VII. Axioms and Their Application

§0. Introduction

In the first section we introduce the κ-e.c.c. (κ-extra chain condition). We prove that if we have an iteration of length $\leq \kappa$ of $(< \omega_1)$-proper forcing notions which do not add reals, and if, moreover each forcing used is $\mathbb{D}$-complete for some simple $\aleph_1$-completeness system $\mathbb{D}$, *then* the limit satisfies the κ-c.c. . This helps us e.g. in iterations of length ω_2 of forcings among which none add reals, but each adds many subsets of $\aleph_1$.

In the second section we deal with forcing axioms; essentially our knowledge is good when we want $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and reasonable when we want $2^{\aleph_0} = \aleph_1$ and even $2^{\aleph_0} = \aleph_2$. In the third section we discuss applications of the forcing axiom which is consistent with CH as just mentioned. In the fourth section we discuss the forcing axiom which is consistent with $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, and in the fifth section we give an example of a CS iteration collapsing $\aleph_1$ only in the limit. See relevant references in the sections.

§1. On the κ-Chain Condition, When Reals Are Not Added

When we prove various consistency results by iterating proper forcings, we often have to check that the $\aleph_2$-chain condition holds.

Remember, we deal with a CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ where $\underset{\sim}{Q}_i$ is a P_i-name of a proper forcing (in V^{P_i}), $P_\alpha = \{f : \mathrm{Dom}(f)$ is a countable subset of α and $i \in \mathrm{Dom}(f)$ implies $\emptyset \Vdash_{P_i}$ "$f(i) \in \underset{\sim}{Q}_i$", i.e. $f(i)$ is a P_i-name of a member of $\underset{\sim}{Q}_i\}$. So, P_α is proper by III 3.2. Here, we concentrate on the case when no real is added, in fact when we have a sufficient condition for it. The case without this restriction will be discussed again in VIII §2 .

Remark. Note that even if $\Vdash_{P^i}$ "$|\underset{\sim}{Q}^1| = \aleph_1$", still may be $|P_2| = 2^{\aleph_1}$, as there may be many P_1-names of elements of Q_1.

1.1. Lemma. If κ is regular, $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$ and $V^{P_\alpha} \models$ "$|Q_\alpha| < \kappa$" *then* P_κ satisfies the κ-chain condition.

Proof. See III 4.1.

1.2. Definition. P satisfies the κ-e.c.c. (κ-extra chain condition) provided that there is a two place relation R on P (usually pRq is intended to mean that "p and q have a least upper bound") such that:

A) for any $p_i \in P$ (for $i < \kappa$) there are pressing down functions $f_n : \kappa \to \kappa$ (i.e. $(\forall \alpha) f_n(\alpha) < 1 + \alpha$) for $n < \omega$ such that: $0 < i, j < \kappa$ and $\bigwedge_{n<\omega}(f_n(i) = f_n(j))$ imply $p_i R p_j$.

B) if in P we have $p_0 \le p_1 \le p_2 \le \dots \le p_n \le p_{n+1} \le \dots p_\omega$ and $q_0 \le q_1 \le q_2 \le \dots \le q_n \le q_{n+1} \le \dots q_\omega$ and $\bigwedge_n p_n R q_n$, *then* there is an r such that $\bigwedge_n r \ge p_n$ & $\bigwedge_n r \ge q_n$.

1.2A Remark. This is very similar to the condition used in [Sh:80] (and similar to a work of Baumgartner, see VIII 1.1, 1.1A(1)). The real difference is the absence of $\aleph_1$-completeness. The fact that there (in [Sh:80], clause (C) there is a parallel to clause (A) here) we use only one function and closed unbounded C, and demand $i, j \in C$, $\mathrm{cf}(i) = \mathrm{cf}(j) = \aleph_1$, is just a variant form which was more convenient to represent there. The role of p_ω, q_ω is just to show that $\{p_n : n < \omega\}$ and $\{q_n : n < \omega\}$, each has an upper bound (so for a $\aleph_1$-complete forcing they are not needed, hence, also $\ell g(\bar{Q}) \leq \kappa$ is not needed). Even closer is Stanley and Shelah [ShSt:154], [ShSt:154a]. We can ask in (A) only that there are p'_i (for $i < \kappa$) such that $P \models p_i \leq p'_i$ and $[0 < i, j < \kappa, \bigwedge_n (f_n(i) = f_n(j)) \Rightarrow p'_i R p'_j]$.

1.3. Lemma. Suppose $V \models 2^{\aleph_0} = \aleph_1$ while $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$ and κ is regular. Suppose further that $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha_0 \leq \kappa \rangle$ is a CS iteration.

In addition:

(a) Each Q_α is $(< \omega_1)$-proper $(= \alpha$-proper for each $\alpha < \omega_1)$;

(b) Each Q_α is $\mathbb{D}$-complete for some $\aleph_1$-completeness system *from* V (see definition in V §7) and, for simplicity, the set of elements of Q_α is a subset of $\lambda = 2^{|P_{\alpha_0}|}$ and $\lambda > \kappa$;

(c) Each Q_α satisfies the κ-e.c.c.

Then P_{α_0} satisfies the κ-c.c. (= κ-chain condition).

1.3A Remark.

1) Compare with Theorem 7.1 from V. We add part (c) to the hypothesis (and $\alpha_0 \leq \kappa$), and get the κ-c.c. of P_{α_0}. In fact, to prove 1.3, we shall repeat the proof of V7.1, after an appropriate preparatory step.

2) We can weaken Definition 1.2 so that the proof of 1.3 still works, e.g. by strengthening the hypothesis on the p_n, q_n in clause (B). For example, we could have demanded that $q_{n+1} \in \mathcal{I}[q_n]$, $p_{n+1} \in \mathcal{I}[p_n]$ where for $r \in P$ we have that $\mathcal{I}[r] \subseteq P$ is a dense subset of P (or even $q_{n+1} \in \mathcal{I}[q_0, q_1, \dots q_n]$, $p_{n+1} \in \mathcal{I}[p_o, \dots, p_n]$).

3) Note that when P is $\aleph_1$-complete, 1.2(B) is satisfied for $R =$ "having a least upper bound". Also 1.2(A) can be weakened by e.g. demanding the conclusion to be true only for $i, j \in A$, for some A in some appropriate filter on $\kappa \times \kappa$, *or* for some A which is not in an appropriate ideal which is precipituous.

Proof. As in the proof of Lemma III 4.1 we can conclude that if the Lemma holds for each $\alpha_0 < \kappa$, then it also holds for $\alpha_0 = \kappa$. So, w.l.o.g. $\alpha_0 < \kappa$. Let $\underset{\sim}{R}_i$ be (a P_i-name of a 2-place relation) exemplifying Definition 1.2 for $\underset{\sim}{Q}_i$. Let $p_\alpha \in P_{\alpha_0}$ ($\alpha < \kappa$) be given. We now define, by induction on $n < \omega$, countable models N^n_α (for all $\alpha < \kappa$ simultaneously) such that:

i) $N^n_\alpha \prec (H(\lambda), \in), P_{\alpha_0} \in N^n_\alpha, \bar{Q} \in N^n_\alpha, p_\alpha \in N^0_\alpha, ||N^n_\alpha|| = \aleph_0$, and $\alpha \in N^0_\alpha$.

ii) $N^n_\alpha \prec N^{n+1}_\alpha$, and the additional conditions below are satisfied.

For $n = 0$, choose any N^0_α satisfying (i).

If we have defined N^n_α for n, let $N^n_\alpha \cap P_{\alpha_0} = \{p^n_{\alpha,\ell} : \ell < \omega\}$, $p^n_{\alpha,0} = p_\alpha$.

For $i < \alpha_0$ and $\ell < \omega$, consider the sequence $\langle p^n_{\alpha,\ell}(i) : \alpha < \kappa\rangle$ (if $i \neq \mathrm{Dom}(p^n_{\alpha,\ell})$ then we are stipulating $p^n_{\alpha,\ell}(i) = \emptyset_{\underset{\sim}{Q}_i}$). In V^{P_i} it is a sequence of length κ of elements of Q_i. But Q_i (in V^{P_i}) satisfies the κ-e.c.c., so there are ω-sequences of functions $\bar{f}^n_{i,\ell} = \langle f^n_{i,\ell,k} : k < \omega\rangle$ exemplifying that condition (A) from Definition 1.2 holds. In V we have P_i-names $\underset{\sim}{\bar{f}}{}^n_{i,\ell}$ for $\bar{f}^n_{i,\ell}$.

We now define N^{n+1}_α such that $N^n_\alpha \prec N^{n+1}_\alpha \prec (H(\lambda), \in)$, and

iii) $\langle\langle \underset{\sim}{\bar{f}}{}^n_{i,\ell} : i < \alpha_0\rangle, \langle p^n_{\gamma,\ell} : \gamma < \kappa\rangle : \ell < \omega\rangle \in N^{n+1}_\alpha, N^n_\alpha \in N^{n+1}_\alpha$ and $\langle N^n_\beta : \beta < \kappa\rangle \in N^{n+1}_\alpha$.

Let $N^\omega_i = \bigcup_{n<\omega} N^n_i$.

Now, using that $(\forall\gamma < \kappa)\gamma^{\aleph_0} < \kappa$ and that κ is regular, we can easily find $i < j < \kappa$, and h such that:

a) h is an isomorphism from N^ω_i onto N^ω_j such that $h(i) = j$.

b) h is the identity on $N^\omega_i \cap N^\omega_j$ (so it maps to themselves: $\langle p_\alpha : \alpha < \kappa\rangle$, $\langle p^n_{\alpha,\ell} : \alpha < \kappa\rangle, \langle\langle \underset{\sim}{\bar{f}}{}^n_{i,\ell} : i < \alpha_0\rangle : \ell < \omega\rangle$).

c) $N^\omega_i \cap \kappa \subseteq j$.

d) $N^\omega_i \cap (\alpha_0 + 1) = N^\omega_j \cap (\alpha_0 + 1) = \{\alpha(\xi) : \xi < \xi_0 < \omega_1\}, \alpha(\xi)$ increasing. Also $N^\omega_j \cap j = N^\omega_i \cap i = N^\omega_j \cap i$.

Now we choose countable $N_\zeta \prec (H(\lambda), \in)$ for $\zeta \leq \xi_0$ such that $h \in N_0$, and $N^\omega_i, N^\omega_j \in N_0$, $\langle N_\alpha : \alpha \leq \zeta\rangle \in N_{\zeta+1}$.

We now repeat the proof of V 7.1, more exactly:

1.4. Claim. Suppose that $0 \leq \xi < \zeta \leq \xi_0$, $N_i (i \leq \xi_0)$, $N_i^\omega \in N_0$, $N_j^\omega \in N_0$, are as above, $r \in P_{\alpha(\xi)}$, $G^* \subseteq (N_i^\omega \cup N_j^\omega) \cap P_{\alpha(\xi)}$, $G^* \cap N_i^\omega$ is generic for $(N_i^\omega, P_{\alpha(\xi)})$ (i.e. $G^* \cap N_i^\omega$ is directed and if $\mathcal{I} \in N_i^\omega$ and $\mathcal{I}$ is pre-dense in $P_{\alpha(\xi)}$, then $\mathcal{I} \cap G^* \cap N_i^\omega \neq \emptyset$), and h maps $G^* \cap N_i^\omega$ onto $G^* \cap N_j^\omega$. In addition, every element of G^* is $\leq r$, r is (N_ε, P_ξ)-generic for any ε satisfying $\varepsilon = 0$ or $\xi \leq \varepsilon < \xi_0, G^* \in N_\xi$ and $p_i \restriction \alpha(\xi)$, $p_j \restriction \alpha(\xi) \in G^*$ and $G^* \in N_{\xi+1}$.

