Chapter XVIII

Vaught and Morley Conjectures for ω-Stable Countable Theories

In this chapter we complete the proof of two of the most important results about the spectrum problem. We prove Vaught's conjecture for countable ω-stable T: A countable ω-stable theory has either countably many or $2^{\aleph_0}$ countable models. Furthermore, we prove Morley's conjecture for countable ω-stable T: If $\aleph_1 \leq \kappa < \lambda$, $I(\kappa, AT) \leq I(\lambda, AT)$.

For Vaught's conjecture the assumption that T is ω-stable is made because it allows us to prove the theorem. The conjecture has not been resolved for any other class in the stability hierarchy although there are partial results by [Lascar 1981] and [Shelah 1978a]. In contrast, for Morley's conjecture the assumption that T is ω-stable is part of a systematic program. If T is not superstable then T has 2^λ models in power λ for all uncountable λ. Although we did not prove this result for singular λ in Section IX.6, we did prove the function was increasing for stable but not superstable T. Thus, the only omissions in our treatment of Morley's conjecture for countable theories are the unstable case and the superstable but not ω-stable case. The first of these cases is handled in the first edition of [Shelah 1978] while both are solved in the second edition.

For each problem we know by Section XVI.3 that T does not have the appopiate version of the dimensional order property. That is, if T has the DOP then T has 2^λ models of power λ for $\lambda > \aleph_0$. In addition, if the ω-stable theory T has the ENI-DOP then T has $2^{\aleph_0}$ countable models. In Chapter XVII, we completed the solution of the spectrum problem in uncountable cardinalities for ω-stable T except for shallow theories with finite depth. For theories with finite depth we did not resolve the difficulties which arise when there are ENI-types on the leaves of the representing tree of a model. The resolution of this difficulty is closely connected to the analysis of depth two types which is necessary to prove Vaught's conjecture for countable ω-stable T.

In Section 1 of this chapter we begin the analysis of types of low depth by considering what it means for one type to support another and some transfer properties of this notion. We justify this study of types with low depth in Section 2 by showing that if there is an ω-stable counterexample to

Vaught's conjecture then it has depth two. We also compute lower bounds on the number of models in uncountable powers under certain conditions on the theory T. In Section 3 we give a detailed analysis of types of depth two which meet the conditions we discussed in Section 2 and prove Vaught's conjecture for ω-stable T. Section 4 contains an example of Shelah which shows the necessity of some of the more complicated steps in the argument. Finally, in Section 5 we complete the proof of Morley's conjecture for ω-stable T.

There are many partial results on Vaught's conjecture. The first significant general result was Morley's proof that a sentence in $L_{\omega_1,\omega}$ has either $2^{\aleph_0}$ or at most $\aleph_1$ countable models. That proof is essentially a result in descriptive set theory. Most arguments in that field would extend to pseudoelementary classes in $L_{\omega_1,\omega}$. But Morley's result does not. Most of the other work not reported here concentrates on particular theories, usually with an ordering of some kind. Thus, Rubin [Rubin 1974] showed that any complete first order theory of a linear order with monadic predicates had either finitely many or $2^{\aleph_0}$ countable models. Further work along this line in [Miller 1981] and [Marcus 1980] culminated in the proof by Steel [Steel 1978] that Vaught's conjecture holds for any $L_{\omega_1,\omega}$ sentence all of whose models are trees. Shelah [Shelah 1978a] shows that if the theory T has Skolem functions and admits a linear order of an infinite set then T has $2^{\aleph_0}$ countable models. Lascar [Lascar 1981] proved Vaught's conjecture for stable theories with Skolem functions. Lascar and Boucaren [Bouscaren & Lascar 1983], [Bouscaren 1983] made substantial progress on the ω-stable case before it was completed by Shelah [Shelah, Harrington, & Makkai 1984].

For ease of reading we adopt the convention of writing single letters to represent finite sequences rather than littering the text with bars.

In this chapter T is a *countable ω-stable theory with NDOP. Moreover, T is shallow and when dealing with Vaught's conjecture T has ENI-NDOP.* Thus, throughout the chapter we consider only the acceptable class **AT** and all notions (e.g. prime, isolation) are with respect to that class.

1. Supportive Types

In this section we work with a type $p \in S(A)$ and a realization a of p. The major task is to analyze the types which 'depend' on $A \cup a$ either directly by being over $A \cup a$ or in the slightly less direct fashion discussed in this section. The next definition describes in more detail the situation that we must analyze. We then prove a number of technical lemmas which allow us to pursue this analysis.

1.1 Definition. The type p *needs* the finite sequence a over the set A, written p needs a/A, if

i) $p \dashv A$ and $p \not\dashv A \cup a$.
ii) p is stationary and strongly regular.
iii) $t(a; A)$ is stationary and has weight one.

Of course, condition i) is the active part of the definition. We differ from [Harrington & Makkai 1985] in not requiring that p be ENI. They make this requirement because for the Vaught conjecture only the ENI case is required. For the Morley conjecture we must discuss persistently isolated types which need specific a. The type $q \in S(A)$ is *supportive* if for some a realizing q and some type p, p needs a/A. In this situation we sometimes say q *supports* p. We say p needs $q \in S(A)$ if for some realization b of q, p needs b/A. The type $q \in S(A)$ is *ENI-supportive* if for some a realizing q and some ENI type p, p needs a/A. We extend these notions to global types by saying a type $p \in S(\mathcal{M})$ is ENI-supportive if for some A, p does not fork over A and $p|A$ is ENI-supportive. Note that if T is an ENI-depth two theory, every ENI-supportive type has depth one and the ENI-types which need supportive types have depth zero. This notion of needing is somewhat more restrictive than what the word need might bring to mind. In particular, needing is not transitive. That is, if $q \in S(c)$, b realizes $p \in S(a)$, q needs b/a, and p needs $a/\emptyset$ it does not follow that q needs $a/\emptyset$.

1.2 Exercise. Show that if p needs a/A then for some finite $A_0 \subseteq A$, p needs a/A_0. (Remember that throughout this chapter we assume that T is ω-stable.)

1.3 Exercise. Show that when T is a depth two theory, every supportive type is not orthogonal to the empty set.

1.4 Exercise. Find the supportive types in the theory of an equivalence relation with infinitely many infinite classes. Do the same for ENI-supportive types if each class contains a model of $\mathrm{Th}(Z, S)$. Show the ENI-depth of the first example is one (the minimum for any theory) while the second has ENI-depth two.

The next two exercises explore the preservation of needing to supersets of A which are independent from a.

1.5 Exercise. Show that if q needs c/A and $c \downarrow_A B$ then q needs c/B.

If a theory does not have the ENI-DOP use the triviality of $\dashv$ to show

1.6 Exercise. If T has ENI-NDOP, q is ENI, q needs c/B and $c \downarrow_A B$ then q needs c/A. Contrast this exercise with Lemma XIII.3.11.

We prove the following results simultaneously for various properties which may or may not be modified by ENI, e.g. NDOP and ENI-NDOP. As usual, this dual purpose is indicated by parentheses around 'ENI'. For Vaught's conjecture we are interested in the ENI-NDOP case while for Morley's conjecture we need both versions. Our principal goal in the following few paragraphs is the analysis of supportive types of depth one. However, the following notation slightly generalizes that situation to the study of the relation between pairs of types whose depth differs by one. This added generality is needed for the study of the Morley conjecture in Section 5.

1.7 Notation. A type s is ENI-constrained by q if eni-dp$(s) \leq$ eni-dp$(q)+1$.

If the definition is made for depth rather than eni-depth one just says q constrains s. Note that r needs p does not imply that r constrains p but there will be some s which constrains p and needs p. If the (ENI)-depth of s is less than or equal one, s is (ENI)-constrained by any q which needs it.

1.8 Theorem. *Suppose T satisfies the (ENI)-NDOP, q is a stationary (ENI)-type, and q (ENI)-constrains s. For any c realizing s if $q \dashv A \cup c$ and $c \rhd_A d$ then $q \dashv A \cup d$.*

Proof. Choose a strongly $|A| + \kappa(T)$-saturated model M with $M \downarrow_A c^\frown d$. Then $d \downarrow_{A \cup c} M \cup c$ so we can choose a copy $M[c]$ of the **S**-prime model over $M \cup c$ with $d \downarrow_{A \cup c} M[c]$. Assume for contradiction that $q \not\dashv A \cup d$. Now, $q \not\dashv A \cup c \cup d$ implies by Theorem XIII.3.11 that $q \dashv M[c]$. Then, clearly, $q \not\dashv M[c][d]$. The models M, $M[c]$, $M[c,d]$ give us the situation of a three model theorem. Since $q \dashv M[c]$, $M[c]$ is a proper subset of $M[c,d]$. If $t(M[c,d]; M[c]) \dashv M$, the depth of $t(c; M)$ is at least two more than the depth of q contrary to hypothesis. Thus, by Theorem XIII.3.4 there is a $b \in M[c,d] - M[c]$ which is independent from c over M. But since $c \rhd_M d$ this implies $b \downarrow_M c^\frown d$. But this is absurd since $b \in M[c,d]$.

We draw a useful conclusion from this result.

1.9 Corollary. *Suppose T is an (ENI)-depth 2 theory which satisfies the (ENI)-NDOP and q is a stationary, (ENI)-type. If $q \dashv M \cup c$ then $q \dashv M[c]$.*

The next exercise provides a simpler proof of a weak version of Theorem XVI.1.11.

1.10 Exercise. Suppose T is an (ENI)-depth two theory satisfying the (ENI)-NDOP and q is a stationary (ENI)-type. Show that if $C = A \cap B$, $A \downarrow_C B$, $q \dashv A$, and $q \dashv B$ then $q \dashv A \cup B$.

1.11 Exercise. Show that requiring eni-dp$(s) \leq$ eni-dp$(q)+1$ is needed to prove Theorem 1.8. (Hint: Consider the theory of two refining equivalence relations with an additive structure on the classes of the coarser equivalence relation.)

We now analyze in more detail the situation when a type q needs b/B and $t(b; B)$ has (ENI)-depth one. We can 'slightly' enlarge both b and B to find a base for a strongly regular type which is not orthogonal to s. The next lemma makes 'slightly' precise and gives a normal form for depth one types.

