $A_+$. If $P \in \mathcal{P}_t$, then by Lemma 2.3, there is an element $Q$ in $\mathcal{P}^t$ such that $P \subseteq Q$. Moreover, there is an element $b$ in $B$ such that $Q \in \mathcal{P}_b$. Hence, the collection $\{\mathcal{P}_b : b \in B\}$ is an open covering of the compact set $\mathcal{P}_t$, and so there exist finitely many elements $b_1, \cdots, b_k$ in $B$ such that $\mathcal{P}_t \subseteq \bigcup_1^k \mathcal{P}_{b_i}$. But $\mathcal{P}^t \subseteq \mathcal{P}_t$, so the former set is compact.

To see that $\mathcal{P}^t$ is a Hausdorff space, let $P$ and $Q$ be distinct elements of $\mathcal{P}^t$, and choose $x \in P \cap A_+$ such that $x \notin Q$, and $y \in Q \cap A_+$ such that $y \notin P$. Then $P \in h(x) \cap \mathcal{P}_y$, and $Q \in h(y) \cap \mathcal{P}_x$. Now

$$(x - y)^+ + y \geqq x, \qquad (x - y)^- + x \geqq y,$$

so

$$\mathcal{P}_x \subseteq \mathcal{P}_y \cup \mathcal{P}_{(x-y)^+}, \quad \text{and} \quad \mathcal{P}_y \subseteq \mathcal{P}_x \cup \mathcal{P}_{(x-y)^-}.$$

Thus, $P \in \mathcal{P}_{(x-y)^-}$, $Q \in \mathcal{P}_{(x-y)^+}$, and these two sets are disjoint. Since $P$ and $Q$ have disjoint neighborhoods in $\mathcal{P}$, they obviously also have disjoint neighborhoods in $\mathcal{P}^t$. This completes the proof of the lemma.

LEMMA 5.2. *If $t$ is a nonzero positive element of the archimedean f-ring $A$, then $\mathcal{P}^t$ is dense in $\mathcal{P}_t$.*

*Proof.* By Lemma 3.2, we have $t^{\perp} = k(\mathcal{P}_t)$, so it suffices to show that $k(\mathcal{P}^t) \subseteq t^{\perp}$. If $x$ is an element of $A_+$ such that $x \notin t^{\perp}$, then by Lemma 3.3, there is an element P $\in \mathcal{P}_t$ such that $(x/t, P) > 0$. Choose a positive rational $m/n$ so that $(x/t, P) > m/n$; then $(mt - nx)^- \notin P$, and so $(mt - nx)^+ \in P$, since $P$ is an $l$-prime ideal. By Lemma 2.3, there is an element $Q \in \mathcal{P}^t$ such that $P \subseteq Q$. Now

$$0 = ((mt - nx)^+/t, Q) = ((mt - nx)/t, Q) \vee 0,$$

or $(x/t, Q) \geqq m/n > 0$. Thus, $x \notin Q$, and so $k(\mathcal{P}^t) \subseteq t^{\perp}$.

The following result has been proved in [5]. The proof given here is based on Theorem 4.2.

THEOREM 5.3. *An archimedean f-ring (resp. f-algebra) $A$ which contains no nonzero nilpotents is isomorphic to an f-ring (resp. f-algebra) of extended functions on some locally compact Hausdorff space.*

*Proof.* Let $S$ be a maximal orthogonal subset of $A_+$, and put $\mathcal{V} = \bigcup\{\mathcal{P}^t : t \in T\}$, where $T = S^2$. By Lemma 5.2, $k(\mathcal{P}^t) = t^{\perp}$, and thus $k(\mathcal{V}) = T^{\perp}$. Lemma 2.4 insures that $T$ is a maximal meet orthogonal set, so $\mathcal{V}$ is dense. By letting $\mathfrak{X} = \bigcup \{\mathcal{V}_t : t \in T\}$, it is clear that $\mathfrak{X} = \mathcal{V}$. By Theorem 4.2, $A$ is isomorphic to an $f$-ring (resp. $f$-algebra) of extended functions on $\mathfrak{X}$. It is an immediate consequence of Lemma 5.1 that $\mathfrak{X}$ is a locally compact Hausdorff space.