Then there is a $G \subseteq (N_i^\omega \cup N_j^\omega) \cap P_{\alpha(\zeta)}$ with $G^* \subseteq G, G \in N_{\zeta+1}$, $G \cap N_i^\omega$ is generic for $(N_i^\omega, P_{\alpha(\zeta)})$ such that h maps $G \cap N_i^\omega$ onto $G \cap N_j^\omega$ and $r \wedge \bigwedge_{q \in G} q \neq \emptyset$ (Boolean intersection). In other words: $r \Vdash_{P_{\alpha(\xi)}}$ "G has an upper bound "in $P_{\alpha(\zeta)} / \underset{\sim}{G}_{\alpha(\xi)}$".

Proof. The proof is as in V 7.1. The only difference is the case $\zeta = \xi + 1$, here we use clause (B) of Definition 1.2: necessarily G^* "tells" us the functions have the same values, as they are pressing down. $\square_{1.4}$

Continuation of the proof of 1.3 From Claim 1.4, it follows that p_i and p_j are compatible in P_{α_0}, for i and j that we fixed earlier.

$\square_{1.3}$

Remark. Note in Lemma 1.1, if the iteration is defined such that we have a support of power $\leq \mu$, and $(\forall \chi < \kappa) \chi^\mu < \kappa, \kappa$ regular, still P_κ satisfies the κ-c.c. (On free limits, see IX §1, 2)

1.6. Lemma. We can replace (in 1.3) "$\aleph_1$-completeness system" by "$\aleph_0$-completeness system".

Proof. Using V 7.2 instead of V 7.1.

1.7. Remark. We can even replace "$\aleph_1$-completeness system" by "2-completeness system", using VIII 4.5, 4.13.

§2. The Axioms

AXIOM I.

1) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and:
2) if $|P| = \aleph_2, P$ proper, $\mathcal{I}_i \subseteq P$ pre-dense (for $i < \omega_1$), *then* there is a directed $G \subseteq P$, $\bigwedge_{i<\omega_1} G \cap \mathcal{I}_i \neq \emptyset$.

2′) Moreover if $|P| = \aleph_2, P$ proper, $\mathcal{I}_i \subseteq P$ pre-dense (for $i < \omega_2$), $P = \bigcup_{i<\omega_2} P_i$ where P_i are increasing and $|P_i| \leq \aleph_1$, *then* there is an $\alpha < \omega_2$, $\mathrm{cf}(\alpha) = \omega_1$ and a directed $G \subseteq P_\alpha$ such that:

$$\bigwedge_{i<\alpha} (G \cap \mathcal{I}_i \neq \emptyset).$$

3) If $|P| = \aleph_1, P$ is proper iff forcing with P does not destroy stationary subsets of ω_1.

AXIOM II.

1) $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$.
2) If $|P| \leq 2^{\aleph_1}$, P is α-proper for every $\alpha < \omega_1$, and P is $\mathbb{D}$-complete for some simple $\aleph_0$-completeness system $\mathbb{D}$, $\mathcal{I}_i \subseteq P$ (for $i < \alpha_0 < 2^{\aleph_1}$) and each $\mathcal{I}_i$ is pre-dense, *then* there is a directed $G \subseteq P$, such that, $\bigwedge_{i<\alpha_0} G \cap \mathcal{I}_i \neq \emptyset$ (we can also define $2'$ like we did in Axiom I.).

AXIOM II′ [S]. ($S \subseteq \omega_1$ stationary, costationary)
Similar to AXIOM II, but $\mathbb{D}$-completeness refers only to those N for which $N \cap \omega_1 \notin S$ (i.e. we have $(\mathbb{D}, S)$-completeness); also in the definition of $(< \omega_1)$-properness we can demand $N_i \cap \omega_1 \notin S$ provided we add properness to the set of hypotheses.

2.1. Theorem. Suppose CON(ZFC + "κ is 2^κ-supercompact"). *Then* CON(ZFC+ Axiom I).

Proof. We start as in III 4.3, defining $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$. Given P_i, define $\underset{\sim}{Q}_i$ by induction on $i < \kappa$ (κ is 2^κ- supercompact in V), as usual.

Case I. If i is not strongly inaccessible, Q_i is the Levy collapse of $(2^{|P_i|})^{V^{P_i}}$ to $\aleph_1$: $Q_i = \{f : |\mathrm{Dom}(f)| = \aleph_0,\ \mathrm{Dom}(f) \subseteq \omega_1,\ \mathrm{Rang}(f) \subseteq (2^{|P_i|})^{V^{P_i}}\}$.

Case II. If i is strongly inaccessible and, in V^{P_i}, there is a proper forcing P, with universe i and $\mathcal{I}_j \subseteq i$ (for $j < i$), each $\mathcal{I}_j$ pre-dense, and $\{j :$ there is a $G \subseteq j$, directed by $\le$ and $\bigwedge_{\xi<j} G \cap \mathcal{I}_\xi \neq 0\}$ is not stationary (subset of i), *then* $Q_i = P$ i.e. Q_i is one of those P's.

Case III. not I nor II – proceed as in Case I.

Now P_κ is proper, has density $\le \kappa$ and satisfies the κ-c.c. (by Lemma 1.1), so in V^{P_κ}, $2^{\aleph_0} = \kappa = \aleph_2$ (Why? $\kappa \le \aleph_2$ by case I, $2^{\aleph_0} \le \kappa$ as P_κ satisfies the κ-c.c., has density κ and κ is strongly inaccessible, $\aleph_2 \le 2^{\aleph_0}$ as clause (2) of the axiom I holds, as is proved below). So clauses (1), (3) of Axiom I hold – as in III 4.5 and clause 2) follows from clause 2′), so we are left with proving clause 2′) of Axiom I.

Suppose in the end that $\underset{\sim}{R}$, $\underset{\sim}{\mathcal{I}}_\alpha(\alpha < \kappa)$ are P_κ-names of a counterexample to 2′).

Let E be an ultrafilter over $\mathcal{S}_{<\kappa}(2^\kappa)$ exemplifying that κ is 2^κ-supercompact; and we code P_κ, $\underset{\sim}{R}$, $\underset{\sim}{\mathcal{I}}_\alpha(\alpha < \kappa)$ on κ. Let $S_0 = \{A \in \mathcal{S}_{<\kappa}(2^\kappa) : \mathcal{P}(A \cap \kappa) \subseteq A$, A is closed under reasonable operation, and $A \cap \kappa$ is a strongly inaccessible cardinal$\}$. Clearly $S_0 \in E$.

Then, for $A \in S_0$, the forcing notions $P_\kappa \cap A = P_{A\cap\kappa}$ and $\underset{\sim}{R}\restriction A$ are proper (see the definition of proper for a discussion of this: in $2^{|P|}$ we can get a witness for properness etc). So, we have proved that $i = A \cap \kappa$, $P_i = P_{A\cap\kappa}$ and, $\underset{\sim}{R} \cap A$, $\langle \underset{\sim}{\mathcal{I}}_\alpha \cap A : \alpha \in A \cap \kappa\rangle$ are candidates for the case II in the definition of Q_i. So, if $A \in S_0$, then $i = A \cap \kappa$ is inaccessible and there are some $P^i, \mathcal{I}^i_j(j < i)$ which we have actually chosen.

By the properties of E, there are such $i(0) < i(1), P^{i(0)} \subseteq P^{i(1)}, \mathcal{I}^{i(0)}_j = \mathcal{I}^{i(1)}_j \cap i(0)$ for $j < i(0)$ and we almost get contradiction to the choice of $P^{i(1)}$

Using such $i(\xi)$, $\xi \le \lambda, i(\lambda) = \lambda < \kappa$ which form a stationary subset of λ we get a contradiction. $\square_{2.1}$

Remark. We essentially use the proof that $\diamondsuit_\kappa$ holds for κ measurable, which is well known.

2.2. Theorem.

1) CON(ZFC + κ is 2^κ-supercompact) implies CON(ZFC + Axiom II + G.C.H.).
2) In both cases (2.1, 2.2(1)) we can relativize to S ($S \subseteq \omega_1$ stationary, costationary). If in Axiom II we assume $|P| < 2^{\aleph_1}$, no large cardinality is needed.

Proof.

1) Similar.
2) Just note that the same iteration works. $\square_{2.2}$

2.3. Discussion. In almost all the applications we need a weaker version of the axioms for whose consistency we do not need a large cardinal.

Usually our task is to show that for every $A \subseteq H(\aleph_1)$ with $|A| = \aleph_1$, there is $B \subseteq H(\aleph_1)$ such that

$$H(\aleph_1) \vDash \varphi(A, B)$$

where for Axiom II, φ is any first order (or L_{ω_1,ω_1}) sentence, and for Axiom I, φ should have quantifications on A, B only.

So we iterate ω_2 times only, each time forcing a B for a given A, till we catch our tail in ω_2 steps (we can have $|A| = \aleph_2$ and visit $A \cap V^{P_\alpha}$ for stationarily many $\alpha < \omega_2, \mathrm{cf}(\alpha) = \omega_1$).

So, only Consis(ZFC) is needed.

For Axiom I part (3), κ inaccessible suffices (see III 4.1,2,3), whether it is necessary is still not clear. We can get Axiom I without inaccessible in the cases above, when we are able, provably, to find Q_A in an intermediate V. For P_{ω_2} to satisfy the $\aleph_2$-c.c. we need $|Q_A| = \aleph_1$. With κ inaccessible, Q_A should just have a cardinality smaller than κ.

For $|P| = \aleph_1$, if $P \in V^{P_{\omega_2}}$, P is proper in $V^{P_{\omega_2}}$, and for arbitrarily large $\beta < \alpha$, $\underset{\sim}{Q}_\beta = \mathrm{Levy}(\aleph_1, 2^{\aleph_1})$, then P is proper in V^{P_α} for every α large enough, (by III 4.2 though not vice versa). So Axiom I 2) for $|P| = \aleph_1$ comes under the previous discussion.

There may be applications where we really have to use information on $V^{P_\alpha}(\alpha < \omega_2)$ to build Q_α, mainly like in Axiom I, when we want to build a forcing giving B for a given A (see above) and we want to use CH or $\diamondsuit_{\aleph_1}$ for the building. But with Axioms I, II (or the versions we can get with an inaccessible) we can use the axiom: collapse $2^{\aleph_0}$, building a forcing and look at the composition (see §3, Application G).

Sometimes we use properties of P_α (like ${}^\omega\omega$-boundedness) which we usually demand from each Q_i, and from P in the axiom, and we have to prove that P_α satisfies it, (see Chapter VI).

However still Axioms I,II look like a reasonable choice. We shall use them, and can remark, for suitable applications, that only CON(ZFC) is needed.