1.12 Lemma. *Suppose $s \in S(B)$ is (ENI)-constrained by q and q needs s over B. There is a finite set D which is atomic over B and a pair of types $\hat{s}$ and s_1 in $S(D)$ which satisfy the following conditions. If b realizes s there is a d realizing s_1 with $b \subseteq d$ and a $c \subseteq d$ which realizes $\hat{s}$ such that*

i) *$t(c;D)$ is strongly regular.*
ii) *$t(d;D\cup c)$ is isolated.*
iii) *For some strongly regular $\hat{q}\in S(D\cup d)$, $q\not\perp\hat{q}$ and $\hat{q}$ needs d/D.*
iv) *$c\rhd_D d$.*

Proof. Suppose q needs b/B. Choose M prime over B with $M\downarrow_B b$. Since q has weight one Lemma XIII.3.11 implies $q\dashv M$. Applying monotonicity, $q\not\dashv B\cup b$ implies $q\not\dashv M[b]$. Choose $c\in M[b]$ with $t(c;M)$ strongly regular. By Exercise XIII.4.8 and since b has weight one, $c\rhd_M M[b]$. By Lemma 1.8 $q\not\dashv M\cup c$. Thus, for some strongly regular $q'\in S(M[c])$, $q\not\perp q'$. Choose $D\subseteq M$ and $d\subseteq M[c]-M$ with $b\cup c\subseteq d$ so that q' is strongly based on $D\cup d$. Let $\hat{q}=q'|D\cup d$. Setting $\hat{s}$ as $t(c;D)$ and s_1 as $t(d;D)$, conditions ii) and iii) are satisfied. We can invoke Lemma VI.3.12 and possibily increase the size of D to guarantee condition iv). This completes the proof.

This result refines Corollary XIII.3.4. There, we showed that if $p\not\dashv M$ and $M\models T$ then $p\not\perp q$ for some strongly regular $q\in S(M)$. While we can not relax the requirement that M be a model to show this for an arbitrary set B we find the strongly regular type by only slightly extending B. Buechler [Buechler 1986] shows that if T is superstable and '1-based' then we can find the regular type in $S(B)$. In the light of Lemma 1.12 we can establish a normal form for each depth one type. Actually, this normal form will depend on q. In this section we exploit the normal form separately for each choice of q. Later we will reduce to the case that s supports only finitely many types. Then we can choose a single s_1 to work for all of them.

1.13 Notation. For any depth one type s and any q which needs s we fix a pair of types $\hat{s}$ and s_1 satisfying Lemma 1.12. Without loss of generality we may assume each of $s,\hat{s},s_1$ have domain D. Now, $\hat{s}$ is strongly regular and if d realizes s_1, there is a $c\subseteq d$ realizing $\hat{s}$ such that $t(d;D\cup c)$ is atomic.

We want to show that if q is nonorthogonal to one realization of s it is nonorthogonal to all realizations which depend on the first. This kind of transfer requires some relation between the depth of s and that of q; for example, q constrains s. An easy counter example to the conclusion of the following lemma but with $\mathrm{dp}(t(c;A))=2$ and $\mathrm{dp}(q)=0$ can be found in the theory REF_2.

1.14 Lemma. *Suppose T satisfies the (ENI)-NDOP and q is a stationary (ENI)-type. Suppose both $t(c;A)$ and $t(c';A)$ are stationary with weight one and that q (ENI)-constrains $t(c';A)$. If $q\dashv A$ and $c\not\downarrow_A c'$ then $q\not\dashv A\cup c$ implies $q\not\dashv A\cup c'$.*

Proof. Let $B=\operatorname{dom} q$ and choose an **S**-model M with $B\cup c\cup c'\downarrow_A M$. Then $q\dashv M$, $q\not\dashv M\cup c$, and $c\not\downarrow_M c'$. Choose an independent set X of realizations of strongly regular types over M so that both $X\cup c$ and $X\cup c'$ are maximal independent sets of realizations in $M[c,c']$ of weight one types. The existence of X is guaranteed by $c\not\downarrow_M c'$. Now, $q\not\dashv M\cup c$ implies two facts. First, $q \not\dashv M[c,c']$. Second, since $c \downarrow_M x$ for each $x \in X$, we have $c \downarrow_M M[X]$. Applying Theorem XIII.3.11, we conclude $q \dashv M[X]$. By the (ENI)NDOP, $q \dashv M[X,c']$ if and only if $q \dashv M[c']$. Now suppose $q \dashv M[X,c']$. Consider the three models $M[X] \subseteq M[X,c'] \subseteq M[c,c']$. If $t(M[c,c'];M[X,c']) \dashv M[X]$ the (ENI)-depth of $t(c';M[X])$ which equals the (ENI)-depth of $t(c';M)$ is greater than the depth of q plus two. Thus, Theorem XIII.4.3 yields a $b \in M[c,c'] - M[c',X]$ which is independent from $M[c',X]$. Because this contradicts the choice of X, $q \not\dashv M[X,c']$; whence, $q \not\dashv M[c']$. By Theorem 1.8, this implies $q \not\dashv M \cup c'$ and thus, since $B \downarrow_{A\cup c'} M[c']$, $q \not\dashv A \cup c'$.

The next lemma is a further transfer property for needing.

1.15 Lemma. *Let T be an ω-stable theory with the (ENI)-NDOP. Suppose $t(b;\emptyset)$ and $t(b';\emptyset)$ have (ENI)-depth ≤ 1. If p needs c/b, p' needs c'/b', and $t(c;b) \perp t(c';b')$ then $p \perp p'$.*

Proof. Suppose first that $c \downarrow_b b'$ and $c' \downarrow_{b'} b$. Then, $p \dashv b \cup b'$ and since $t(c;b) \perp t(c';b')$, $c \downarrow_{b\cup b'} c'$. Thus, by Theorem XIII.3.11, $p \dashv b \cup b' \cup c'$. But, if $p \not\perp p'$ then $p' \not\dashv b' \cup c'$ implies by Theorem XIII.2.24 iii) that $p \not\dashv b' \cup c'$. So $p \not\dashv b \cup b' \cup c'$ and we have the result. Thus, the major task of the proof is to replace b, c, b', c' by $\hat{b}, \hat{c}, \hat{b}', \hat{c}'$ which satisfy the following conditions.

i) $stp(c^\frown b;\emptyset) = stp(\hat{c}^\frown \hat{b};\emptyset)$ and $stp(\hat{c}'^\frown \hat{b}';\emptyset) = stp(c'^\frown b';\emptyset)$.
ii) $t(\hat{c};\hat{b}) \perp t(\hat{c}';\hat{b}')$.
iii) p needs $\hat{c}/\hat{b}$ and p' needs $\hat{c}'/\hat{b}'$.
iv) $\hat{c} \downarrow_{\hat{b}} \hat{b}'$ and $\hat{c}' \downarrow_{\hat{b}'} \hat{b}$.

For this, choose first $\hat{b}, \hat{b}'$ so that $\{b^\frown c^\frown b', \hat{b}, \hat{b}'\}$ is independent over the empty set. Let $q = t(c;b)$ and $q' = t(c';b')$. Since $t(b;\emptyset)$ has (ENI)-depth at most one, $q \not\dashv \emptyset$. Writing $\hat{q}$ for $q_{\hat{b}}$, we have by Corollary VI.2.22 that $q \not\perp \hat{q}$. Let M be an S-model containing $\{b, \hat{b}', \hat{b}\}$ with $c \downarrow_{b\hat{b}'\hat{b}} M$; witness the nonorthogonality of q and $\hat{q}$ by choosing $\hat{c}$ realizing $\hat{q}$ with $\hat{c} \downarrow_{\hat{b}} M$ and $\hat{c} \not\downarrow_M c$. By transitivity of independence, $c \downarrow_b M$. Thus, p needs c/b implies p needs c/M. By Lemma 1.14 since $\hat{c} \not\downarrow_M c$, p needs $\hat{c}/M$ and thus p needs $\hat{c}/\hat{b}$. Since $\hat{b}' \in M$, $\hat{c} \downarrow_{\hat{b}} \hat{b}'$. Similarly, we can choose $\hat{c}'$ realizing the nonforking extension to $S(M)$ of $q'_{\hat{b}'}$ and verify that p' needs $\hat{c}'/\hat{b}'$ and $\hat{c}' \downarrow_{\hat{b}'} \hat{b}$. Then we complete the proof as outlined.

The following theorem plays a key role in the proof of Vaught's conjecture (cf. Lemma 3.3).

1.16 Theorem. *Let $A \subseteq M$ and suppose $s \in S(A)$ is a trivial (ENI)-depth one type. If $q \in S(M)$ needs s then there is a $b \in M$ such that q needs b/A.*

Proof. By definition of needing s, q needs d/A for some realization d of s. Suppose for contradiction that $d \downarrow_A M$. By Theorem XIII.3.11, this implies $q \dashv M \cup d$ contrary to the choice of d. Thus, $d \not\downarrow_A M$. Applying Theorem 1.12, choose $c' \subseteq d$ which realizes the strongly regular type $\hat{s} \in S(A)$ and so that d is atomic over $A \cup c$. Since nonorthogonality preserves triviality, $\hat{s}$ is also trivial. By Theorem XVI.2.12 and the triviality of $\hat{s}$, $\hat{s}$ is realized in M by some c with $c \not\downarrow_A c'$. Now, choose $b \subseteq M$ so that $t(b; A \cup c) = t(d; A \cup c')$. Since $b \not\downarrow_A d$, Lemma 1.14 implies q needs b/A.

1.17 Historical Notes. Most of this material is from the first two sections of [Shelah, Harrington, & Makkai 1984]. We have benefited greatly from discussions about this material with Buechler, Harrington, and Saffe.

2. Toward the Vaught and Morley Conjectures

In this section we obtain sufficient conditions on the type structure of a theory to compute certain lower bounds on the number of models of T. We prove first that if the countable ω-stable theory T has fewer than $2^{\aleph_0}$ countable models then it has ENI-depth at most two. We refine this analysis to show that T has only finitely many nonorthogonality classes of types in each of the following categories.

i) R^e: ENI-types which are orthogonal to the empty set.
ii) S^e: Types which support ENI-types.
iii) Q^e_b: ENI-types which need a fixed realization b of a supportive type.

As a step toward the Morley conjecture, we then make a similar analysis of the types which depend on an (ENI)-depth two type p over a set X. We introduce the notion of a frame $\langle R_p, S_p, Q_{p,s}\rangle$ for p. A frame is a triple of sets of types which form a basis for those types which need a specific realization of p over X or need some type which needs that realization over X. We prove a lower bound for the number of models of T in an uncountable power if for some p and s, R_p, S_p, or $Q_{p,s}$ is infinite.

We begin now to restrict the possibilities for a countable ω-stable theory which has fewer than $2^{\aleph_0}$ countable models. At first glance the following theorem seems to be too weak. It is easy to see that what appears to be a typical example of an ENI-type which is orthogonal to the empty set (an equivalence relation with infinitely many classes, each a model of (Z, S)) (cf. XVII.3.8) leads immediately to $2^{\aleph_0}$ models. Thus, one might hope to prove that any countable ω-stable theory with fewer than $2^{\aleph_0}$ models has ENI-depth one. However, the example in Section 4 shows this intuition is incorrect.

2.1 Theorem. *Let T be a countable ω-stable theory without ENI-DOP. If T has ENI-depth greater than 2 then T has $2^{\aleph_0}$ countable models.*

Proof. If T has ENI-depth greater than 2 then there is a type q with ENI-depth 2. Without loss of generality, we may assume q is over the empty set and choose a to realize q. Then there exist finite sequences $a \subseteq b \subseteq c$ and a nonisolated $p \in S(c)$ such that $p \dashv b$ and $t(c; b) \dashv a$; moreover p can be chosen stationary and strongly regular while $r = t(b; a)$, and $t(c; b)$ are stationary weight one types. Thus, both p and r have weight one. Without loss of generality we can absorb a into the language and work over the empty set.