## 6. Zero-dimensional spaces

Recall that each minimal $l$-prime module in an $f$-ring $A$ is a ring ideal (Lemma 2.2), and that the set of all minimal $l$-prime modules in any lattice-ordered group is a dense zero-dimensional Hausdorff space (Lemma 2.1). Applying Theorem 4.2 with $\mathcal{V}$ the set of all minimal $l$-prime modules in $A$, we obtain

THEOREM 6.1. *An archimedean f-ring (resp. f-algebra) which contains no nonzero nilpotents is isomorphic to an f-ring (resp. f-algebra) of extended functions on some zero-dimensional Hausdorff space.*

Let $\mathfrak{M}$ denote the space of all minimal $l$-prime modules in the lattice-ordered group $G$, and let $\mathfrak{C} = \{\mathfrak{M}_a : a \in G_+\}$. For each pair of elements $a$, $b$ in $G_+$, we have $\mathfrak{M}_{a \wedge b} = \mathfrak{M}_a \cap \mathfrak{M}_b$, and $\mathfrak{M}_{a \vee b} = \mathfrak{M}_a \cup \mathfrak{M}_b$. Thus, $\mathfrak{C}$, when partially ordered by set inclusion, is a (distributive) lattice.

A positive element 1 in $G$ is called a *weak order unit* if the set $\{1\}$ is maximal meet orthogonal. If $G$ has a weak order unit, then Lemma 2.1(a) implies that $\mathfrak{M} = \mathfrak{M}_1$.

A *component* of the weak order unit 1 is an element $e$ such that $e \wedge (1 - e) = 0$. It is known that the set of all components of 1 is a Boolean algebra when equipped with the lattice operations which it inherits from $G$.

An $l$-module $I$ is called a *direct summand* of $G$ if there is an $l$-module $J$ such that $I \cap J = \{0\}$, and $G = I + J$. If $I$ is a direct summand of $G$ with complementary direct summand $J$, then it is easy to see that $J = I^{\perp}$.

The following lemma is a consequence of certain of the developments in [1] and [6], but for the sake of completeness, we give an independent proof.

LEMMA 6.2. *Let $G$ be a lattice-ordered group with a weak order unit* 1, *and suppose that for each $a \in G_+$, the $l$-module $a^{\perp}$ is a direct summand of $G$. Then $\mathfrak{C}$ is a Boolean algebra, and $\mathfrak{M}$ is homeomorphic to the Boolean space of $\mathfrak{C}$. Moreover, $\mathfrak{C}$ and the Boolean algebra $B$ of all components of* 1 *are isomorphic.*

*Proof.* For $e$ in $B$, put $\beta(e) = \mathfrak{M}_e$. It is clear that the mapping $\beta$ preserves finite infima and suprema. To show that this mapping is onto $\mathfrak{C}$, let $G = a^{\perp} + a^{\perp\perp}$, where $a \in G_+$, and write $1 = u + v$, where $u \in a^{\perp}$ and $v \in a^{\perp\perp}$. Since $1 = u \vee v$, and since $G$ is a distributive lattice, we have $a \wedge 1 = a \wedge v$, so $\mathfrak{M}_a = \mathfrak{M}_{a \wedge 1} = \mathfrak{M}_{a \wedge v} = \mathfrak{M}_a \cap \mathfrak{M}_v$. Now $v \in a^{\perp\perp}$ implies that $a^{\perp} \subseteq v^{\perp}$. By Lemma 3.2, $k(\mathfrak{M}_a) \subseteq k(\mathfrak{M}_v)$, and by Lemma 2.1(c), $\mathfrak{M}_v \subseteq \mathfrak{M}_a$. Thus, $\mathfrak{M}_a = \mathfrak{M}_v$, and $v$ is obviously a component of 1.