As we mentioned (see III 4.3) CON(ZFC+ κ inaccessible) implies the consistency of

AXIOM I_a:
If $|P| = \aleph_1, P$ does not destroy stationary subsets of ω_1 and $\mathcal{I}_i \subseteq P$ pre-dense (for $i < \omega_1$), *then* there is a directed $G \subseteq P$, such that

$$\bigwedge_{i<\omega_1} (\mathcal{I}_i \cap G \neq \emptyset).$$

We can ask whether we can get something like Axiom I for $2^{\aleph_0} = \aleph_3$. Roitman (see [B]) proved that this is difficult, by proving that:

2.4. Theorem. (Roitman)

1) If $\bar{Q} = \langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$ is a CS iteration, $\underset{\sim}{Q}_n$ nontrivial (i.e. above every element there are two incompatible ones) *then* Lim$\bar{Q}$ does not satisfy the $2^{\aleph_0}$-chain condition.
2) If $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \omega_1 \rangle$ is a CS iteration, $\underset{\sim}{Q}_i$ nontrivial, $2^{\aleph_0} > \aleph_1$, then Lim$\bar{Q}$ collapses $2^{\aleph_0}$ to $\aleph_1$.

2.5. Question.

1) What kind of axioms can we get with:

   A) $2^{\aleph_0} = \aleph_3$?

   B) with $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$?

2) Can we define properness so that it works for higher cardinals (e.g, for SH.)?

3) Similar to (1), but we ask about iterations.

   For 1), a solution for particular problems appears in [AbSh:114].

2.6. More Discussion. In connection with the beginning of the previous discussion, there is no problem if for $A \subseteq H(\aleph_1)$, $|A| = \aleph_1$, we need to force by some Q_A of large cardinality to get some $B \subseteq \mathrm{Ord}$. We iterate up to the first larger cardinal. Baumgartner and Mekler have improved Theorem 2.1 as follows.

2.7 Theorem.

1) (Mekler) In 2.2(1) we can weaken the hypothesis on κ to π^2_2-indescribability (the advantage of this property is that if κ satisfies it in V, it satisfies it in L).

2) (Baumgartner) If κ is supercompact in some forcing extension $V^\dagger$ of V, $\aleph_1^{V^\dagger} = \aleph_1^V, \aleph_2^{V^\dagger} = \kappa$, $2^{\aleph_0} = \aleph_2$ and if

   $(*)$ P is a proper forcing, and $\underset{\sim}{S}_i (i < \omega_1)$ are P-names such that $\Vdash_P$ "$\underset{\sim}{S}_i$ is a stationary subset of ω_1",

   then there is a directed $G \subseteq P$, and stationary $S_i \subseteq \omega_1$ $(i < \omega_1)$ such that for every j, $i < \omega_1$, for some $p \in G, p \Vdash$ "$j \in \underset{\sim}{S}_i$" and $j \in S_i$, or $p \Vdash$ "$j \notin \underset{\sim}{S}_i$" and $j \notin S_i$.

Proof.

1) For simplicity assume $V = L$.

We define the CS iteration $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$, such that $|P_i| < \kappa$, and each $\underset{\sim}{Q}_i$ is proper, and $\underset{\sim}{Q}_i$ is the first P_i-name of a counterexample for "$V^{P_i} \models$ "$\underset{\sim}{Q}_i$ satisfies the axiom"". If P_κ is not as required look at $\{\langle P, \underset{\sim}{Q}, \langle \underset{\sim}{\mathcal{I}}_i : i < \omega_1 \rangle \rangle :$

$p \in P_\kappa, \underset{\sim}{Q}, \underset{\sim}{\mathcal{I}}_i$ are P_κ names, $p \Vdash \underset{\sim}{Q}, \underset{\sim}{\mathcal{I}}_i$ (for $i < \omega_1$) form a counterexample$\}$. Choose the first (by the canonical order of L) member $\langle p, \underset{\sim}{Q}, \bar{\underset{\sim}{I}} \rangle$. Now use indescribability.

2) By Laver [L], w.l.o.g. κ remains supercompact if we force by any κ-complete forcing. Force by $\{\bar{Q} : \bar{Q}$ is a CS iteration of proper forcing of power $< \kappa$, $\ell g(\bar{Q}) < \kappa\}$, ordered by being an initial segment. The generic object is, essentially such $\bar{Q}$ of length κ, so force by $\lim \bar{Q}$. $\square_{2.7}$

Probably better is the following:

2.8 Definition. Let κ be a supercompact cardinal. We call $f : \kappa \to H(\kappa)$ a *Laver diamond* if for every cardinal λ and $x \in H(\lambda)$, there is a normal fine ultrafilter D on $\mathcal{S}_{<\kappa}(H(\lambda))$ such that the set

$$A_D(x) \overset{\text{def}}{=} \{a \in \mathcal{S}_{<\kappa}(H(\lambda)) : x \in a, a \cap \kappa \in \kappa, \text{ and in the Mostowski collapse } MC_a \text{ of } a,\ x \text{ is mapped to } f(a \cap \kappa)\}$$

is in D.

By Laver $[L]$, if κ is a supercompact cardinal, we can assume that a Laver diamond for it exists.

2.9 Lemma. Suppose κ is a supercompact and f^* is a Laver diamond for it. Define $\underset{\sim}{Q}_i$ by induction on $i < \kappa$, as follows:

If $f^*(i)$ *is a* P_i*-name,* $\Vdash_{P_i}$ "$f^*(i)$ *proper*", i *limit,* then $\underset{\sim}{Q}_i = f^*(i)$.

Otherwise $\underset{\sim}{Q}_i = \text{Levy}(\aleph_1, 2^{2^{\aleph_1}})$.

Then $\Vdash_{P_\kappa}$ "(*) of 2.7(2) , i.e. $Ax_{\omega_1}[\text{proper}]$ " (see Definition 2.10 below).

Proof. By the properness iteration lemma, P_κ is proper, and also it satisfies the κ-c.c. Let $\underset{\sim}{Q}$ be a P_κ-name for a proper forcing, and λ a regular cardinal such that $\underset{\sim}{Q} \in H(\lambda)$; without loss of generality $\Vdash_{P_\kappa}$ "$2^{|\underset{\sim}{Q}|} < \lambda$". We use the formulation of Definition 2.10: $Ax_{\omega_1}[\text{proper}]$. Let $\underset{\sim}{\mathcal{I}}_i (i < \omega_1)$ and $\underset{\sim}{S}_\beta (\beta \leq \omega_1)$ be given as in Definition 2.10 (i.e., they are P_κ-names of such objects). Apply

Definition 2.8 to $x = \underset{\sim}{Q}$ and λ such that $\underset{\sim}{Q} \in H(\lambda)$ and even $2^{|P_\kappa * \underset{\sim}{Q}|} < \lambda$, and get D as there. Choose $a \in A_D(x)$ such that $\langle \underset{\sim}{\mathcal{I}}_i : i < \omega_1 \rangle$ and $\langle \underset{\sim}{S}_\beta : \beta \leq \omega_1 \rangle$ belong to a, $(a, \in)$ is isomorphic to some $(H(\chi), \in)$ and letting MC_a be the Mostowski Collapse of a (i.e. the unique isomorphism from $(a, \in)$ onto $(H(\chi), \in)$ and $\mu \stackrel{\text{def}}{=} a \cap \kappa \in \kappa$, we have $f(\mu) = MC_a(\underset{\sim}{Q})$. Note $MC_a(\underset{\sim}{Q})$ is a P_μ-name of a proper forcing etc. Easily, $\underset{\sim}{Q}_\mu = f^*(\mu)$ in V^{P_μ}, and $\underset{\sim}{Q}_\mu$ is isomorphic to $a \cap \underset{\sim}{Q}$, so we can finish.

$\square_{2.9}$

2.10 Definition. (1) Let $\alpha \leq \omega_1, \varphi$ a property of forcing notion, λ a cardinal. Then $Ax_{\alpha,\beta}[\varphi, \lambda]$ means:

if

(i) P is a forcing notion satisfying φ and $P \in H(\lambda)$.

(ii) $\mathcal{I}_i$ is a pre-dense subset of P for $i < i^* \leq \beta$.

(iii) $\underset{\sim}{S}_i$ is a P_i-name of a stationary subset of ω_1, for $i \leq \alpha$.

then there is a G such that:

G is a directed subset of P,

G is not disjoint to $\mathcal{I}_i$ for $i < i^*$

$\underset{\sim}{S}_i[G] = \{\zeta < \omega_1 :$ for some $p \in G$ we have $p \Vdash_P$ "ζ in $\underset{\sim}{S}_i$"$\}$,

is a stationary subset of ω_1 for each $i < \alpha$.

(2) If $\lambda = \aleph_2$ we omit it. If $\beta = \omega_1$ we may omit it. If $\alpha = 0$, we omit it. $Ax^+[\varphi, \lambda]$ is $Ax_1[\varphi, \lambda]$.

§3. Applications of Axiom II (so CH Holds)

3.1 Application A. Axiom II implies SH($=$ SH$_{\aleph_1}$), in fact – every Aronszajn tree is special. (See V. 6.1.) Alternatively, see Application F.

3.2 Application B. On isomorphisms of Aronszajn trees on a closed unbounded set of levels, etc, see U. Abraham and S. Shelah [AbSh:114].

3.3 Application C: Uniformization. Axiom II implies: If η_δ is an increasing ω-sequence converging to δ, for any limit $\delta < \omega_1$ *then* for every $F : \omega_1 \to \omega$ there is a $g : \omega_1 \to \omega$ such that: $(\forall \delta < \omega_1)[\delta \text{ limit} \to (\exists n)[n \geq F(\delta) \ \& \ n = g(\eta_\delta(k))$ for all but finitely many $k < \omega]]$

Proof. Let η_δ for limit $\delta < \omega_1$ be given. Let $P_F = \{f : \mathrm{Dom}(f)$ is an ordinal $< \omega_1, \mathrm{Rang}(f) \subseteq \omega$, and for every limit $\delta \leq \mathrm{Dom}(f)$ the condition above holds$\}$. The order on P_F is the extension of functions. Now, α-properness is very easy. We prove $\mathbb{D}$-completeness for a simple $\aleph_0$-completeness system $\mathbb{D}$.

We have to note that if N is a countable elementary submodel of $(H(\lambda), \in)$, $P_F \in N$, $N \cap \omega_1 = \delta$, $p \in N$ and $\eta^1_\delta, F^1(\delta), \ldots, \eta^n_\delta, F^n(\delta)$ are n "candidates" for η_δ, $F(\delta)$, *then* we can choose $\alpha_\ell < \delta$ (for $\ell = 1, \ldots, n$) and $m > \mathrm{Max}_{\ell=1,n} F^\ell(\delta)$, such that $A = \bigcup_\ell (\mathrm{Rang}(\eta^\ell_\delta) \setminus \alpha_\ell)$ has order type ω and is disjoint to $\mathrm{Dom}(f)$, and find a $q \geq p$, q is (N, P_F)-generic, $q{\restriction}A$ is constantly m.