For each $X \subseteq \omega$ we define a model M_X as follows. Let $B = \{b^n : n \in X\}$ be an independent sequence of realizations of $t(b; \emptyset)$. For each $n \in X$, choose a sequence $C^n = \{c_i^n : i < n\}$ of sequences which are independent over b^n such that $c_i^n \frown b^n$ realizes $t(c \frown b; \emptyset)$. Since $t(c; b) \dashv \emptyset$, $\{C^n : n \in X\}$ is an independent sequence over $\emptyset$. Let p_i^n denote $p_{c_i^n}$ and $\mathcal{P}$ denote the collection of all the p_i^n.

Use the choice of the C^n, the fact that $t(c; b)$ is stationary and Corollary VI.2.22 to show

2.2 Exercise. For $n \neq m$ or $i \neq j$, $p_i^n \perp p_j^m$.

Returning to the proof of Theorem 2.1, since the c_i^n for $i < n$ are independent over b^n, if $\tilde{p}_i^n$ denotes the nonforking extension of p_i^n to $S(C^n)$ then for any $M \supset C^n$, $\dim(\tilde{p}_i^n, M) = \dim(p_i^n, M)$. By Theorem XIV.2.4, there is a model M_X such that for each $n \in X$ and each $i < n$, $\dim(\tilde{p}_i^n, M_X) < \omega$ and if q is ω-irrelevant to $\mathcal{P}$ then $\dim(q, M_X)$ is infinite.

Since $r = t(b; \emptyset)$ has weight one, forking is mildly transitive on realizations of this type. Thus forking is an equivalence relation on $r(\mathcal{M})$. By a class $\mathcal{B}$ or $[b]$ we mean an equivalence class under this relation. To show that $M_X \approx M_Y$ implies $X = Y$ we will assign an integer $m([b])$ to each class $[b]$ and prove that $\{m([b]) : b \in r(M_X)\} = X$. Namely, $m(\mathcal{B})$ is the supremum over all $b' \in \mathcal{B}$ of the $m \leq \omega$ such that there exists an independent sequence c'_i for $i < m$ over b' with $t(c'_i \frown b'; \emptyset) = t(c \frown b; \emptyset)$ and with $\dim(p_{c'_i}, M_X) < \omega$.

The following extension of Exercise 2.2 is the key to rest of the argument. It is an immediate consequence of Lemma 1.15

2.3 Exercise. Consider any b_0, b_1, e_0, e_1 such that b_0, b_1 realize $t(b; \emptyset)$ and $t(e_i \frown b_i; \emptyset) = t(c \frown b; \emptyset)$ for $i = 0, 1$, and $b_0 \downarrow_\emptyset b_1$. Then $p_{e_0} \perp p_{e_1}$.

2.4 Exercise. Prove the preceding exercise directly rather than deriving it from Lemma 1.15.

We use this fact to compute $m(\mathcal{B})$ in several cases. If $\hat{c} \frown \hat{b}$ realizes $t(c \frown b; \emptyset)$ and $\dim(p_{\hat{c}}; M_X) < \omega$ then for some p_i^n, $p_{\hat{c}} \not\perp p_i^n$. Applying the contrapositive of Exercise 2.3, $(\hat{b} \not\downarrow b^n; \emptyset)$ for some n. Thus, if $\mathcal{B} \neq [b^n]$ for some $n \in X$, we have $m(\mathcal{B}) = 0$.

Now we show $m([b^n]) = n$. By the construction, $m([b^n]) \geq n$. Suppose $\hat{b} \in [b^n]$ and for some k that $\langle \hat{c}_i : i < k \rangle$ is an independent sequence over $\hat{b}$ with $t(\hat{c}_i \frown \hat{b}; \emptyset) = t(c \frown b; \emptyset)$. For every $m \neq n$, $\hat{b} \downarrow_\emptyset b^m$ so by Exercise 2.3, letting $\hat{p}_i$ denote $p_{\hat{c}_i}$, $\hat{p}_i \perp p_i^m$ if $m \neq n$. But $\dim(\hat{p}_i, M_X) < \omega$ implies there is some $p \in \mathcal{P}$ with $\hat{p}_i \not\perp p$. So for each $i < k$ there is an $\hat{i} < n$ with $\hat{p}_i \not\perp p_{\hat{i}}^n$. Since the p_i^n are pairwise orthogonal and nonorthogonality is transitive on strongly regular types, the map $i \mapsto \hat{i}$ must be $1 - 1$ and we conclude $m([b^n]) = n$.

Thus, the isomorphism type of M_X determines X and we finish the proof of Theorem 2.1.

We have shown any ω-stable theory with fewer than $2^{\aleph_0}$ countable models has ENI-depth at most two. We need a slightly stronger observation than the assertion that R^e is finite. The following lemma is an immediate consequence of Theorem XV.3.21 ii).

2.5 Lemma. *If T has less than $2^{\aleph_0}$ countable models then for any finite set A, there are only finitely many nonorthogonality classes of ENI-strongly regular types which are not orthogonal to A.*

This shows that R^e is finite. Moreover, for each $s \in S^e$ and each realization b of s, Q^e_b is finite. The following theorem shows we may assume S^e is finite.

2.6 Theorem. *Let T have the ENI-NDOP. If there are infinitely many pairwise orthogonal ENI-supportive types then T has $2^{\aleph_0}$ countable models.*

Proof. If there are infinitely many ENI-supportive types $\langle s_n : n < \omega \rangle$ which are pairwise orthogonal then we will normalize them as in the proof of Theorem XV.3.25. Let s_n be strongly based on b_n, let c_n realize s_n and suppose $q_n \in S(d_n)$ needs c_n/b_n. Without loss of generality, we can assume that all conditions of Lemma 1.12 are satisfied. In particular, $b_n \subseteq c_n \subseteq d_n$ and q_n is not isolated. Construct an independent sequence $\langle b_n{}^\frown c_n{}^\frown d_n : n < \omega \rangle$ realizing these types. As in Paragraph XV.3.4 we can require that for $i \neq j$ either b_i and b_j have the same type over the empty set or every conjugate of s_i is orthogonal to s_j. (If some conjugate $\alpha(s_i)$ is not orthogonal to s_j, then α^{-1} of s_j could be chosen as s_j.) Since none of the s_i are orthogonal to the empty set and $s_i \perp s_j$, the first alternative implies b_i and b_j realize distinct strong types over the empty set (Corollary VI.2.23). But, since T is ω-stable $t(b_i;\emptyset)$ has finite multiplicity. Thus, we can find a subsequence $S = \langle s_i : i < \omega \rangle$ such that if $i \neq j$ then every conjugate of s_i is orthogonal to s_j and $t(b_i;\emptyset) \neq t(b_j;\emptyset)$.

By Theorem XIV.2.4, for any $X \subseteq \omega$ there is a model M_X such that if $n \in X$, $\dim(q_n, M_X) < \omega$ and if $n \notin X$, $\dim(q_n, M_X) = \omega$. We claim $n \in X$ if and only if there is a triple $\langle b, c, d \rangle$ realizing $t(b_n{}^\frown c_n{}^\frown d_n;\emptyset)$ and with $\dim(q_d, M_X) < \omega$, where q_d is a copy of q_n over d. Clearly, if $n \in X$, we have the claim by construction. Fix n such that $\langle b, c, d \rangle$ realizes $t(b_n{}^\frown c_n{}^\frown d_n;\emptyset)$. For any $m \neq n$, the choice of S guarantees $t(c;b) \perp t(c_m;b_m)$. Since q_d needs c/b and q_m needs c_m/b_m, Lemma 1.15 implies $q_d \perp q_m$. Thus, if $n \notin X$, $q_d \perp q_m$ for every $m \in X$. By Theorem XIV.2.4, $\dim(q_d, M_X) = \omega$ as required. Since we can recover X from the isomorphism type of M_X, if $X \neq X'$ then $M_X \not\approx M_{X'}$. We have proved the theorem.

This completes the contribution of this section to Vaught's conjecture. We turn to the Morley conjecture. We now want to provide lower bounds for the number of models in power κ of an ω-stable theory which has finite depth. We describe these bounds in terms of the height of a type.

2.7 Definition. Let $p \in S(B)$ be strongly regular. If $p \not\perp \emptyset$ then $\mathrm{ht}(p) = 0$. If for some a and A, p needs a/A with $p' = t(a; A)$ and $\mathrm{ht}(p') = n$ then $\mathrm{ht}(p) \geq n + 1$. Finally, $\mathrm{ht}(p)$ is the least n such that $\mathrm{ht}(p) \not\geq n + 1$. We say p' witnesses that the height of p is at least $n + 1$.

Note that without loss of generality one may assume that $A \cup a$ is contained in $\mathrm{dom}\, p$. Thinking of p as occuring in a representation of a model M, the height of p is the height of a realization of p, not the height of the domain of p.

A major technical difficulty in establishing lower bounds arises from the problem of ensuring that nonorthogonal types in different representations of the same model have the same dimension. We solved this problem in Chapter XVII by using $\aleph_1$-ample trees to guarantee that, at least often enough, the types we were comparing could be based on the same set. The trouble is that at the top levels we may need to preserve finite dimensions but in a model prime over an $\aleph_1$-ample tree of height m there may be no types with finite dimension. The solution is to construct trees which are $\aleph_1$-ample up to height $m - 2$ or $m - 1$ and to label the top nodes of these trees with the dimensions that we want to preserve at the top level. Using the fact that the shorter trees are $\aleph_1$-ample, we obtain an isomorphism between the shorter trees. By applying the results of Chapter XIV and Section 1 of this chapter, we are able to preserve the labels on the top nodes.

Thus, the combinatorial principal underlying the counting below is the following lemma, which is proved by induction as similar results were earlier. In order to treat the problem uniformly in $\aleph_\beta$, we must allow the number of labels to depend on the subscript β.

2.8 Lemma. *Let $\lambda = \lambda(\beta)$. If $\beta \geq 1$, there are at least $\beth_m(\lambda)$ partially λ-labeled trees of power $\aleph_\beta$ which have height m and are $\aleph_1$-ample.*

Although Lemma 2.8 is proved by a routine induction, there is one important feature. In the first step, we do not try to calculate how many times the various labels are realized. Rather, we observe that there are 2^λ subsets of λ and that any set X of labels can have the property that each member of X is realized at least once and therefore at least $\aleph_1$ times while labels which are not in X are not realized. This is essentially the same idea as the proof of Lemma XVII.4.11

The following definition defines the context for the detailed analysis of depth two types. We formalize a situation analogous to the discussion of R^e, S^e, and Q_b^e earlier in this section. For the study of Morley's conjecture we must consider types which are not ENI; thus, the superscript e's disappear. The word set in this definition is not used in our technical sense (smaller than $|\mathcal{M}|$). In fact, if $\tilde{S}_a$, for example, is infinite then it will be a class in the technical sense since it contains all depth one types based in $\mathcal{M}$ which need a/X.