To see that $\beta$ is one-to-one, let $\mathfrak{M}_e = \mathfrak{M}_f$, where $e$ and $f$ are elements of $B$. By Lemma 3.2, we have $e^{\perp} = f^{\perp}$; therefore, $(1 - e) \wedge f = 0$, and $(1 - f) \wedge e = 0$. Now $1 = e \vee (1 - e)$ implies that $f = f \wedge 1 = f \wedge e$; and $1 = f \vee (1 - f)$ implies that $e = e \wedge 1 = e \wedge f$.

For $P \in \mathfrak{M}$, let $f(P) = \{U \in \mathfrak{C} : P \notin U\}$. It is easy to see that $f(P)$ is a prime ideal in the Boolean algebra $\mathfrak{C}$, and that the mapping $f$ is a homeomorphism of $\mathfrak{M}$ into the Boolean space of $\mathfrak{C}$. To show that this mapping is onto, let $Q$ be a prime ideal in $\mathfrak{C}$, and let $P = \{x \in G : \mathfrak{M}_x \in Q\}$. If $|x| \leqq |y|$, and $y \in P$, then the identity $\mathfrak{M}_z = \mathfrak{M}_{|z|}$ implies that $x \in P$; if $x$ and $y$ are in $P$, then the inequality $|x + y| \leqq |x| + |y|$, the previous identity, and the identity $\mathfrak{M}_{u+v} = \mathfrak{M}_u \cup \mathfrak{M}_v$, which holds for $u, v$ in $G_+$, imply that $x + y \in P$. Thus, $P$ is an $l$-module. It is easy to see that an $l$-module $I$ is an $l$-prime module if, and only if, $x \wedge y = 0$ implies that $x \in I$ or $y \in I$. Hence, if $x \wedge y = 0$, then $\mathfrak{M}_x \cap \mathfrak{M}_y = \mathfrak{M}_0 = \emptyset$, and so $\mathfrak{M}_x \in Q$ or $\mathfrak{M}_y \in Q$, i.e., $x \in P$ or $y \in P$. To show that $P$ is minimal, let $x \in P$; since $\mathfrak{C}$ is a Boolean algebra, $\mathfrak{M}_x$ has a complement, say $\mathfrak{M}_y$, in $\mathfrak{C}$. We may assume that both $x$ and $y$ are in $G_+$. Because $Q$ is a proper ideal, $\mathfrak{M}_y \notin Q$, i.e., $y \notin P$. Now $\mathfrak{M}_x \cap \mathfrak{M}_y = \mathfrak{M}_{x \wedge y} = \emptyset$, and since $\mathfrak{M}$ is dense, we must have $x \wedge y = 0$. By Lemma 2.1(a), $P \in \mathfrak{M}$. The inclusion $f(P) \subseteq Q$, and the fact that prime ideals in a Boolean algebra are maximal imply that $f(P) = Q$. This completes the proof of the lemma.

It is clear that a ring unit 1 in an $f$-ring $A$ is a weak order unit. With the notation as in Theorem 4.2, let $\mathcal{V} = \mathfrak{M}$, the space of all minimal $l$-prime modules in $A$; taking $T = \{1\}$, we have $\mathfrak{X} = \mathfrak{M}$. An archimedean $f$-ring with a ring unit has no nonzero nilpotents. (A proof of this can be found in [3].) From these remarks and Lemma 6.2, we get

THEOREM 6.3. *Let $A$ be an archimedean f-ring* (*resp. f-algebra*) *with ring unit* 1, *and suppose that for each element $a \in A_+$, the l-module $a^\perp$ is a direct summand of $A$. Then $A$ is isomorphic to an f-ring* (*resp. f-algebra*) *of extended functions on the Boolean space of the Boolean algebra of components of* 1.

It is known (see, e.g., [7]) that the $l$-module $a^\perp$ is a direct summand of $G$ for each positive element of a $\sigma$-complete lattice-ordered group $G$. Thus, the previous result yields the representation theorem of Birkhoff and Pierce which was quoted in the introduction.

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