More formally, (see V. 5.2, 5.3), we shall define $\mathbb{D}_{\langle N, P\cap N, p\rangle}$, as a filter on $A_0 = \{G \subseteq P \cap N : p \in G, G$ is directed not disjoint to any $\mathcal{I} \in N, \mathcal{I} \subseteq P$, $\mathcal{I}$ pre-dense$\}$ such that it depends only on the isomorphism type of (N, P, p). Let $N \cap \omega_1 = \delta$.

The filter will be generated by $A^n_{\eta,p}$, where for $n < \omega, \eta$ an ω-sequence converging to δ, $A^n_{\eta,p} = \{G \subseteq A_0 :$ for some $k < \omega$, and $k \geq n$, for every $q \in G$ and $\ell < \omega$, if $\eta(\ell) \in (\mathrm{Dom}(q) \setminus \mathrm{Dom}(p))$, then $q(\eta(\ell)) = k\}$. $\square_{3.3}$

A conclusion is the following:

3.4 Application D. G.C.H. $\not\to \Phi^{\aleph_0}_{\aleph_1}$. (For a definition - see below).
But we know $(2^{\aleph_0} < 2^{\aleph_1}) \Rightarrow \Phi^2_{\aleph_1}$ (Devlin and Shelah [DvSh:65], or here AP §1).

3.4A Definition. The statement Φ^κ_λ is defined as: For every $G : {}^{\lambda>}2 \to \kappa$ there is an $F : \lambda \to \kappa$ such that for every $g \in {}^\lambda 2$ we have $\{i < \lambda : G(g{\restriction}i) = F(i)\}$ is stationary.

Question. Does G.C.H. $\to \Phi^3_{\aleph_1}$? (See the next chapter.)

3.5 Application D*. Axiom II implies: If $\langle \eta_\delta : \delta < \omega_1, \text{ limit } \rangle$ is as above, and $c_\delta \in {}^\omega\omega$, *then* there is an $f : \omega_1 \to \omega$ such that $(\forall\delta)(\exists^\infty n)[f(\eta_\delta(n)) = c_\delta(n)]$ (this was proved in U. Abraham, K. Devlin and S. Shelah [ADSh:81]).

There an application of this to a problem of Hajnal and Mate on the coloring number of graphs is given.

Proof. Easy by now. $\square_{3.5}$

3.6 Application E. Fleissner showed:

$\Phi^{\aleph_0}_{\aleph_1} \Rightarrow$ (topological statement A) $\Rightarrow$ not $\Big[$there is a tree $\Phi = \langle \eta_\delta : \delta < \omega_1 \rangle$, with $\mathrm{Dom}(\eta_\delta) = \omega$, for each δ, and $\eta_\delta(n)$ (for $n < \omega$) are increasing with $\delta = \bigcup_n \eta_\delta(n)$, such that:

$$(\forall h : \omega_1 \to \omega)(\exists f : \omega_1 \to \omega)(\exists h' : \omega_1 \to \omega)(\forall \text{ limit } \delta < \omega_1)$$

$$(\exists\, m_\delta < \omega)(\forall n > m_\delta)[h(\delta) \leq f(\eta_\delta(n)) \leq h'(\delta)].\Big].$$

For a definition of the topological statement A see 3.20(3), 3.25A, 3.25B.

Clearly we can choose any Φ and then use application C to see that this Φ satisfies the conclusion above:

So Axiom II $\Rightarrow$ not (topological statement A) and Axiom II is consistent with ZFC + G.C.H, so

3.7 Conclusion. G.C.H. $\not\Rightarrow$ (topological statement A) (again CON(ZFC) suffice).

3.8 Application F. Fleissner asks about the consistency of the following with G.C.H. $(*_E)$ there is a special Aronszajn tree T such that letting $T^\dagger \stackrel{\text{def}}{=} \{t \in T : \text{ht}\,(t) \text{ limit}\}$ (where $\text{ht}\,(t)$ is the level or height of t) we have (we may assume that $T = \langle \omega_1, <_T \rangle$ with i-th level $[\omega i, \omega i + \omega)$, and we may write $x < y$ instead $x <_T y$):

$$\begin{aligned}\otimes_T \ \ (\forall f: T \to \omega)\Big[&(\forall t \in T^\dagger)(\exists x < t)(\forall y)[x < y < t \to f(y) \le f(t)] \Rightarrow \\ &(\exists g: T \to \omega)(\forall t \in T^\dagger)(\exists x < t)(\forall y)[x < y \le t \to \\ &\qquad\qquad\qquad f(y) \le g(y) = g(x)]\Big]\,.\end{aligned}$$

(This is sufficient for the existence of some examples in general topology, see the end of the application and 3.25 below; we can add $(\forall y)(f(y) \le g(y))$.) Again CON(ZFC) suffices (as Claim 3.19 F10 holds), i.e. we prove:

Claim. [AxII] Every Aronszajn tree T satisfies $\otimes_T$ (remembering that AxII is consistent with G.C.H. and implies that every Aronszajn tree is special, we get the desired answer).

The proof is quite similar to that of Application A, see V 6.1. However, here we have to do more; an incidental point is that here we have to find a $g: T \to \omega$, not from a club of levels but from all of them.

Let T be an Aronszajn tree and $f^*: T \to \omega$ be such that:

$$(\forall t \in T^\dagger)(\exists x < t)(\forall y)(x < y < t \to f^*(y) \le f^*(t)).$$

Let $F = \{(g, C)$: for some countable ordinal i we have $\mathrm{Dom}(g) = T_{\le i} \overset{\text{def}}{=} \bigcup_{\alpha \le i} T_\alpha$, $\mathrm{Rang}(g) \subseteq \omega$ (and, if you like, $(\forall x \in \mathrm{Dom}(g))[f^*(x) \le g(x)]$), $C \subseteq (i+1)$, C closed, $i \in C$, and

$$(\forall t \in T^\dagger \cap T_{\le i})(\exists x < t)(\forall y)[x < y \le t \to f^*(y) \le g(y) = g(x)]\}.$$

For $(g, C) \in F$, let $i(g)$ be the unique i such that $\mathrm{Dom}(g) = \bigcup_{\alpha \le i} T_\alpha$. We order F by

$$(g_1, C_1) \le (g_2, C_2) \ \text{ iff } \ g_1 \subseteq g_2, C_1 = C_2 \cap (i(g_1) + 1).$$

A generic subset of F gives g as required, but F not only is not necessarily $\aleph_1$-complete but may collapse $\aleph_1$, or add reals.

So what do we do? We add obligations.

3.9 F1 Definition. I (more formally (I, C, m)) is called an *obligation for* (T, f^*) if $C = C(I)$ is a closed unbounded subset of ω_1, $m = m(I) < \omega$ and:

a) $I \subseteq \bigcup\{(T_\alpha)^{m(I)} : \alpha \geq \mathrm{Min}(C)\}$ and

$$(a_1, \ldots, a_{m(I)}) \in I \Rightarrow \bigwedge_{k \in [1, m(I)]} \bigwedge_{\ell < k} (a_\ell \neq a_k),$$

b) if $\mathrm{Min}(C) \leq \alpha < \beta$, and $(a_1, \ldots, a_{m(I)}) \in (T_\beta)^{m(I)} \cap I$, then

$$(a_1 {\restriction} \alpha, \ldots, a_{m(I)} {\restriction} \alpha) \in (T_\alpha)^{m(I)} \cap I$$

(of course for $a \in T_\beta$, $a {\restriction} \alpha$ is the unique $b \in T_\alpha$ such that $b <_T a$),

c) if $\alpha = \mathrm{Min}(C)$, then $(T_\alpha)^{m(I)} \cap I$ has $\aleph_0$ pairwise disjoint members,

d) if $\alpha \in C, (a_1^*, \ldots, a_{m(I)}^*) \in (T_\alpha)^{m(I)} \cap I$ and $\beta \in C, \alpha < \beta$, then

$$\{(a_1, \ldots, a_{m(I)}) \in (T_\beta)^{m(I)} \cap I : \bigwedge_{\ell=1}^{m(I)} a_\ell {\restriction} \alpha = a_\ell^*\}$$

contains $\aleph_0$ pairwise disjoint members,

e) there are $n_\ell(I) < \omega$ (for $\ell = 1, \ldots, m(I)$) such that: *if* $\bar{a} \in I \cap (T_\alpha)^{m(I)}$, $\bar{b} \in I \cap (T_\beta)^{m(I)}$ and $\alpha, \beta \in C, \alpha < \beta$ and $\bar{a} \leq \bar{b}$ (see below), *then* $n_\ell(I) \geq \mathrm{Max}\{f^*(t) : a_\ell \leq t < b_\ell\} < \omega$.

Notation. If $\bar{a} \in (T_\alpha)^{m(I)}$, then $\alpha(\bar{a}) \stackrel{\text{def}}{=} \alpha$.

3.10 F2 Definition. For $\bar{a}, \bar{b} \in I, \bar{a} \leq \bar{b}$ holds *if* $\bar{a} \in (T_{\alpha(\bar{a})})^{m(I)}$, $\bar{b} \in (T_{\alpha(\bar{b})})^{m(I)}$, $\alpha(\bar{a}) \leq \alpha(\bar{b})$, and

$$\bigwedge_{\ell=1}^{m(I)} [a_\ell = b_\ell {\restriction} \alpha(\bar{a})].$$

Then we say $\bar{b}$ extends $\bar{a}$ or $\bar{b}$ is an extension of $\bar{a}$.

Let $\alpha_\xi(I)$ be the ξ-th element of $C(I)$, in the increasing enumeration of $C(I)$.

3.11 F3 Definition. (g, C) fulfills the obligation I if a), b) and c) below hold, where:

a) $i(g) \in C(I)$, $i(g) \geq \alpha_1(I)$ and $C \setminus \alpha_0(I) \subseteq C(I)$.

Subdefinition (F3i). We say that g is I-good for $\bar{a} \in I$ *if* $\alpha(\bar{a}) \leq i(g)$ and $(\forall \ell)(\forall y)[1 \leq \ell \leq m(I)\ \&\ a_\ell \restriction \alpha_0(I) \leq y \leq a_\ell \rightarrow f(y) \leq g(y) = g(a_\ell \restriction \alpha_0(I))]$,
moreover $(\forall \ell)(\forall y)[1 \leq \ell \leq m(I)\ \&\ a_\ell \restriction \alpha_0(I) \leq y \leq a_\ell \rightarrow g(y) = n_\ell(I)]$.

b) There are $\bar{a}_{k,\ell} \in I \cap (T_{\alpha_1(I)})^{m(I)}, \bar{a}_k \in I \cap (T_{\alpha_0(I)})^{m(I)}, (k < \omega, \ell < \omega)$ such that: $\bar{a}_k \leq \bar{a}_{k,\ell}$; $\{\bar{a}_k : k < \omega\}$ are pairwise disjoint and for each k we have: $\{\bar{a}_{k,\ell} : \ell < \omega\}$ are pairwise disjoint and g is I-good for each $\bar{a}_{k,\ell}$.

Subdefinition (F3ii). We say (g, C) is I-very good for $\bar{a}$ *if* $\alpha(\bar{a}) \in C \cap C(I)$, and for any β, $\alpha(\bar{a}) < \beta \in C \cap (i(g)+1)$, $\bar{a}$ has $\aleph_0$ pairwise disjoint extensions in $I \cap (T_\beta)^m$ for which g is I-good.

c) If $\alpha < \beta < \gamma$ are in $C \cap C(I)$, $\bar{a} \in I \cap (T_\alpha)^{m(I)}$ and, (g, C) is I-very good for $\bar{a}$ *then* $\bar{a}$ has $\aleph_0$ pairwise disjoint extensions in $I \cap (T_\beta)^{m(I)}$ for which (g, C) is I-very good (γ appear just to make β not of maximal level as then I-very good is meaningless).