2.9 Definition. Let $p \in S(X)$ have depth two and let a realize p. A *frame* at a for p consists of three sets of types: $\tilde{R}_a$ $\tilde{S}_a$ and $\tilde{Q}_a$:

i) $\tilde{R}_a$ is the set of depth zero types which need a/X.
ii) $\tilde{S}_a$ is the set of depth one types which need a/X.
iii) For each $s \in \tilde{S}_a$, and any realization b of s, $\tilde{Q}_{a,b}$ is the set of types that need $b/\operatorname{dom} s$. $\tilde{Q}_{a,s}$ denotes an arbitrary conjugate over $X \cup a$ of $\tilde{Q}_{a,b}$. We write $\tilde{Q}_a$ for $\cup\{\tilde{Q}_{a,s} : s \in \tilde{S}_a\}$
iv) Let R_a, S_a, and for each $s \in S_a$ and each b realizing s, $Q_{a,b}$ be maximal pairwise orthogonal subsets of $\tilde{R}_a$, $\tilde{S}_a$, and $\tilde{Q}_{a,b}$ respectively.

Although this definition refers to a specific realization a of p, there is clearly an automorphism fixing X and taking the frame at a to the frame at b for any other realization b of p. To emphasize the independence of these notions from the specific realization of p, we write $\langle \tilde{R}_p, \tilde{S}_p, \tilde{Q}_p \rangle$ to denote an arbitrary conjugate over X of $\langle \tilde{R}_a, \tilde{S}_a, \tilde{Q}_a \rangle$ and $Q_{p,s}$ to denote an arbitrary conjugate over X of $Q_{a,s}$.

We can simplify the study of the frame for p by replacing $\tilde{R}_p$, $\tilde{S}_p$, and for each $s \in \tilde{S}_p$, $\tilde{Q}_{p,s}$ by maximal pairwise orthogonal subsets R_p, S_p, and for each $s \in S_p$, $Q_{p,s}$. As in the original case we write Q_p for the union of the $Q_{p,s}$ over $s \in S_p$. We may sometimes fix a realization a of p and speak, for example, of S_a. In this section, since we are looking for lower bounds we can work directly with $\langle R_p, S_p, Q_p \rangle$. In Section 5, to compute upper bounds, we must rely on the results of Section 1 to show that controlling $\langle R_p, S_p, Q_p \rangle$ allows one to control $\langle \tilde{R}_p, \tilde{S}_p, \tilde{Q}_p \rangle$.

In the next definition we distinguish three kinds of depth two type. In the following theorem we give, for each kind, lower bounds on the number of models of T if T has a depth two type with height r of that kind.

2.10 Definition. Let $p \in S(A)$ have depth two.

The type p has *kind I* if there is an $s \in S_p$ such that $Q_{p,s}$ is infinite.
The type p has *kind II* if S_p is infinite.
The type p has *kind III* if R_p is infinite.

Recall that in the construction of models over trees for a countable ω-stable theory we are always considering types over countable sets. Thus, the A in the following theorem can be taken to be countable.

2.11 Theorem. *Let $p \in S(A)$ have height r and depth two.*

i) *If p has kind I then $I(\aleph_\beta, AT) \geq \beth_{r+1}(|\beta + 1|^{|\omega|})$.*
ii) *If p has kind II then $I(\aleph_\beta, AT) \geq \beth_{r+1}(|\beta + \omega|)$.*
iii) *If p has kind III then $I(\aleph_\beta, AT) \geq \beth_r(|\beta + 1|^{|\omega|})$.*

Proof. For each of the three cases we will describe a $1-1$ correspondence between $\aleph_1$-ample labeled trees of height r or $r+1$ (depending on the case) and models of T. The exact labeling depends upon the kind of p, but the construction of a core tree is the same in each case.

For any $\aleph_1$-ample tree of height r, X, form a tree of models, $\bar{A}_X$, as follows. Choose p_i for $i < r$ by induction so that $p = p_0$ and p_{i+1} witnesses that the height of p_i is $r - i$. Note that these types are pairwise orthogonal.

Now let the elements of $\bar{A}_X$ with height i be prime models over realizations of conjugates of p_{r-i}. Depending upon the kind of p, we describe below the method for completing the tree $\bar{A}_X$ to a tree $\hat{A}_X$ by adding elements of height $r+1$ and $r+2$. In discussing kind I, we will label the nodes of height $r+1$. For kinds II and III we will label the top nodes (i.e. nodes of height r) of X.

Let M_X be prime over $\hat{A}_X$. Now, suppose $M_X \approx M_Y$. Then, we know by Theorem XVII.4.7 that $X \approx_q Y$ and since X and Y are $\aleph_1$-ample that $X \approx Y$, as trees. It remains to show that this isomorphism preserves the labels. However, this will be clear when we explain the assignment of labels in each case. Then in each case we complete the proof of the theorem by counting the labels and thus the labeled trees.

i) When p has kind I, we make the trees of height $r+1$ be $\aleph_1$-ample as well. Thus, if $M_X \approx M_Y$ there is an isomorphism α between the subtrees of elements with height at most $r+1$ of $\hat{A}_X$ and $\hat{A}_Y$. Since p has kind I, there is an element B of $\hat{A}_X$ which has height $r+1$ and has infinitely many pairwise orthogonal strongly regular types in $S(B)$. That is, B is $X[a][b]$ where b is a realization of the $s \in S_p$ which supports infinitely many types. In fact, without loss of generality, we can require all models of height $r+1$ to have this form. But now, for each $q_i \in S(B)$, $\dim(q_i, M_X) = \dim(\alpha(q_i), M_Y)$. Thus, we can label a node B of height $r+1$ by $f \in |\beta+1|^{|\omega|}$ if for each $i < \omega$, there are i mutually orthogonal types in $S(B)$ which are orthogonal to B^- and satisfy: $\dim(q_i, M_X) = f(i)$. We force this last condition to hold by putting $f(i)$ independent realizations of q_i into $\hat{A}_X$. By Lemma 2.8, we have $\beth_{r+1}(|\beta+1|^{|\omega|})$ distinct models of power $\aleph_\beta$ as required.

To see the labels are preserved, suppose $\hat{A}_X$ and $\hat{A}_Y$ represent M. Then there is an isomorphism between $(\hat{A}_X)^1$ and $(\hat{A}_Y)^1$ induced by nonorthogonality as in Theorem XVII.4.7. Moreover, continuing the notation of that theorem, if $q \in S(B)$ and $\mathrm{ht}(B) = r+1$ then $\alpha(q) \not\perp q$ and both are strongly based on $M_{B,\alpha(B)}$. But then $\dim(q, M) = \dim(\alpha(q)M)$.

ii) If p has kind II, S_p is infinite. We will label the nodes of height r in $\bar{A}_X$. Each such node corresponds to a realization a of p. We label the node by telling for each $s \in S_a$, the number of realizations of s which support a type of dimension $\aleph_\gamma$ for $\omega \le \gamma \le \beta$. There are then

$$\left(|\beta+\omega|^{|\beta+1|}\right)^{\omega} = |\beta+\omega|^{|\beta+\omega|}$$

labels. If α is an isomorphism between the elements of height at most r in $\hat{A}_X$ and $\hat{A}_Y$, then a and $\alpha(a)$ have the same label. Applying Lemma 2.8, there are at least $\beth_r(|\beta+\omega|^{|\beta+\omega|}) = \beth_{r+1}(|\beta+\omega|)$ models of power $\aleph_\beta$ as required.

iii) If p has kind III, for each a realizing p and for each $r \in R_a$, there are at least $|\beta+1|$ possible dimensions for r. Since R_a is infinite, there are $|\beta+1|^{\omega}$ possible labels for a. Thus, by Lemma 2.8 again, there are at least $\beth_r(|\beta+1|^{|\omega|})$ models of T as required.

Apply Theorem XVII.3.19 to show the following exercise.

2.12 Exercise. If T has depth $r+2$ and some p of height r has kind I then $I(\aleph_\beta, AT) = \beth_{r+1}(|\beta+1|^{|\omega|})$.

In this section we have computed lower bounds on $I^*(\aleph_\beta, \mathbf{AT})$ if one of R_p, S_p, or $Q_{p,s}$ is infinite. When studying Vaught's conjecture, β is zero. By Theorem 2.1, we may assume that T has ENI-depth two. In that case we found that if one of R^e, S^e or Q^e_s is infinite the lower bound is 2^ω. Thus, for further analysis of Vaught's conjecture in Section 3 we may assume each of R^e, S^e, and Q^e_s is finite. In Section 5 we improve the lower bounds in uncountable cardinalities for this case and consider the upper bounds for all cases.

2.13 Historical Notes. The discussion of Vaught's conjecture relies on [Shelah, Harrington, & Makkai 1984] and [Bouscaren 1984]. The material on Morley's conjecture appeared in [Baldwin & Harrington]. I was greatly aided by discussions with Buechler and with Saffe, who gave the first proof of the finite depth case of the Morley conjecture [Saffe 1983].

3. The Vaught Conjecture for ω-Stable Countable Theories

In this section we complete the proof of Vaught's conjecture for an ω-stable countable theory. With this in hand we finish in Section 5 the finite depth case of the proof, outlined in Chapter XVII, that the spectrum function of a countable ω-stable theory is increasing on uncountable cardinals. Throughout this section we assume T is a countable ω-stable theory with fewer than $2^{\aleph_0}$ countable models. The following notion of a frame for T formalizes the ideas introduced at the beginning of Section 2 but since T has ENI-depth two it is less complex than the analogous notion of a frame for p introduced in Definition 2.9.

3.1 Definition. A *frame* for T is a triple $\langle R^e, S^e, Q^e\rangle$ satisfying the following conditions. R^e is a maximal set of pairwise orthogonal, ENI-depth zero, ENI-types which are each nonorthogonal to the empty set. S^e is a nonorthogonality basis for the ENI-supportive types. For each $s \in S^e$ and each b realizing s, Q^e_b is a nonorthogonality basis for the ENI-types which need b over dom s. We write Q^e_s for an arbitrary representative of the conjugacy class of the $Q^e_{s,b}$ and Q^e for the union of the Q^e_s over $s \in S$.

Let M_0 be the prime model of T. Then, by Theorem XIII.3.4 each $p \in S^e \cup R^e$ is nonorthogonal to some strongly regular type which is strongly based on M_0. By Lemma 2.5 and Theorem 2.6, R^e and S^e are finite. Thus, by choosing the right members of each nonorthogonality class we can assume all members of $R^e \cup S^e$ are based on the same finite subset A of M_0. Moreover, we can assume the members of R^e are nonisolated and strongly regular.

By Lemma 2.5 again there are only finitely many mutually orthogonal ENI-types which are not orthogonal to A. Thus, we can choose a finite set D with $A \subseteq D \subseteq M$ so that if $r \not\perp A$, there is an $\tilde{r} \in S(D)$ which is strongly regular and not orthogonal to r. Perhaps enlarging our original choice, we let R^e denote a nonorthogonality basis for the set of all ENI-depth zero types which are not orthogonal to D. For each realization b of $s \in S^e$, we denote by Q_b a nonorthogonality basis for the types which need b/A. As in the previous two cases, we can assume that all members of Q_b are in $S(A \cup b')$ where b' is a finite sequence which is atomic over $A \cup b$. There is one slippery point in this specialization. There may be realizations of some $s \in S^e$ which depend on D over A. Since there are only finitely many such points we will see, in the proof Theorem 3.13, that they are harmless.