3.12 F4 Definition. $P_{(T,f^*)} = \{(g, C, B) : (g, C) \in F, B$ a countable family of obligations for (T, f^*) which (g, C) satisfies$\}$.

For short, we may write P_T for $P_{(T,f^*)}$.

3.13 (F5) Claim. $\mathcal{I}_i \stackrel{\text{def}}{=} \{(g, C, B) \in P_{(T,f^*)} : i \leq i(g)\}$ is a dense subset of $P_{(T,f^*)}$ (for each $i < \omega_1$).

Proof. Let $(g, C, B) \in P_{(T,f^*)}$ and $i < \omega_1$.

As B is countable, $\cap_{I \in B} C(I)$ is a closed unbounded subset of ω_1. So we can find a ξ such that: $\xi > i$, $i(g)$ and $\xi \in C(I)$ for every $I \in B$. We let $C^\dagger = C \cup \{\xi\}$, $B^\dagger = B$.

Now we shall define g' (such that $(g', C^\dagger, B^\dagger) \geq (g, C, B)$ and $(g', C^\dagger, B^\dagger) \in \mathcal{I}_i$ will exemplify the desired conclusion).

The nontrivial part in defining g' is to satisfy Definition (F3)(c). There, the nontrivial case (and it implies the others) is $\beta = i(g), \gamma = \xi$. So, for each $\alpha < \beta$ $(= i(g))$ and $I \in B$, $\alpha \in C(I) \cap C$ and $\bar{a} \in (T_\alpha)^{m(I)} \cap I$ such that (g, C) is I-very good for $\bar{a}$, we have to provide $\aleph_0$ pairwise disjoint $\bar{b} \geq \bar{a}$ such that $\bar{b} \in (T_\beta)^{m(I)} \cap I$ and $(g', C^\dagger)$ is I-very good for $\bar{b}$. (Hence $(g', C^\dagger)$ will be I-very good for $\bar{a}$.)

Let $\{\langle I_k, \alpha_k, \bar{a}_k\rangle : k < \omega\}$ be a list of all triples as above. As (g, C) is I_k-very good for $\bar{a}_k$, there is a set of pairwise disjoint sequences $\{\bar{b}_{k,\ell} : \ell < \omega\} \subseteq (T_\beta)^{m(I_k)} \cap I_k$ such that g is I_k-good for $\bar{b}_{k,\ell}$ and $\bar{a}_k < \bar{b}_{k,\ell}$.

Now we can easily find an infinite $S_k \subseteq \omega$ (for $k < \omega$) such that $\ell_1 \in S_{k_1}$, $\ell_2 \in S_{k_2}$, $k_1 \neq k_2 \Rightarrow \bar{b}_{k_1,\ell_1} \cap \bar{b}_{k_2,\ell_2} = \emptyset$ (more exactly, the intersection of their ranges is empty). Also for $k, \ell < \omega$ we can choose $\bar{c}^m_{k,\ell} \in (T_\gamma)^{m(I_k)} \cap I_k$ for $m < \omega$ such that: $\bar{b}_{k,\ell} < \bar{c}^m_{k,\ell}$ and $m(1) \neq m(2) \Rightarrow \mathrm{Rang}(\bar{c}^{m(1)}_{k,\ell}) \cap \mathrm{Rang}(\bar{c}^{m(2)}_{k,\ell}) = \emptyset$; remember $\gamma = \xi$.

Let $\bar{b}_{k,\ell} = \langle b_{k,\ell,e} : 1 \leq e \leq m(I_k)\rangle$ and $\bar{c}^m_{k,\ell} = \langle c^m_{k,\ell,e} : 1 \leq e \leq m(I_k)\rangle$. Let $\gamma^{m(1),m(2)}_{k,\ell,e} \stackrel{\text{def}}{=} \mathrm{Max}\{\zeta : c^{m(1)}_{k,\ell,e}\restriction\zeta = c^{m(2)}_{k,\ell,e}\restriction\zeta\}$, so by Ramsey theorem and basic properties of trees for any k, $\ell < \omega$, $e \in [1, m(I_k)]$, either $(\exists\gamma)(\forall m)[\gamma^m_{k,\ell,e} = \gamma]$ or $\gamma^{m(1),m(2)}_{k,\ell,e}$ does not depend on $m(2)$ and is strictly increasing in $m(1)$.

Now we have to define $g'\restriction(\bigcup\{T_\zeta : i(g) < \zeta \leq \xi\})$. If $t \in \bigcup\{T_\zeta : i(g) < \zeta \leq \xi\}$, $b_{k,\ell,e} < t \leq c^m_{k,\ell,e}$, then note that by clause (e) of 3.9

$$\mathrm{Max}\{f(s) : b_{k,\ell,e} \leq s \leq t\} \leq n_e(I_k),$$

in which case we let $g'(t) = n_e(I_k)$ (by the above choice of $\bar{b}_{k,\ell}$ there is no contradiction, as $b_{k,\ell,e} = b_{k_1,\ell_1,e_1} \Rightarrow k = k_1 \,\&\, \ell = \ell_1 \,\&\, e = e_1$).
By the assumption on the $\bar{c}^m_{k,\ell,e}$'s, if $t \in \bigcup\{T_\zeta : i(g) < \zeta \leq \xi\}$ and $g'(t)$ is not yet defined *then:* $g'(y)$ is not yet defined for any large enough $y < t$ or $t \notin T^\dagger$.

Let $\{t_n : n < \omega\}$ be a list of all $t \in \bigcup\{T_\zeta : i(g) < \zeta \leq \xi\}$, such that for arbitrarily large $s < t$ we have that $g'(s)$ is still undefined and the height of t is a limit ordinal. We can easily define by induction on n, sets $A_n \subseteq \bigcup\{T_\zeta : i(g) < \zeta \leq \xi\}$, such that: on every $t \in A_n$, $g'(t)$ is undefined, A_n are pairwise disjoint, each A_n is linearly ordered, each A_n is convex (i.e,

$x, y \in A_n \wedge x < z < y \Rightarrow z \in A_n$), and for every t_n, for some $x_n < t_n$, and $n' \leq n$ we have $\{y : x_n \leq y \leq t_n\} \subseteq A_{n'}$, and $\mathrm{Sup}_{x \in A_n} f^*(x) < \omega$ (see the hypothesis on f^*; A_n may be empty).

Define $g'(t)$ for $t \in A_n$ as $\mathrm{Max}_{x \in A_n} f^*(x)$. Complete g' to its required domain $\bigcup_{\alpha \leq \xi} T_\alpha$ by $g'(t) = f^*(t)$ if $g'(t)$ not already defined. It is easy to check $(g', C^\dagger) \in F$, $(g', C^\dagger, B^\dagger) \in P_{(T,f^*)}$ and $(g, C, B) \leq (g', C^\dagger, B^\dagger)$. $\square_{3.13}$

3.14 F6 Claim. If $(g, C, B) \in P_{(T,f^*)}$, $\xi > i(g)$, $\xi \in \bigcap_{I \in B} C(I)$ is a limit ordinal and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are branches of $\bigcup_{\alpha<\xi} T_\alpha$, and $y_1, \ldots y_n$ are $<_T$-incomparable, $y_\ell \in \mathbf{t}_\ell$, $y_\ell \notin \mathrm{Dom}(g)$, $n_\ell \geq \mathrm{Max}\{f^*(x) : x = z_\ell$ or $y_\ell \leq x \in \mathbf{t}_\ell\}$, *then* there is a $(g^\dagger, C^\dagger, B^\dagger) \in P_T$ such that $i(g^\dagger) = \xi, (g, C, B) \leq (g^\dagger, C^\dagger, B)$ and $\bigwedge_{\ell=1}^n (\forall x \in \mathbf{t}_\ell)(x \notin \mathrm{Dom}(g)$ & $y_\ell \leq x \rightarrow g'(x) = n_\ell)$.

3.15 Remarks. By the demand on f^* we know that n_ℓ, y_ℓ always exist, if t_ℓ have distinct an upper bound in T (in particular the Max is well defined).

Proof. Same proof, assuring $b_{k,\ell,e} \notin t_\ell$. $\square_{3.14}$

3.16 F7 Claim. If λ is large enough ($2^{\aleph_1}$ should be o.k.), $\{P_{(T,f^*)}, T, f^*\} \in N \prec (H(\lambda), \in)$, $||N|| = \aleph_0, \delta \stackrel{\mathrm{def}}{=} \omega_1 \cap N$ and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are (distinct) δ-branches of $N \cap T = \bigcup_{\alpha<\delta} T_\alpha$, $(g, C, B) \in P_T \cap N, \mathcal{J} \in N$ a maximal antichain of P_T, $y_\ell \in \mathbf{t}_\ell$, $y_1, \ldots, y_n$ are pairwise $<_T$-incomparable, $y_\ell \in \bigcup_{i \leq i(g)} T_i$ and for every ℓ and z, $y_\ell \leq z \in \mathbf{t}_\ell$ we have

$$n_\ell \geq \mathrm{Max}\{f^*(z) : y_\ell \leq z \in \mathbf{t}_\ell, \mathrm{ht}\,(z) > i(g)\}$$

and

$$n_\ell = g(y_\ell) = g(z) \text{ when } y_\ell \leq z \in \mathbf{t}_\ell, \mathrm{ht}\,(z) \leq i(g)$$

(remember, $\mathrm{ht}\,(z)$ is the height of z in the tree), *then* there is a $(g', C^\dagger, B^\dagger) \in P_T \cap N$ such that $(g', C^\dagger, B^\dagger) \geq (g, C, B)$ and $(g', C^\dagger, B^\dagger)$ is above some member of $\mathcal{J}$ and $(\forall x \in \mathrm{Dom}(g'))[(x \in \mathbf{t}_\ell \setminus \mathrm{Dom}(g)) \Rightarrow g'(x) = n_\ell]$.

Proof. Suppose not. Define

$$
\begin{aligned}
I_0 \stackrel{\text{def}}{=} \{(a_1, \ldots, a_m) : & \text{ for some } \beta < \omega_1, \beta > i(g), a_1, \ldots, a_m \in T_\beta, \\
& a_\ell \geq \mathbf{t}_\ell \restriction i(g) = \text{ the unique } x \in \mathbf{t}_\ell \cap T_{i(g)}, \\
& (\text{so necessarily } a_1, \ldots, a_m \text{ are distinct}), \beta \in \bigcap_{I \in B} C(I), \\
& \bigwedge_\ell (\forall x)(x \leq a_\ell \wedge x \notin \mathrm{Dom}(g) \Rightarrow f^*(x) \leq n_\ell) \\
& \text{and there is no } (g', C^\dagger, B^\dagger) \geq (g, C, B) \text{ such that:} \\
& \quad \text{a) } i(g') \leq \beta \\
& \quad \text{b) } (g', C^\dagger, B^\dagger) \text{ is above some member of } \mathcal{J} \\
& \quad \text{c) } \bigwedge_{\ell=1}^{m} (\forall x)[x \leq a_\ell \,\&\, \mathrm{ht}\,(x) > i(g) \,\&\, \mathrm{ht}\,(x) \leq i(g') \\
& \qquad\qquad \Rightarrow g'(x) = n_\ell]\}.
\end{aligned}
$$

Clearly $I_0 \in N$, and if $a_\ell \in \mathbf{t}_\ell \cap T_\beta$ (for $\ell = 1, \ldots, m$) are distinct (and for $\beta \in [i(g), \delta)$ there are such a_ℓ) (a_ℓ is determined by $\mathbf{t}_\ell, \beta$), then $(a_1, \ldots, a_m) \in I_0$, provided that $i(g) < \beta \in \bigcap_{I \in B} C(I)$. Note: $\bar{a} \in I_0 \Rightarrow \alpha(\bar{a}) \in \bigcap_{I \in B} C[I]$

But in N, the set $\bigcap_{I \in B} C(I) \in N$ is unbounded below δ.

So $N \models$ "for arbitrarily large $\beta < \omega_1$ there is an $\bar{a} \in I_0 \cap (T_\beta)^m$". So by N's choice this really holds.

We now define, by induction on $\varepsilon < (2^{\aleph_1})^+$, a set I_ε. I_0 was already defined. For limit ε, we set $I_\varepsilon = \bigcap_{\zeta < \varepsilon} I_\zeta$, and if $\varepsilon = \zeta + 1$ then $\bar{a} \in I_\varepsilon$ iff $\bar{a} \in I_\zeta$ and for arbitrarily large $\gamma < \omega_1$ we have: there is $\bar{b} \in (T_\gamma)^m$ such that $\bar{a} < \bar{b} \in I_\zeta$. Clearly, if $\bar{a} < \bar{b}$ are in I and $\bar{b} \in I_\varepsilon$ then $\bar{a} \in I_\varepsilon$; also for some $\varepsilon(*) < (2^{\aleph_1})^+$ we have $\varepsilon(*) \leq \varepsilon < (2^{\aleph_1})^+ \Rightarrow I_\varepsilon = I_{\varepsilon(*)}$. For $\bar{a} \in I_0$ let $\varepsilon(\bar{a}) = \mathrm{Min}\{\varepsilon : \varepsilon = \varepsilon(*)$, or $\bar{a} \notin I_{\varepsilon+1}\}$. Returning to $\mathbf{t}_\ell$'s, easily $\varepsilon(\langle \mathbf{t}_\ell \restriction \beta : \ell = 1, \ldots, n \rangle)$ is nonincreasing with β for $\beta \in (i(g), N \cap \omega_1)$, hence eventually constant, hence is constantly $\varepsilon(*)$. So $\langle \mathbf{t}_\ell \restriction \beta : \ell = 1, \ldots, n \rangle$ is in $I_{\varepsilon(*)}$ for $\beta \in (i(g), \delta)$, hence $I_{\varepsilon(*)} \neq \emptyset$.

By the proof of Theorem III 5.4 (as in V §6), there is a closed unbounded $C^* \subseteq \bigcap_{I \in B} C(I) \setminus (i(g)+1)$ such that there are $\aleph_0$ pairwise disjoint members of $I_{\varepsilon(*)}$ in $(T_\delta)^m$, moreover there are $\aleph_0$ pairwise disjoint members of $I_{\varepsilon(*)}$ in $(T_\delta)^m$ which are above $\bar{a}$, if $\bar{a} \in (T_\beta)^m$, $\bar{a} \in I_0$, $\beta < \delta$, $\delta \in C^*$ and $\bar{a} \in (T_\beta)^m \cap I_{\varepsilon(*)}$.

Define $I \overset{\text{def}}{=} \{\bar{a} :$ for some γ we have $\gamma > \beta \geq \mathrm{Min}(C^*),\ \bar{a} \in (T_\beta)^m, \bar{b} \in I_{\varepsilon(*)} \cap (T_\gamma)^m,\ \gamma \in C^*,\ \bar{a} < \bar{b}\}$.

Then I (more formally (I, C^*, m)) is an obligation for (T, f^*). By a variant of 3.14 F6 we can find a $(g', C^\dagger, B^\dagger)$, where $g' \geq g, i(g) = \alpha_1(I) =$ second element of C^*, $C^\dagger = C \bigcup \{\alpha_0(I), \alpha_1(I)\}$, $B^\dagger = B \cup \{I\}$, such that for infinitely many pairwise disjoint $\bar{a} \in I \cap (T_{\alpha_0(I)})^m$, for infinitely many pairwise disjoint $\bar{b}$ we have $\bar{a} < \bar{b} \in I \cap (T_{\alpha_1(I)})^m$ and $\bigwedge_{\ell=1}^m (\forall x)[x \leq b_\ell \wedge \mathrm{ht}\,(x) > i(g) \rightarrow g'(x) = n_\ell]$. So $(g, C, B) \leq_{P_{(T,f^*)}} (g', C^\dagger, B^\dagger)$.

So, there is a $(g'', C'', B'') \geq (g', C^\dagger, B^\dagger)$ in $P_{(T,f^*)}$ which is above a member of $\mathcal{J}$. As $I \in B^\dagger$, clearly $i(g'') \in C^* = C(I)$, and there is an $\bar{a} \in I \cap (T_{i(g'')})^m$ for which g'' is I-good. This contradicts the definition of I_0, I.

$\square_{3.16}$

3.17 F8 Claim.

1) P_T is proper

2) P_T is α-proper for every $\alpha < \omega_1$.

Proof.

1) If N is as in 3.16 F7 and $\delta \overset{\text{def}}{=} N \cap \omega_1$, while $(g_0, C_0, B_0) \in P_{(T,f^*)} \cap N$, let $\langle \mathcal{J}_n : n < \omega \rangle$ be a list of maximal antichains of P_T which belong to N. We define $(g_n, C_n, B_n) \in P_{(T,f^*)} \cap N$ which are increasing, (g_n, C_n, B_n) is above a member of $\mathcal{J}_{n-1}$ (when $n > 0$) and "on the side" we all the time have more comitments of the form that appear in 3.16F7. More specifically, together with $(g_n,\ C_n,\ B_n)$ we have $\mathbf{t}_\ell$, y_ℓ, m_ℓ, for $\ell = 1, \ldots, k_n$, such that $\mathbf{t}_\ell$ is a branch of $T \cap N$ with an upper bound in T_δ, $y_\ell \in \mathbf{t}_\ell$, $y_\ell \in \bigcup_{i \leq i(g)} T_i$, $\langle \mathbf{t}_\ell {\restriction} i(g_n) : \ell = 1, \ldots, k_n \rangle$ are pairwise distinct, $y_\ell \leq y \in \mathbf{t}_\ell \cap \bigcup_{i \leq i(g)} T_i \Rightarrow g_n(y) \leq n_\ell$ and $n_\ell \geq \mathrm{Max}\{f^*(s) : y_\ell \leq s \in \mathbf{t}_\ell\}$ and if $t_\ell \in T_\delta$ is the upper bound of $\mathbf{t}_\ell$ then $n_\ell \geq f^*(t_\ell)$. We can continue by 3.16 F7 in order that in the end we get a condition. We have two kinds of tasks:

A) for every $t \in T_\delta$, there is an $x \in \bigcup_{j<\delta} T_j, x < t$ such that

$$(\forall y)(x < y < t \rightarrow f^*(y) \leq g(y) = g(x) \ \&\ f^*(t) \leq g(x))$$

If our promise until now is $\mathbf{t}_1, n_1, y_1 \ldots, \mathbf{t}_\ell, n_\ell, y_\ell$ we let $\mathbf{t}_{\ell+1} = \{x : x <_T t\}$, $n_{\ell+1} = \mathrm{Min}\{n :$ for some $z \in \mathbf{t}_{\ell+1}$, for every y, $z \le y \le t \Rightarrow f(y) \le n\}$ and then choose $y_{\ell+1}$ appropriately.

B) for every $I \in B_n$, $\alpha \in C_n \cap C(I), \bar{a} \in (T_\alpha)^m \cap I, (g_n, C_n)$ is I-very good for $\bar{a}$, and $k < \omega$, we want that there will be k pairwise disjoint $\bar{b}$'s in $(T_\delta)^m \cap I$, so that $(\cup g_n, \cup C_n)$ will be I-good for $\bar{b}$.

(As we do this eventually for every k, we can get $\aleph_0$ pairwise disjoint $\bar{b}$'s.) This is again quite easy.

We have only $\aleph_0$ tasks, so there is no problem. Having defined $\langle (g_n, C_n, B_n) : n < \omega \rangle$ it is straightforward to find an upper bound $(g^*, C^*, B^*) \in P_{(T,f^*)}$, with $i(g^*) = \delta$, which is $(N, P_{(T,f^*)})$-generic and is above (g_0, C_0, B_0).

2) A similar proof. $\square_{3.17}$

3.18 F9 Claim. P_T is $\mathbb{D}$-complete for some simple $\aleph_1$-completeness system $\mathbb{D}$.

Proof. As in the proof of 3.17 F8 above, we have to prove that *if* for one N (with $\delta \stackrel{\text{def}}{=} N \cap \omega_1$) we are given countably many possible pairs

$\langle T_\delta, \langle I \cap (T_\delta)^m : I \in N$ an obligation for $(T, f^*)\rangle\rangle$

then we can define a sequence $\langle (g_n, C_n, B_n) : n < \omega \rangle$ which is appropriate for all of them at once. This is trivial as in the proof of 3.17 F8 we do not actually need to assume that $\mathbf{t}_\ell$ has an upper bound in T_δ $(\ell = 1, \ldots, k_n)$, just that it is well defined as a δ-branch of T_δ with $\sup\{f^*(\mathbf{t}_\ell{\restriction}\alpha) : \alpha < \delta\} < \omega$. $\square_{3.18}$

The following is not needed for applying AxII, but is needed if we want to use the weaker variant equiconsistent with ZFC.

3.19 F10 Claim. P_T satisfies the $\aleph_2$-e.c.c. (see §1).

Proof. Trivial: define $h : P_T \to \omega_1$ such that

$$h(g, C, B) = h(g', C^\dagger, B^\dagger) \textit{ iff } g = g' \text{ and } C = C^\dagger$$

(this is possible as $2^{\aleph_0} = \aleph_1$). Let $(g, C, B)R(g', C^\dagger, B^\dagger)$ mean $(g, C) = (g', C^\dagger)$ and in Definition 1.2(B) let

$$f(i) = h(p_i).$$

Remember that (g, C, B), $(g, C, B^\dagger)$ have a lub: $(g, C, B \bigcup B^\dagger)$. $\square_{3.19,3.8}$

3.20 Discussion.

1) The proof here is appropriate for Application A; the small gain is that we directly find a function specializing T rather than finding one specializing a closed unbounded set of levels, and then using a theorem saying this is equivalent.
2) In applications A and F, compared to Jensen's CON(ZFC + G.C.H. + SH), we can get also CON(ZFC + CH + SH + $2^{\aleph_1}$=anything of cofinality $> \aleph_1$) using Lemma 1.3, and iterating ω_2 times, each time specializing all Aronszajn trees. The same proof works for all the relevant cases.
3) *Fleissner's Question.* It is unknown whether ZFC $\vdash$ " there is a countably paracompact non normal Moore space". (Equivalently, there is a countably paracompact not hereditarily countably paracompact Moore space).
   Such spaces can be constructed by Wager's technique from normal non-metrizable Moore spaces.
   Application E gives the first example of such a space not constructed in this way.
   Application F shows it can even be a Jones road space – a more traditional space than the space constructed in §3 E; see more in 3.25.