We now fix a particular model M and describe a 'basis' for M with respect to a specific frame $\langle R^e, S^e, Q^e \rangle$ for T. Recall that each $s \in S^e$ is associated with a strongly regular $\hat{s}$ given in Notation 1.13.

3.2 Notation. (Fig. 1). An ENI-*extended basis* or *set of reference points* for M relative to $\langle A, D, R^e, S^e, Q^e \rangle$ consists of

i) A basis J_r of $r(M)$ for each $r \in R^e$.
ii) A basis I_s of $s(M)$ for each $s \in S^e$.
iii) For each $b \in I_s$ and each $q \in Q^e_b$, a basis $J_{b,q}$ for $q(M)$.

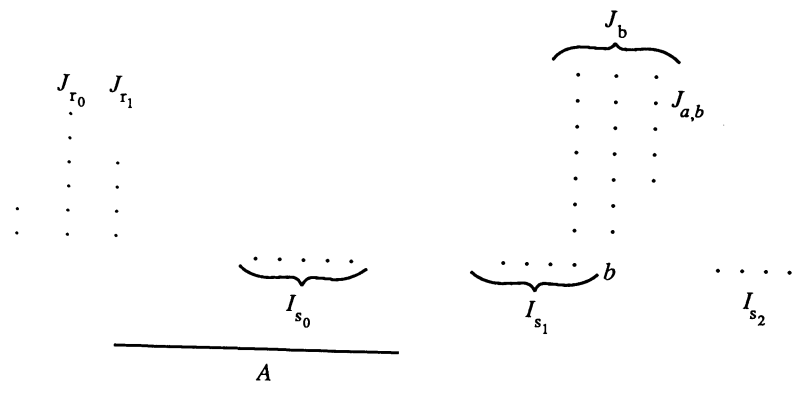

Fig. 1. An extended basis

Since Q^e_s is finite we are able to identify the s_1 from Notation 1.13 with s. Associated with I_s is a set $I_{\hat{s}}$, a basis for $\hat{s}(M)$, such that for each $b \in I_s$ there is an $a \subseteq b$ with $a \in I_{\hat{s}}$ and $t(b; A \cup a)$ is isolated.

We write J_b for $\bigcup_{q \in Q^e_b} J_{q,b}$ and J_s for $\bigcup_{b \in I_s} J_b$; we call J_b a basis for Q^e_b in M. We write I^e for $\bigcup_{r \in R} J_r \cup \bigcup_{s \in S^e} I_s \cup \bigcup_{s \in S^e} J_s$; we call I^e an extended basis for M.

Of course, some J_r and I_s may be empty. If I_s is empty, then no Q^e_b for b realizing s is even defined. Note that I is independent over A relative to the partial ordering which places c above b if $c \in J_b$ and otherwise the elements of I^e are incomparable.

The next lemma justifies the terminology 'extended basis'.

3.3 Lemma. *If I^e is an ENI-extended basis for M then M is atomic over $D \cup I^e$. Thus, if M is countable, M is prime over $D \cup I^e$.*

Proof. Let N be a maximal atomic model with $I^e \cup D \subseteq N \subseteq M$. The existence of N is guaranteed by Zorn's lemma since the union of atomic models is atomic and if $E \subseteq N$ is atomic over $D \cup I^e$ then there is a model N' which is atomic over $D \cup I^e$ with $I^e \cup D \cup E \subseteq N' \subseteq M$.

We claim $N = M$. In the ensuing two paragraphs we eliminate each possibility for a strongly regular type $q \in S(N)$ which might be realized in $M - N$. The definition of a frame provides a classification of the possible q.

First, we show that no type $q \in S(N)$ which satisfies either of the following two conditions can be realized in $M - N$. i) q is ENI. ii) q is nonorthogonal to a type which has a realization $b \in M$ such that b supports an ENI-type. For, if so, either $q \not\perp r$ for some $r \in R^e$ or $q \not\perp s$ for some $s \in S^e$ or q needs b/A for some $b \in I_s$. By Theorem 1.16 if q needs b/A, $q \not\perp q_0$ for some $q_0 \in Q^e_b$. Now in any case, since the approppropriate r, s, or q_0 is strongly regular, and not orthogonal to q, one of r^N, s^N, or ${q_0}^N$ is realized in $M - N$ contradicting the maximality of I^e.

Now suppose some persistently isolated $q \in S(N)$ is realized by some $c \in M - N$. We will show $N \cup c$ is atomic over $I^e \cup D$ and thus contradict the maximality of N. Let E be an arbitrary finite subset of N and without loss of generality assume q is strongly based on E and that $D \subseteq E$. (That is, if necessary, expand E a little.) Then $q|E$ is isolated. Since $c \downarrow_E N$ and $t(c; E)$ is stationary, $t(c; E) \vdash t(c; N)$. Applying transitivity of atomicity (Lemma IX.1.11) twice, $N \cup c$ is atomic over $E \cup I^e$ and then over $D \cup I^e$.

The next step in the proof is a solution of the difficulty we encountered in Chapter XVII. The decomposition tree describes J_b for only one member b from each of the dependency classes under forking of the realizations of the supportive types. But J_b and $J_{b'}$ may vary drastically even when b and b' are dependent. Rather than choosing a representative from each such class (by choosing a basis), it is necessary to describe J_b for each member of the class. Since in this case we are dealing with depth one types, this is not too difficult.

For the present purposes of Vaught's conjecture the β in the following definition and lemmas can be set at zero. However, it doesn't clutter the definition or excessively complicate the proofs. So since we apply the more general notion in Section 5 we obtain its properties at this time.

3.4 Definition. Let $s \in S(A)$ have depth one and let $A \subseteq M$ with $M \models T$ and $|M| = \aleph_\beta$.

i) For $c \in s(M)$ the *local β-configuration* of s at c is $\overline{v}_{c,M}$, the vector of dimensions in M of the types $q \in Q_c^e$.
ii) For $b \in s(M)$, the *β-configuration* of s at b is $\overline{v}_{[b],M} = \{\overline{v}_{c,M} : c \not\perp_A b\}$. If β is fixed we speak of an s-configuration.

The following definitions will clarify the effect of a particular type $s \in S^e$.

iii) The model M is *s-simple* if M is prime over A, a basis for s in M and for each b in this basis, a basis for Q_b^e in M.

Here is an even simpler model.

iv) The model M is *s-trite* if M is prime over $A \cup b \cup J_b$ where b realizes s and J_b is a basis for Q_b^e.

A β-configuration of s at b in M is said to be trite or simple if M is trite or simple.

Each β-configuration is thus a set (not a sequence) of countable (finite in the Vaught conjecture case) sequences of cardinals less than or equal $\aleph_\beta$. We write configuration for 0-configuration. Vectors denoted by $\overline{s}$, $\overline{t}$ range over local configurations; vectors denoted by $\bar{S}$, $\bar{T}$ range over configurations.

In Section 5 we will rely on an obvious variant on the notion of an s-configuration to include types $s \in S_a - S_a^e$. This variant is obtained by allowing the types q in the definition of a local β-configuration to range over Q_c rather than just Q_c^e (i.e. we include isolated types.)

Given the dimension of each type in R^e and for each $s \in S^e$ and for each vector $\overline{t}$, the number of members, b, of a basis for s for which $\overline{v}_{b,M}$ should equal $\overline{t}$, we may determine a model M of T. However, there may be more than one such sequence which yields the same model. For, if $\overline{v}_{b,M} = \overline{t}$ and $\overline{v}_{c,M} = \overline{s}$ and $b \not\perp c$, we *may* obtain the same model by choosing a basis element with associated dimensions $\overline{s}$ as by choosing one with associated dimensions $\overline{t}$. This situation is illustrated by the example in Section XVIII.4.

We now show that the β-configurations which occur in trite models are either equal or disjoint and then that any configuration is a union of trite β-configurations. This analysis is based on [Bouscaren 1984].

3.5 Proposition. Let b, c realize s and suppose $b \not\perp_A c$. For any $q \in Q_c$, there is a $\tilde{q} \in Q_b$ with $q \not\perp \tilde{q}$.

Proof. Since q needs c/A, Lemma 1.14 implies q needs b/A. But, Q_b is a nonorthogonality basis for the types which need b/A so we finish.

The following lemma employs the notation of 3.2. It is here that we make crucial use of the choice of the strongly regular type $\hat{s}$.

3.6 Lemma. *Let M be s-simple. Suppose $b \in s(M)$ and J_b is a basis for Q_b^e in M. Let $M_1 \subseteq M$ be prime over $A \cup b \cup J_b$. For any $c \in s(M)$ with $c \not\perp_A b$, there is a $c' \in M_1$ with $\overline{v}_{c,M} = \overline{v}_{c',M_1}$.*

Proof. We first show that for each $c \in s(M_1)$ which depends on b over A, $\overline{v}_{c,M} = \overline{v}_{c,M_1}$. Clearly, each component of $\overline{v}_{c,M_1}$ is less than the corresponding component of $\overline{v}_{c,M}$. For the converse, choose any $q \in Q^e_c$. By Proposition 3.5 there is a $q' \in Q^e_b$ with $q \not\perp q'$. Now if q is realized in $M - M_1$, so is $(q')^{M_1}$. But this contradicts the assumption that J_b is a basis for Q^e_b in M.

Note that $\overline{v}_{[b],M_1}$ is the same for every M_1 prime over $A \cup b \cup J_b$ (by the uniqueness of prime models). Thus, it suffices to show that for every $c \in s(M)$ with $c \not\downarrow_A b$, there is a copy of M_1 which contains a c' with $\overline{v}_{c',M_1} = \overline{v}_{c,M}$. By the first paragraph of this proof, we need only find an M_1 and a $c' \in M_1$ so that $\overline{v}_{c',M} = \overline{v}_{c,M}$. Recall that by Lemma 1.12 there is a strongly regular type $\hat{s}$ such that there is an $a \subseteq b$ which realizes $\hat{s}$ and such that i) $a \rhd_A b$ and ii) $t(b; a \cup A)$ is isolated. Let $d \subseteq c$ realize $\hat{s}$ and satisfy that i) $d \rhd_A c$ and ii) $t(c; d \cup A)$ is isolated.

We now use the s-simplicity of M to find a copy of M_1 which contains an appropriate c'. Let I'_s denote $I_s - \{a\}$, $I'_{\hat{s}}$ denote $I_{\hat{s}} - \{b\}$, and J'_s denote $J_s - \{J_b\}$. Making essential use of the strong regularity of $\hat{s}$ we show

$$d \underset{A \cup b \cup J_b}{\downarrow} I'_{\hat{s}}.$$

For, $c \not\downarrow_A b$ implies $c \not\downarrow_A b \cup J_b$. Since $d \rhd_A c$, $d \not\downarrow_A b \cup J_b$. Now, by regularity, $t(d; A \cup b \cup J_b) \perp \hat{s}^{A \cup b \cup J_b}$. Thus, $d \downarrow_{A \cup b \cup J_b} I'_{\hat{s}}$.

Recall that for each $f \in I_s$, there is an $g \in I_{\hat{s}}$ such that $g \rhd_A f$ and that $f \rhd_A J_f$. This implies that $I'_{\hat{s}} \rhd_A (I'_s \cup J'_s)$. The displayed formula and the independence of $I'_{\hat{s}}$ from $b \cup J_b$ imply $d \cup b \cup J_b \downarrow_A I'_{\hat{s}}$. From domination we get $d \cup b \cup J_b \downarrow_A I'_s \cup J'_s$. Now monotonicity yields $d \downarrow_{A \cup b \cup J_b} I_s \cup J_s$. Since M is atomic over $A \cup I_s \cup J_s$, by the open mapping theorem $t(d; A \cup b \cup J_b)$ is isolated. Choose a copy of M_1 prime over $A \cup b \cup J_b$ and containing d and choose $c' \in M_1$ to realize $stp(c; A \cup d)$.

Now, both Q^e_c and $Q^e_{c'}$ form nonorthogonality bases for the ENI types which need d/A. Thus, $\not\perp$ defines a $1-1$ correspondence between Q^e_c and $Q^e_{c'}$ sending, say, q to $\tilde{q}$. Since any member of Q^e_c or $Q^e_{c'}$ is not orthogonal to $A \cup d$ while c and c' realize the same isolated type over $A \cup d$, Theorem XIV.2.9 yields $\dim(q, M) = \dim(\tilde{q}, M)$. Thus $\overline{v}_{c',M} = \overline{v}_{c,M}$.

3.7 Corollary. i) *Every simple s-configuration is a trite s-configuration.*
ii) *If $\bar{S}$ and $\bar{T}$ are trite s-configurations and $\bar{S} \cap \bar{T} \neq \emptyset$ then $\bar{S} = \bar{T}$.*

Proof. In Lemma 3.6 we showed that any configuration realized in the s-simple model M is also realized in the s-trite model M_1. Thus we have i). For ii), suppose two trite configurations, $\bar{S}$ and $\bar{T}$, intersect in a tuple $\bar{t}$. There are trite models M and N and an element b with $\overline{v}_{b,M} = \overline{v}_{b,N} = \bar{t}$, but $\overline{v}_{[b],M} = \bar{S}$ and $\overline{v}_{[b],N} = \bar{T}$. Let the model M_1 be prime over $A \cup b \cup J_b$. By Lemma 3.6, $\bar{S} = \overline{v}_{[b],M} = \overline{v}_{[b],M_1} = \overline{v}_{[b],N} = \bar{T}$.

Two easy lemmas complete our analysis of s-configurations. It remains open whether, assuming for instance that T is ω-stable with fewer than $2^{\aleph_0}$ countable models, one can prove that every s-configuration is trite.

3.8 Lemma. *If $\bar{S}$ is a β-configuration for s then $\bar{S}$ is a union of trite β-configurations for s.*

Proof. Let $M \supset A$ and suppose $\overline{v}_{[b],M} = \bar{S}$. For any $c \in s(M)$ which depends on b, let M_c be prime over $c \cup J_c$ where J_c is a basis for Q^e_c in M. Then, since s has weight one, if $d \in s(M_c)$, $d \not\downarrow_A b$, so $\overline{v}_{[c],M_c} \subseteq \bar{S}$ as required.

3.9 Lemma. *For every depth one type $s \in S$ with $|Q^e_s| < \omega$, there are either $|\beta + \omega|$ or for some natural number k, $|\beta + k|$ distinct trite β-configurations of s.*

Proof. Let C be the set of cardinals less than $\aleph_\beta$ and suppose $|Q^e_s| = n$. Then each trite configuration $\bar{W}$ is a subset of C^n. For each $i < n$, there is at most one infinite cardinal λ such that for some $v \in \bar{W}$, $v_i = \lambda$. For, $v_i = \dim(q, M)$ for some $q \in Q_b$. But, if $c \not\downarrow_A b$ and $\tilde{q} \in Q_c$ with $q \not\perp \tilde{q}$, $\dim(q, M) = \dim(\tilde{q}, M) \bmod \aleph_0$. Thus, if $w = \overline{v}_{c,M}$, $w_i = \lambda$ as well. Thus, there are at least $|\beta + 1|$ distinct trite β-configurations. There are at most $|\beta + \omega|$ since this number is attained when all finite dimensions can be discriminated.

Since each β-configuration is a union of trite configurations the number of configurations will be $2^{|\beta+k|}$ or $2^{|\beta+\omega|}$. When β is finite the first case of Lemma 3.9 gives $2^{|\omega|}$ configurations; the second finitely many. The following definition isolates the difficult case. In the next section we show that such pathology actually exists.

3.10 Definition. For any B and any $s \in S(B)$ with depth one, let Q^e_s be the collection of depth zero types which are ENI and need some realization of s. Suppose Q^e_s is non-empty. If there are $|\beta + \omega|$ distinct β-configurations of s realized in models of T which have power $\aleph_\beta$ we say s is *normal.* If not, s is *abnormal.*

3.11 Definition. If T has no ENI-depth 2 types, the *full configuration* of a model M is

$$\{k_r : k_r = \dim(r, M), r \in R^e\} \cup$$
$$\{l_{\bar{T},s} : l_{\bar{T},s} = |\{b : b \in s(M) \text{ and } \overline{v}_{[b],M} = \bar{T}\}|, s \in S^e\}.$$

Recall that $\bar{T}$ ranges over configurations.

The key point of the following proof is the selection of an infinite set of trite 0-configurations for a type s such that any subset of them can be realized.

3.12 Theorem (Vaught's Conjecture for ω-stable T). *A countable ω stable theory has either at most $\aleph_0$ or $2^{\aleph_0}$ countable models.*

Proof. The successive reductions in this chapter have shown that if T has fewer than $2^{\aleph_0}$ countable models then each model is uniquely determined by its full configuration. Let $\langle R^e, S^e, Q^e \rangle$ be the ENI-frame for T. Suppose some $s \in S^e$ is normal. Let $\langle \bar{T}_i : i < \omega \rangle$ be a set of distinct trite configurations for s. Since $D - A$ is finite for each $s \in S^e$, given any infinite independent set of realizations of s, all but finitely many are independent from D over A. For any $W \subseteq \omega$, let M_W be prime over an extended basis such that if b realizes s and $b \downarrow_A D$, $|J_b| \in \bar{T}_i$ for some $i \in W$. Then $M_U \approx M_W$ implies $W = U$ so T has $2^{\aleph_0}$ countable models.

If every $s \in S^e$ is abnormal, every countable model of T is prime over one of the countably many ENI-full configurations and we finish. There are infinitely many full configurations since the full configuration counts the number of times each s-configuration is realized.

3.13 Historical Notes. The proof of Vaught's conjecture first appears in [Shelah, Harrington, & Makkai 1984]. The proof here depends greatly on the analysis of [Bouscaren 1984]. Numerous conversations with Steve Buechler further aided the preparation of this section. Bouscaren [Bouscaren 1984] has shown that the analysis expounded here yields Martin's conjecture for ω-stable countable T. To state Martin's conjecture we must define a certain extension of first order logic. The language L^* is the fragment of $L_{\omega_1,\omega}$ generated by the consistent types of T. Now, Martin's conjecture asserts that if T has fewer than $2^{\aleph_0}$ countable models then for every countable model M of T, the L^* theory of M is $\aleph_0$-categorical. The next problem is to show that an ω-stable countable theory T satisfies the strong Martin conjecture: either T has fewer than $2^{\aleph_0}$ countable models and for every countable model M of T, the L^* theory of M is $\aleph_0$-categorical or T has $2^{\aleph_0}$ distinct complete extensions in $L^*(T)$.

4. The Existence of Abnormal Types

In this section we describe Shelah's example of a countable theory which has ENI-depth two but only $\aleph_0$ countable models. This phenomenon arises from the existence of abnormal types. Thus, this example justifies the attention paid to abnormal types in the proof of the Vaught and Morley conjectures for ω-stable theories. This completes the survey in Section XVII.3 of prototypic depth 2 theories.

Locally the pathology we are investigating is manifested by the existence of a type which is abnormal (Definition 3.10). From a more global and less formal perspective the situation is this. We will find a countable ω-stable theory which has only countably many countable models but which has a non-principal type which is orthogonal to the empty set. At first blush, we hope to be able to pick the dimensions of infinitely many copies of such a type independently. In the example at hand we can not.

There are three parts to our description of the example. First, we will informally describe the axioms of the theory and show they induce certain definable relations in each model of the theory. Then, we will show that any theory whose models admit these definable relations provides the required example. Finally, we describe the prime model of the required theory.

4.1 Definition. Let the language L contain unary relation symbols P_i for $i < \omega$, binary relations symbols E, R, and a 5-ary relation symbol G.

The first part of the description (through Paragraph 4.5) contains an informal partial axiomatization of the theory. On each model M of T, the relation E will define an equivalence relation and R will be an asymmetric relation on the structure M/E. We will denote the E-equivalence classes in M by α, β etc. and write $R(\alpha, \beta)$ if for one (and hence any) $a \in \alpha$, $b \in \beta$, $R(a, b)$ holds. By a cycle in M/E we mean a finite sequence of points $\langle \alpha_0, \ldots \alpha_{n-1} \rangle$ with $n \geq 4$ which are distinct except that $\alpha_0 = \alpha_{n-1}$ and for each $i < n$ either $R(\alpha_i, \alpha_{i+1})$ or $R(\alpha_{i+1}, \alpha_i)$. If $M \models T$ then M/E shall contain no cycles. For each $i < \omega$, there is a unary predicate P_i which intersects each E-equivalence class in an infinite set. The P_i shall be disjoint; if $P_i(a)$ holds, we say a has order i. We say a has finite order to mean $P_i(a)$ holds for some $i < \omega$ without specifying the i. An element which satisfies none of the P_i is said to have infinite order.

The relation between two R-adjacent classes α and β in a model of T is determined by a map $\gamma_{\alpha,\beta}$ which depends on a point a from α and a subset $\{b, c\}$ from β. The graph of the function $\gamma_{\alpha,\beta}$ is defined by the formula $g(a, b, c, x, y)$. The relation G is symmetric in the second and third coordinates; that is, $G(a, b, c, x, y) \leftrightarrow G(a, c, b, x, y)$.