3.21 Application G. Ax II implies: There are no Kurepa trees. Moreover, every $\aleph_1$-tree (a tree of height ω_1 with all levels countable) is essentially specialized, i.e. there is an $f : T \to \mathbb{Q}$ (rationals) such that: $t \leq s \Rightarrow f(t) \leq f(s)$, and $t \leq s_1, t < s_2, f(t) = f(s_1) = f(s_2) \Rightarrow (s_1 \leq s_2$ or $s_2 \leq s_1)$. (Why does this imply that T is not a Kurepa tree? On any ω_1-branch B, f is eventually constant, so choose the minimal $x \in B$, $f \restriction \{y \in B : y \geq x\}$ is constant, call it $x(B)$. Then $B_1 \neq B_2$ ω_1-branches $\Rightarrow x(B_1) \neq x(B_2)$, so T has $\leq \aleph_1$ branches.)

3.22 Remark. Here Con(ZFC $+\exists$ inaccessible) is sufficient (and necessary).

Proof. First, let P_0 be a Levy collapse of $\aleph_2$ to $\aleph_1$ (which is $\aleph_1$-complete). In V^P, by Silver (or see III 6.1), T has at most $\aleph_1$ many ω_1-branches. Let $\{B_i : i < \omega_1\}$ be a list of the ω_1-branches of T, w.l.o.g. they are pairwise disjoint (choose $B_i^\dagger \subseteq B_i$ pairwise disjoint end segments by induction).

Now Q is a forcing (in V^P) which specializes T as in Application A, *but* on each B_ℓ the function is constant. The proof is the same.

So $P * \underset{\sim}{Q}$ essentially specializes T and so guarantees that T has $\leq \aleph_1$ branches. A directed $G \subseteq P * \underset{\sim}{Q}$ defines the function f which essentially specializes T if it meets the following $\aleph_1$ dense sets:

$\mathcal{I}_t = \{p : p \Vdash \text{“}f(t) = q\text{”} \text{ for some } q \in \mathbb{Q}\}$ for $t \in T$.

So by Axiom II there is a G as required, provided that $P * \underset{\sim}{Q}$ is $(< \omega_1)$-proper and $\mathbb{D}$-complete for a simple $\aleph_1$-completeness system $\mathbb{D}$.

For $(< \omega_1)$-properness: P obviously is, $\underset{\sim}{Q}$ – as in Application A, (in V^P) and so by III 3.2. applied to $(< \omega_1)$-properness, $P * \underset{\sim}{Q}$ is $(< \omega_1)$-proper. The task of checking for $\mathbb{D}$-completeness is left to the reader. $\square_{3.21}$

3.23 History for G. Baumgartner Malitz Reinhart [BMR] prove MA $+\neg$ CH $\Rightarrow$ every Aronszajn tree is special.

Silver proves that by Levy collapsing κ=first (strongly) inaccessible to $\aleph_2$ one obtains that there are no Kurepa trees (this includes the lemma we quote). Devlin proved: CON(ZFC + MA + $2^{\aleph_0} = \aleph_2$+ no Kurepa trees).

Shelah [Sh:73] used essential specialization function as the function above and showed that we can use $\aleph_1$-c.c. forcings to essentially specializes $\aleph_1$-trees with few branches (the proof uses more particularly unnecessary information).

Baumgartner [B3], using proper forcing, defines essential specialization and strengthens Devlin's result to: CON(ZFC + MA + every tree of power $\aleph_1$ and height $\aleph_1$ is essentially special). Independently, Todorcevic proved these results too.

It is well known that for such a consistency result an inaccessible cardinal is necessary.

3.24 Application H. Assume Axiom II′[S]. For $\delta \in S$ let η_δ be an ω-sequence converging to δ. Then $\langle \eta_\delta : \delta \in S \rangle$ has the $\aleph_0$-uniformization property. See [Sh:64], and [Sh:98]; this result should be an exercise to a reader who arrives here (but you may want more refined results as in [Sh:98], then proofs there are still of interest).

3.25 On Countable Paracompactness. Some general topologists consider suspiciously application F. So, let us give the derivation of the solution of the original problem.

3.25A Problem. Is the existence of a countably paracompact regular space which is not normal consistent with G.C.H.?

3.25B Definition. A topological space X is countably paracompact if *for every* family of open sets U_n of X (for $n < \omega$), which forms a cover (i.e. satisfies $X = \bigcup_{n<\omega} U_n$), *there are* open U'_n $(n < \omega)$ which refine U_n $(n < \omega)$ (i.e. $\bigwedge_n \bigvee_m U'_n \subseteq U_m$) and form a cover of X (i.e. $X = \bigcup_{n<\omega} U'_n$) which is locally finite (i.e. for every $x \in X$, $\{n : x \in U'_n\}$ is finite).

3.25C Definition. We shall consider a tree T as a topological space as follows: the set of points of the space is the set of nodes of T, for $t \in T$ its neighbourhood basis is:

$$\Big\{\{t\}\Big\} \text{ if ht}\,(t) \text{ non limit.}$$
$$\Big\{\{y : x <_T y \leq_T t\} : x <_T t\Big\} \text{ if ht}\,(t) \text{ is a limit ordinal.}$$

3.25D Fact. For a ω_1-tree T we have:
$(*)$ as a topological space, T is Hausdorff (use the normality of the tree) and even regular. $\square_{3.25D}$

3.25E Claim. If T is an ω_1-tree satisfying $\otimes_T$ (the conclusion of 3.8(=Application F)) above, *then*, as a topological space, T is countably paracompact.

Proof. Let $U_n \subseteq T$ be open with $T = \bigcup_{n<\omega} U_n$. Define a function f from T to ω by: $f(t) = \text{Min}\{n : t \in U_n\}$.

First check that f satisfies the antecedent of $\otimes_T$, i.e. $(\forall t \in T^\dagger)(\exists x < t)(\forall y)[x < y \le t \Rightarrow f(y) \le f(t)]$ (the order is of the tree).

So let $t \in T^\dagger$, i.e. $t \in T$, $t \in T_\delta$, δ a limit ordinal. For some n, $n = f(t)$ so $t \in U_n$, hence for some $x < t$, $\{y : x < y \le t\} \subseteq U_n$, hence $x < y \le t \Rightarrow f(y) \le n = f(t)$, as required. So by $\otimes_T$ there is a function $g : T \to \omega$ satisfying the conclusion of $\otimes_T$, i.e. $(\forall t \in T^\dagger)(\exists x < t)\forall y[x < y \le t \to f(y) \le g(y) = g(x)]$. Now, without loss of generality as asid above we can force it, still we derive it. [Why?

$(*)$ $f(t) \le g(t)$ for every $t \in T$.

Let $t \in T^\dagger$, so for some $x_1 < t$ we have:

$(*)_1$ $x_1 < y < t \Rightarrow f(y) \le f(t)$

(this is true, as we have verified the antecedent of $\otimes_T$)

and for some $x_2 < t$

$(*)_2$ $x_2 < y \le t \Rightarrow f(y) \le g(y) = g(x_2)$

(this is possible by the choice of g).

Now if $y \in A_t \overset{\text{def}}{=} \{y : y < t, x_1 < y, x_2 < y\}$, then $f(y) \le f(t)$ (by $(*)_1$), $f(t) \le g(t)$ (by $(*_2)$, by the "$f(y) \le g(y)$" there) and $g(t) = g(x_2)$ (by $(*)_2$, by the "$g(y) = g(x_2)$" there) and $g(x_2) = g(y)$ (by $(*)_2$, by the "$g(y) = g(x_2)$" there). Together, $y \in A_t \Rightarrow f(y) \le g(y)$. So, $\{t : f(t) > g(t)\}$ is necessarily a set of isolated points with no accumulation point. Hence we can change the values of g on it while not harming the conclusion of $\otimes_T$].

Define $U'_{n,\ell} = \{x \in U_n : g(x) = \ell\}$ (for $n \le \ell < \omega$). First, clearly $U'_{n,\ell} \subseteq U_n$. Second, each $U'_{n,\ell}$ is also an open set: if $t \in T^\dagger \cap U'_{n,\ell}$, let $x_2 < t$ be such that $x_2 < y \le t \Rightarrow f(y) \le g(y) = g(x_2) = g(t)$, so $x_2 < y \le t \Rightarrow g(y) = \ell = g(x_2)$ (there is such an x_2 by the choice of g). Let $x_1 < t$ be such that $x_1 < y \le t \Rightarrow y \in U_n$ (there is such an x as we have verified the antecedent of $\otimes_T$) and choose an $x < t$, $x > x_1, x > x_2$, clearly $\{y : x \le y \le t\} \subseteq U_{n,\ell}$. If $t \in T \setminus T^\dagger$, obviously $\{t\} \subseteq U'_{n,\ell}$.

Third, $T = \bigcup_{n \le \ell < \omega} U'_{n,\ell}$ because if $t \in T$, then for some n, we have that $t \in U_n \setminus \bigcup_{m<n} U_m$, hence by the choice of f, $f(t) = n$ and for some ℓ we have $g(t) = \ell$, so $t \in U'_{n,\ell}$ and $n \le \ell$ as $n = f(t) \le g(t)$.

Fourth, $\{U_{n,\ell} : n \leq \ell < \omega\}$ is locally finite: if $t \in T$, $\ell^* = g(t)$ then $\{U_{n,\ell} : n \leq \ell < \omega, t \in U_{n,\ell}\} \subseteq \{U_{n,\ell} : n \leq \ell = \ell^*\}$ which has $\ell^* + 1$ members. $\square_{3.25E}$

It was proved in [DvSh:85] (using the weak diamond) that

3.25F Claim. (CH) No special Aronszajn-tree is normal (as a topological space, in the topology we considered). $\square_{3.25F}$

So we can solve Watson's problem:

3.26 Conclusion. AxII (which is consistent with G.C.H.) implies that every Aronszajn ω_1-tree T is special and $\otimes_T$ holds. Hence, we have a countably paracompact, non normal, regular topological space which is an Aronszajn tree. In fact, it suffices to use a weaker version of AxII, for whose consistency (even with GCH), CON(ZFC) suffices. $\square_{3.26}$

§4. Applications of Axiom I

4.1. Claim. P is proper and even α-proper for every $\alpha < \omega_1$ if at least one of the following holds:

1. P satisfies the $\aleph_1$-c.c.

2. P is $\aleph_1$-complete (then P is even strongly proper and ${}^\omega\omega$-bounding)

3. P is Sacks forcing, or Silver forcing, or Gregorief forcing, or a product with countable supports of such forcings (then P is even strongly proper and ${}^\omega\omega$-bounding) for definitions see Lemma VI 2.14(2); Remark VI 4.1A; Definition VI 4.1(1) and VI 4.1A, Definition IX 2.6, Definition V 4.1 respectively).