4.2 Definition. (Fig. 2). The E equivalence classes α and β are *properly linked* via $\langle a, b, c \rangle$ with $a \in \alpha$ and $b, c \in \beta$ if there exists a unique map $\gamma_{\alpha,\beta}$ such that

i) $\gamma_{\alpha,\beta}$ is a $1-1$ onto map from $\alpha - \{a\}$ to $\beta - \{b, c\}$.

ii) For every $i < \omega$, $\bigwedge_{j<i} \neg P_j(a) \rightarrow \bigwedge_{j<i} (\neg P_j(b) \wedge \neg P_j(c))$.

iii) For every $i < \omega$, $P_i(a) \rightarrow (P_i(b) \vee P_i(c))$.

iv) For every $j < \omega$,

$$(\forall x)[(\neg P_j(x) \wedge P_j(\gamma_{\alpha,\beta}(x)) \rightarrow (P_j(a) \wedge [(P_j(b) \wedge \neg P_j(c)] \vee [\neg P_j(b) \wedge P_j(c)])].$$

v) For every $j < \omega$,

$$(P_j(a) \wedge P_j(b) \wedge \neg P_j(c)) \rightarrow ((\exists! x)[\neg P_j(x) \wedge P_j(\gamma_{\alpha,\beta}(x))]).$$

v′) For every $j < \omega$,

$$(P_j(a) \wedge P_j(c) \wedge \neg P_j(b)) \rightarrow ((\exists! x)[\neg P_j(x) \wedge P_j(\gamma_{\alpha,\beta}(x))]).$$

vi) For every $i, j < \omega$,

$$(\forall x)((\neg P_i(x) \wedge P_i(\gamma_{\alpha,\beta}(x)) \wedge P_i(b)) \rightarrow [P_j(c) \leftrightarrow P_j(x)]).$$

vi′) For every $i, j < \omega$,

$$(\forall x)((\neg P_i(x) \wedge P_i(\gamma_{\alpha,\beta}(x)) \wedge P_i(c)) \rightarrow [P_j(b) \leftrightarrow P_j(x)]).$$

It easily follows from ii) that if a has infinite order so do both b and c. It is possible for b or c to have infinite order while a does not. If both b and c have

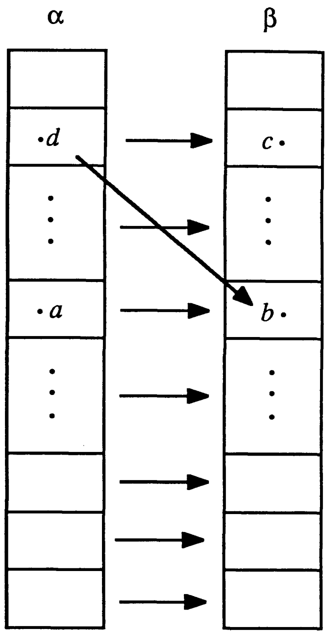

Fig. 2. α and β are properly linked; a, b, and c have finite order.

infinite order so does a. These axioms imply for all j, $P_j(x) \leftrightarrow P_j(\gamma_{\alpha,\beta}(x))$ holds with at most one exception. Moreover, this exception can occur only if a and thus one of b or c has finite order. Note that the conditions in Definition 4.2 are symmetric in b and c.

4.3 Uniqueness of Proper Linkings. i) There is a unique proper link between α and β if they are R-adjacent.

ii) If $G(a, b, c, x, y)$ defines a nonempty relation on $\alpha \times \beta$ then α and β are properly linked via $\langle a, b, c\rangle$.

By condition 4.3 ii), the fact that α and β are properly linked by $\langle a, b, c\rangle$ can be expressed by a single first order sentence. Note however that the properties of a proper linking described in Definition 4.2 can only be expressed by an infinite conjunction.

Each model of T is partitioned by the connected components of M/E. In order to calculate the number of models we will analyze the possible structure of a component. For this we require the following notation.

4.4 Notation. i) For any equivalence class α, q_α denotes the type of an element of α which is in no P_i (has infinite order).

ii) We write $\dim(q_\alpha, M)$ for the cardinality of the set of elements of infinite order in α.

When the choice of M is clear we write $\dim(q_\alpha)$ for $\dim(q_\alpha, M)$.

After the fact we will justify the notation in Definition 4.4 ii) by showing that for any model M of T and any α, the realizations of q_α in M, denoted $q_\alpha(M)$, form a set of indiscernibles. This indiscernibility will be a special case of the proof that the isomorphism type of any component of a model of T is determined by the dimensions of the equivalence classes in that component. In fact, we will show that if $\dim(q_\alpha) = \dim(q_\beta)$ then any isomorphism between α and β extends to an isomorphism of the components containing them. Such an isomorphism can be easily constructed by going back and forth between the components once the existence of enough proper linkings is stipulated. We so stipulate in the next family of axioms.

4.5 Existence of Proper Linkings. 1) If α is R-adjacent to β then α and β are properly linked.

2) We must specify that each α has many R-successors.

i) For each equivalence class α, and each $a \in \alpha$ there exist infinitely many β which are R-successors of α and for each β unique elements $b, c \in \beta$ such that α and β are properly linked by $\langle a, b, c\rangle$.

ii) When a has finite order i, we must make this requirement more specific.

a) There are infinitely many β and for each β unique b, c each having order i so that α and β are properly linked by $\langle a, b, c\rangle$.

b) For each $d \in \alpha$ with order $j > i$, there exist infinitely many β which are R-successors of α and for each β unique elements $b, c \in \beta$ such that α and β are properly linked by $\langle a, b, c\rangle$ and $\gamma_{\alpha,\beta}(d)$ has order i.

3) We must specify that each β has many R-predecessors.

i) For each equivalence class β and each $b, c \in \beta$ there exist infinitely many α which are R-predecessors of β and for each α a unique element $a \in \alpha$ such that α and β are properly linked by $\langle a, b, c\rangle$.

ii) If b has finite order i, we must make this requirement more specific. For every $c \neq b$ with order greater than or equal to i, and for every $e \neq b \in \beta$ with order i, there are infinitely many α and for each α a unique a with order i so that α and β are properly linked by $\langle a, b, c\rangle$ and $\gamma^{-1}_{\alpha,\beta}(e)$ has the same order as c.
To achieve this effect with a single sentence even in the case that the order of c differs from that of b note that in view of Definition 4.2 it suffices to assert $\neg P_i(\gamma^{-1}_{\alpha,\beta}(e)))$. The symmetry between b and c makes it unneccessary to repeat condition 3 substituting c for b.

The calculation of the number of countable models follows easily from the following result.

4.6 Claim 1. *Each countable component, C, of a model, M, of T falls into exactly one of the following three classes.*

i) *For each* $\alpha \subset C$, $\dim(q_\alpha) = 0$.
ii) *For each* $\alpha \subset C$, $\dim(q_\alpha) = \aleph_0$.
iii) *For each* $\alpha \subset C$, $1 \leq \dim(q_\alpha) < \aleph_0$.

In the terminology of Definition 3.4, Claim 1 asserts that the unique type over the empty set sets admits only three 0-configurations: $\{0\}, \{\aleph_0\}, \omega$. The claim follows from the following more specific assertion.

4.7 Claim 2. *Suppose* $R(\alpha, \beta)$ *and* α *and* β *are properly linked by* $\langle a, b, c\rangle$.

i) $\dim(q_\alpha) \leq \dim(q_\beta) \leq \dim(q_\alpha) + 1$.
ii) *If* $\dim(q_\alpha) = 0$ *or* $\dim(q_\beta) = 0$ *then* $\dim(q_\alpha) = \dim(q_\beta)$.
iii) *If* $\dim(q_\alpha) \geq 1$ *then for some* β *properly linked to* α, $R(\alpha, \beta)$ *holds and* $\dim(q_\beta) = \dim(q_\alpha) + 1$.
iv) *If* $\dim(q_\beta) > 1$ *then for some* α *properly linked to* β, $R(\alpha, \beta)$ *holds and* $\dim(q_\beta) = \dim(q_\alpha) + 1$.

Proof. We must examine the cases depending on which of a, b, c have infinite order.

a) All three have infinite order. Note that this case occurs if a has infinite order. In this situation $\gamma_{\alpha,\beta}$ is a $1-1$ onto map from $q_\alpha(M) - \{a\}$ onto $q_\beta(M) - \{b, c\}$. Thus, $\dim(q_\beta) = \dim(q_\alpha) + 1$.

b) Suppose a and b have finite order, i, while c has infinite order. By Definition 4.2 v) we can choose a d with $\neg P_i(d) \wedge P_i(\gamma_{\alpha,\beta}(d))$. By vi) d has infinite order. In this case $\gamma_{\alpha,\beta}$ is a $1-1$ onto map from $q_\alpha(M) - \{d\}$ onto $q_\beta(M) - \{c\}$. Thus, $\dim(q_\beta) = \dim(q_\alpha)$.

b′) Suppose a and c have finite order, i, while b has infinite order. Now, the analysis is just as in case b) interchanging the roles of b and c with appeal to v′) and vi′) instead of v) and vi). We find $\dim(q_\beta) = \dim(q_\alpha)$.

c) All three have finite order. If they all have the same order condition 4.2 iv) implies that $\gamma_{\alpha,\beta}$ preserves the P_i and $\dim(q_\beta) = \dim(q_\alpha)$. If b and c have different orders we proceed as in case b) or b′) depending on which order is larger. In either case $\gamma_{\alpha,\beta}$ is a $1-1$ onto map from $q_\alpha(M)$ onto $q_\beta(M)$. Thus, $\dim(q_\beta) = \dim(q_\alpha)$.

In all three cases conclusion i) holds. If $\dim(q_\alpha) = 0$ or $\dim(q_\beta) = 0$ it is impossible (invoking Definition 4.2 ii) in the second case) to choose a to realize q_α. Thus, conclusion ii) holds. If $\dim(q_\alpha) \geq 1$ and $\dim(q_\beta) > 1$, we can choose a, b and c to have infinite order. Applying the appropriate one of the existence conditions (4.5) we conclude iii) and iv).

4.8 Exercise. Show that if $\dim(q_\beta) = 1$ then for any R-predecessor α of β, $\dim(q_\alpha) = 1$.

4.9 Theorem. *Let* T *satisfy the axioms given in this section. Then* T *is complete and has countably many countable models.*

Proof. It is easy to conclude Claim 1 from Claim 2. The isomorphism class of a component is determined by the category from Claim 1 in which it fits. In fact, if α and α' are two equivalence classes with the same dimension any $1-1$ map between α and β which preserves the P_i can be extended to an isomorphism between the components which contain α and α'. This isomorphism is constructed equivalence class by equivalence class using the existence conditions 4.5 and inducting on the distance in M/R from α. But then a countable model of T is determined by the number of components of each of the three kinds it has. Thus T has only countably many countable models.

It remains to show that there is a consistent theory satisfying the conditions described above. For those with sufficient intuition, the prescription in those conditions will lead directly to the model. This path is smoothed if one thinks of M/R as a tree where any class is chosen as a base point and whether an R-adjacent point is a successor or a predecessor is indicated by coloring the edge connecting them.

For a more concrete representation, we will explicitly construct the prime model of the theory. The first difficulty in such a construction is to explicitly describe a partial order (A, R) which is isomorphic to the required structure $(M/E, R)$. The following notation will enable us to overcome this problem. The structure we define here is the solution to the proportion

$$\omega : Z = \omega^{<\omega} : x.$$

4.10 Notation. Let V_ω denote the set of all partial functions with domain an initial segment of the integers and range contained in ω. Now $(V_\omega, \subseteq)$ is the required partial order.