4. P is Laver forcing ($P = \{T : T \subseteq {}^{\omega>}\omega$, T non empty closed under initial segments, no XX-minimal element and for some $\eta \in T$ such that: $T \cap {}^{\ell g(\eta)}\omega = \{\eta\}$ and $\eta \trianglelefteq \nu \in T \Rightarrow (\exists^* n)(\nu{}^\frown\langle n\rangle \in T)\}$ ordered by inverse inclusion).

Proof. An exercise. $\square_{4.1}$

4.2. Discussion. Baumgartner [B3], independently of the author's work on proper forcing, and at about the same time, introduced Axiom A forcing defined below. It covers a large part of the application of proper forcing, but to many it seems easier to handle.

P satisfies Axiom A if there are partial orders $\leq_n$ on P such that:

i) $\leq_0$ is the usual order $\leq$,

ii) $x \leq_{n+1} y \Rightarrow x \leq_n y$,

iii) if $\bigwedge_n x_n \leq_n x_{n+1}$ then $\{x_n : n < \omega\}$ has an upper bound in $\leq$,

iv) if $\underset{\sim}{\tau}$ is a name of an ordinal, $p \in P, n < \omega$, then there are $q \in P, p \leq_n q$ and a countable $A \subseteq \mathrm{Ord}, A \in V$ such that $q \Vdash_P$ "$\underset{\sim}{\tau} \in A$"

Baumgartner proved "P satisfies *Axiom* $A \Rightarrow P$ is proper"; in fact "P satisfies $A \Rightarrow P$ is $(< \omega_1)$-proper". Which forcing notions are equivalent to ones satisfying Axiom A? See XIV 2.4.

4.3. Discussion. Baumgartner [B3] found many applications for proper forcing: new ones and simplified proofs for the old ones (see proofs there). It is a matter of taste whether to deduce them from Axiom I, or build a forcing doing it. Some of them are:

4.3A. P_a, the forcing giving finite information on the enumeration of a closed unbounded subset of ω_1, is proper.

Remark. But P_a is not ω-proper, so this shows proper $\neq$ ω-proper.

This club does not include any old infinite sets. So if we iterate ω_2 times (by Axiom I or any variant) we obtain that for any $\aleph_1$ infinite subsets of ω_1 there is a club which does not include any of them. So this statement is consistent with $2^{\aleph_0} = \aleph_2$. In fact $2^{\aleph_0}$ can be anything, since if P_{ω_2} is the forcing for doing the above, its product with adding κ Cohen generic is also O.K.

U. Abraham proved the consistency of a similar assertion for $\aleph_2$ (with CH).

4.3B. Every tree of height $\aleph_1$ and power $\aleph_1$ is essentially special.

See §3 Application G. Only before forcing with P, add a Cohen real. Here the forcing specializing T consists of finite information, but the first part is the same (see history there, more exactly in 3.23).

4.3C. CON(ZFC + no $\aleph_2$-Aronszajn trees) (originally proved by Mitchell and Silver). We use κ weakly compact.

4.3D. Laver's consistency of Borel's conjecture: Baumgartner iterates $\aleph_2$ times the forcing $P = \{A : A \subseteq \omega \text{ is infinite}\}$, ordered by $A <_P B$ iff $B \leq_{ae} A$ for adding a Ramsey ultrafilter, and Mathias forcing for those ultrafilters (note that we can use the preservation of the Laver property (see Definition VI 2.9 and Conclusion 2.12)).

4.3E. CON(ZFC + "there are no $\aleph_2$ subsets of $\aleph_1$ of power $\aleph_1$, with pairwise countable intersection"), originally proved by Baumgartner.

4.4. On isomorphism of Aronszajn trees see Abraham and Shelah [AbSh:114].

§5. A Counterexample Connected to Preservation

5.1 Example. If we iterate forcing which does not destroy stationary subsets of ω_1 we may destroy $\aleph_1$.

Proof. For every $\alpha < \omega_2$, let $\alpha = \bigcup_{i<\omega_1} A_i^\alpha$ where A_i^α are countable, increasing and continuous in i and let $h_\alpha(i) =$ the order type of A_i^α.

Let $\mathcal{D} = \mathcal{D}_{\omega_1}$ (the filter of closed unbounded subsets of ω_1). Then $g_1 <_\mathcal{D} g_2$ means $\{\alpha < \omega_1 : g_1(\alpha) < g_2(\alpha)\} \in \mathcal{D}$.

Then $\alpha < \beta \Rightarrow h_\alpha <_\mathcal{D} h_\beta$. Suppose

(*) $(\forall \alpha < \omega_2)[h_\alpha <_\mathcal{D} g]$ for some $g : \omega_1 \to \omega_1$ (g exists e.g., if $V = L$).

Define $P_g = \Big\{(h, s, F, T) : h$ is a function from some $i < \omega_1$ to ω_1, i is a successor ordinal, s is a characteristic function of a closed subset of i, and $h <_s g$, which means $s(j) = 1 \Rightarrow h(j) < g(j)$, F is a countable subset of $\{h_\alpha : \alpha < \omega_2\}$, T is a function from F to the set of closed subsets of i such that $h_\alpha \in F \,\&\, j \in T(h_\alpha) \Rightarrow h_\alpha(j) < h(j)\Big\}$.
We call i the domain of the condition.

The partial ordering is obvious. P_g is not necessarily $\aleph_1$-closed.

5.1A Fact. If CH, then P_g satisfies the $\aleph_2$-chain condition (moreover there is a function $\mathbf{h} : P_g \to \omega_1$ such that if $\mathbf{h}(p_0) = \mathbf{h}(p_1)$ then p_0, p_1 and a lub).

Proof. For the first phrase find a Δ-system then take the union.
(For the second let $\mathbf{h}((h, s, F, T))$ code (h, s).) $\square_{5.1A}$

5.1B Fact. Forcing with P_g does not destroy stationary subsets of ω_1 and does not add reals.

Proof. Let $S \subseteq \omega_1$ be stationary, $p \in P_G$ and $\underset{\sim}{C}$ a P_g-name of a closed unbounded subset such that $p \Vdash_{P_g}$ "$\underset{\sim}{C}$ is disjoint to S".

First take $N \prec (H(\aleph_7), \in, P, \underset{\sim}{C}, S, p)$, $||N|| = \aleph_1$ such that $\omega_1 + 1 \subseteq N$.

Let $N \cap \omega_2 = \varepsilon$, so $\varepsilon = \{\zeta(i) : i < \omega_1\}$, and without loss of generality N is such that $N = \bigcup_{i<\omega_1} N_i, N_i \prec N, N_i$ are increasing continuous, $||N_i|| = \aleph_0$, and $\langle N_j : j \leq i\rangle \in N_{i+1}$.

Now $h_\varepsilon <_{\mathcal{D}} g$ by the assumption on g. So let C_0 be a closed unbounded set such that $i \in C_0 \Rightarrow h_\varepsilon(i) < g(i)$.

For each $i < \omega_1, \zeta(i) < \epsilon$ so $h_{\zeta(i)} <_{\mathcal{D}} h_\varepsilon$. So let C^i be a closed unbounded subset of ω_1 such that $j \in C^i \Rightarrow h_{\zeta(i)}(j) < h_\varepsilon(j)$.

Let $C_1 \subseteq \omega_1$ be closed unbounded such that

$$i \in C_1 \Rightarrow N_i \cap \varepsilon = \{\zeta(j) : j < i\}.$$

Now

$$C \stackrel{\text{def}}{=} \{i \in C_0 \cap C_1 : (\forall j < i) i \in C^j\}$$

is known to be closed unbounded subset of ω_1. Choose a $\delta \in C$ and let $\delta = \bigcup_n j_n, j_n < \delta$.

Now, by induction on $n < \omega$, define $p_n \in P_g \cap N_\delta, p = p_0, p_n \leq p_{n+1}$ such that $j_n \subseteq \mathrm{Dom}(h^{p_n}), p_{n+1} \Vdash$ "$\gamma_n \in \underset{\sim}{C}$" for some γ_n, $j_n < \gamma_n < \delta$.

Now $\bigcup\limits_{n<\omega} p_n$ can be extended to a condition $p^* \in P_g$ by adding δ to the domain because $\delta \in C$ and $h_\varepsilon(\delta)$ can serve as the value of $h(\delta)$ i.e. $p^* = (h^{p^*}, s^{p^*}, F^{p^*}, T^{p^*})$ is defined by: $\mathrm{Dom}(h^{p^*}) = \delta + 1$, $h^* \restriction \delta = \bigcup\limits_{n<\omega} h^{p_n}$ (well defined as $j_n \subseteq \mathrm{Dom}(h^{p_n})$), and $h^{p^*}(\delta) = h_\varepsilon(\delta)$, s^{p^*} is a function with domain $\delta + 1$, $s^{p^*} \restriction \delta = \bigcup\limits_{n<\omega} s^{p_n}$ and $s^{p^*}(\delta) = 1$, $F^{p^*} = \bigcup\limits_{n<\omega} F^{p_n}$ (which is $\subseteq N$ as each p_n belongs to N) and lastly if $h_\alpha \in F^{p^*}$ let $n(\alpha) = \mathrm{Min}\{n : h_\alpha \in F^{p_n}\}$ and let $T^{p^*}(h_\alpha) = (\bigcup\limits_{n \in [n(\alpha),\omega)} T^{p_n}(h_\alpha)) \cup \{\delta\}$. Why $p^* \in P_g$? As $\delta \in C_0$, we have $h_\varepsilon(\delta) < g(\delta)$ and $\zeta \in N \cap \omega_2 \Rightarrow \zeta = \zeta(i)$, for some $i < \delta$, hence $h_\zeta(\delta) < h_\varepsilon(\delta)$ (as $\delta \in C$) but $h_\varepsilon(\delta) < g(\delta)$ so $h_\zeta(\delta) < g(\delta)$. Now $p^* \Vdash$ "$\delta \in \underset{\sim}{C}$" as $\underset{\sim}{C}$ is closed and $\delta = \bigcup_n j_n = \bigcup_n \gamma_n$ and $p_{n+1} \Vdash$ "$\gamma_n \in \underset{\sim}{C}$". But $\delta \in S$, so p cannot force that $\underset{\sim}{C}$ is disjoint to S. Also, if $\underset{\sim}{r} \in N_0$ is a P_g-name of a real then we can arrange that p_{n+1} forces a value to $\underset{\sim}{r}(n)$ hence $p^* \Vdash$ "$\underset{\sim}{r} = r$" for some old real r. $\square_{5.1B}$

P_g in general is not proper. Now if $G \subseteq P_g$ is generic, we can find a generic function h such that

$$(\forall \alpha < \omega_2) h_\alpha <_{\mathcal{D}} h <_{\mathcal{D}} g\,.$$

So P_h is well defined.

If we iterate ω-times (taking any kind of limit) we necessary destroy ω_1. Why?

We get $h_0, h_1 \ldots$ functions from ω_1 to ω and $h_{n+1} <_{\mathcal{D}} h_n$ (any closed unbounded subset of ω_1 remains closed unbounded), contradiction. (Note: the kind of limit we take at ω is irrelevant.) $\square_{5.1}$