In the following definition we require several auxilliary functions. Let δ be a map from $\omega \times \omega$ onto the direct product of ω^5 with the set A of subsets of ω with one or two elements. We require the following properties of δ.

$$\text{For each } n, (\lambda x)\delta(n, x) \text{ maps } \omega \text{ bijectively onto } A \times \omega^5$$

$$\text{For each } m, (\lambda x)\delta(x, m) \text{ maps } \omega \text{ bijectively onto } A \times \omega^5$$

If $\delta(m, n)$ picks out a two element subset of ω we write $\delta_l(m, n)$ (respectively $\delta_h(m, n)$) for the lesser (greater) of the two element subset. If $\delta(m, n)$ picks out a one element subset of ω we just let $\delta_l(m, n) = \delta_h(m, n)$. The 5-tuple is denoted $\langle \delta_0(m, n), \delta_1(m, n), \delta_2(m, n), \delta_3(m, n), \delta_4(m, n) \rangle$. The function δ can easily be constructed by analogy with the enumeration of the rational numbers.

We will define a structure M so that E and R are the equivalence relation and graph described after Definition 4.1 and the P_i are the unary relations. Suppose α and β are properly linked by $\langle a, b, c \rangle$, the order of a is r, and the order of c is s. Let g be the partial function whose graph is obtained by fixing three elements and then projecting G on the last two coordinates. Then $(\lambda x)g(a, b, c, x)$ is $\gamma_{\alpha,\beta}$.

Let $W = \omega \times \omega$. The universe of M is $V_\omega \times V_\omega \times W$. In the following we let s, t range over elements of V_ω and $w = \langle m, n \rangle$ range over W.

- $E(\langle s, t, w \rangle, \langle s', t', w' \rangle)$ iff $s = s' \wedge t = t'$.
- $R(\langle s, t, w \rangle, \langle s', t', w' \rangle)$ iff for some $m, n \in \omega$, $s' = s^\frown m$ and $t = t'^\frown n$.

- $P_i(\langle s,t,\langle m,n\rangle\rangle)$ iff $m=i$.

We say that the E class of $\langle s,t,w\rangle$ is indexed by $\langle s,t\rangle$.

Finally we define the partial function g which connects those equivalence classes which are properly linked by $\langle a,b,c\rangle$. Suppose that the two classes are indexed by $\langle s,t\rangle$ and $\langle s',t'\rangle$. Let $s'=s^\frown m$ and $t=t'^\frown n$. The elements a,b,c,d, and $e=\gamma_{\alpha,\beta}(d)$ have the following explicit representations.

$$\begin{aligned}
a &= \langle s,t,\delta_l(m,n),\delta_0(m,n)\rangle\\
b &= \langle s',t',\delta_l(m,n),\delta_1(m,n)\rangle\\
c &= \langle s',t',\delta_h(m,n),\delta_2(m,n)\rangle\\
d &= \langle s,t,\delta_h(m,n),\delta_3(m,n)\rangle\\
e &= \langle s',t',\delta_l(m,n),\delta_4(m,n)\rangle
\end{aligned}$$

The relation G will only hold when the first three coordinates are the a,b,c which properly link two successive E-classes. Then, $G(a,b,c,x,y)$ holds exactly when $x=\langle s,t,u,v\rangle$ and $v=\langle s',t',u,v\rangle$ with the following exceptions.

i) There is no y so that $G(a,b,c,a,y)$ holds.
ii) There is no x so that $G(a,b,c,x,b)$ or $G(a,b,c,x,c)$ holds.
iii) For any $i,j,k\in\omega$ let $\sigma_{i,j,k}$ be a bijection between $\omega-\{i\}$ and $\omega-\{j,k\}$. Then if x has the form $\langle s,t,\delta_l(m,n),v\rangle$, $G(a,b,c,x,y)$ holds if $y=\langle s',t',\delta_l(m,n),\sigma_{\delta_0(m,n),\delta_1(m,n),\delta_4(m,n)}(v)\rangle$.
iv) $G(a,b,c,d,e)$ holds.

Now checking that $[a]E$ and $[b]E$ are properly linked by a,b,c and then that M has the properties described in Paragraphs 4.1 through 4.5 completes the formal proof that T is consistent.

To give a little more insight into the example consider the following proposition.

4.11 Proposition. *For any model M of T and any α, the realizations of q_α in M, denoted $q_\alpha(M)$, form a set of indiscernibles.*

This proposition can be deduced as a special case of the general argument for extending maps which preserve the P_i from one equivalence class to the entire component. The following alternative argument also displays some of the structure of the automorphism group of M. For this we require some further notation.

4.12 Notation. Let L_n denote the language containing E, R, G, and the P_i for $i<n$.

4.13 Lemma. *Fix $n<\omega$. Let β be an E-class of M. If π is a permutation of β which fixes $P_i(M)$ as a set for $i<n$ then π extends to an automorphism $\hat\pi$ of $M|L_n$.*

Proof. We will define the extension of π to the R-predecessors of β. Suppose α precedes β with α indexed by $\langle s,t\rangle$ and β indexed by $\langle s',t'\rangle$. Suppose $s' = s^\frown m$ and $t = t'^\frown n$. Let

$$\pi(b) = \langle s', t', l_0, l_1\rangle$$

and

$$\pi(c) = \langle s', t', k_0, k_1\rangle.$$

Choose n' so that

$$\delta_l(m, n') = \min(l_0, k_0)$$
$$\delta_h(m, n') = \max(l_0, k_0)$$
$$\delta_1(m, n') = l_1$$
$$\delta_2(m, n') = k_1.$$

Now our strategy is to map α to the class α' indexed by $\langle s, t'^\frown n'\rangle$. We extend the domain of the partial automorphism π to α by defining

$$\hat{\pi}(a) = \langle s, t^\frown n', \delta_l(m, n'), \delta_0(m, n')\rangle$$

and

$$\hat{\pi}(\langle s,t,u,v\rangle) = \langle s, t^\frown n', \gamma_{\alpha',\beta}{}^{-1} \circ \pi \circ \gamma_{\alpha,\beta}(\langle s,t,u,v\rangle)\rangle.$$

There is no conflict between the two clauses of the definition since a is not in the domain of $\gamma_{\alpha,\beta}$. There are in fact infinitely many possible choices for n'.

We have seen how to extend π to the R-predecessors of β. The situation for the R-successors is similar. An easy induction using the same procedure extends π to an automorphism of M.

Lemma 4.13 shows that if we fix pointwise a finite number of the sets $P_i(M) \cap \beta$, then the automorphism group of M acts transitively on the remainder of β. This ensures that there are no definable relations on $q_\alpha(N)$ for any $N \models T$ and any α and demonstrates Proposition 4.11.

To show that T is ω-stable we require another observation about the automorphism group of M.

If two elements, d and e, of M are in the same P_i, say P_n, then there is an automorphism taking one to the other. For, the structure of M/E (ignoring G and the P_i) is the structure of a directed graph without cycles such that each element has infinitely many predecessors and infinitely many successors. It is easy to construct inductively automorphisms which guarantee that the automorphism group of such a structure acts transitively on it. To find an automorphism of M taking d to e, choose an automorphism π_0 of M/E taking the E-class of d to the E-class of e. Then π_0 induces an automorphism of M which we also denote by π_0. Let π_1 stabilize $P_i(M) \cap \beta$ for $i \leq n$ and map $\pi_0(d)$ to e. By Lemma 4.13 we extend π_1 to an automorphism of M. Then $\pi_1 \circ \pi_0$ is the required automorphism of M mapping

d to e. This yields that there are only countably many 1-types over the empty set.

4.14 Theorem. *The theory* $T = \mathrm{Th}(M)$ *is* ω*-stable.*

Proof. As we have just argued that there are only countably many one types over the empty set, it suffices to show that each type over the empty set has only countably many extensions over the countably saturated model of T. There are, in fact, just three basic kinds of non-algebraic type. For each of them, there are only countably many completions to an element of $S(M)$. To verify this in detail one must consider each of the a, b, c, d, e associated with a particular realization.

i) $p \supseteq q = \{E(x, m_0) : \text{some } m_0 \in M\} \cup \{x \neq m_0 : m_0 \in M\}$.
ii) $p \supseteq q = \{\neg E(x, m) : m \in M\} \cup \{R(x, m_0) : \text{some } m_0 \in M\}$.
iii) $p \supseteq q = \{\neg E(x, m) : m \in M\} \cup \{\neg R(x, m) : m \in M\}$.

This completes the description of the basic example. One other necessary feature of the example follows from the proof by Bouscaren (Proposition 4 of [Bouscaren 1983]) that if $U(t(\overline{a}; \emptyset))$ is finite then each $q \in S(\overline{a})$ is normal.

4.15 Exercise. Complete the proof of Theorem 4.14.

4.16 Exercise. Verify directly that in this example any $p \in S(\emptyset)$ has infinite U rank.

4.17 Exercise. Show that T has ENI-depth 2.

The example given does not show that the abnormal types affect the computation of the spectrum in uncountable powers as T has $2^{\aleph_1}$ models of power $\aleph_1$. Harrington suggested the following variation to solve this difficulty. Add to the language unary functions which guarantee that all the P_i have the same cardinality. Observe the following.

4.18 Theorem. *If a theory* T' *is an expansion of theory satisfying conditions 4.2, 4.3 and 4.5 and all the* P_i *have the same cardinality then* T *has only countably many models of power* $\aleph_1$.

Proof. There are now two categories of component: those in which each P_i is countable and those in which each P_i is uncountable. The first category divides into the four cases discussed in Claim 1. In the second category, there is the additional case that all the q_α have dimension $\aleph_1$.

4.19 Historical Notes. This example was discovered by Shelah in 1981. Lascar and Bouscaren had solved the Vaught conjecture for countable ω-stable T modulo the assumption that T admits no abnormal types. Moreover, they had given sufficient conditions, e.g., α_T is finite, for there to be no abnormal types. In the summer of 1984 Harrington and Cherlin presented the example at the Forking Festival. This writeup differs from that presentation primarily in the choice of description. The example was described as the model completion of a theory satisfying certain conditions;

we have specified the prime model. In addition to discussions with Buechler, Hrushovsky, and Harrington, we owe a great debt to Alan Mekler for correcting errors in the penultimate version.

5. *The Morley Conjecture for ω-Stable Theories*

We turn now to the proof of Morley's conjecture for countable ω-stable T. In Chapter XVII we reduced to the case that T has NDOP and has finite depth, say m. Thus, if $M \models T$, M can be decomposed by a normal tree of countable models $\langle \bar{A}, < \rangle$ with depth m. We can assume that T has fewer than 2^λ models of power λ for each λ. Thus, by Theorem XV.2.12 we can assume every type with positive depth is trivial. The analysis in Chapter XVII showed that M was completely determined by the associated ω-labeled tree where $t(a; A^-)$ labels $A \in \bar{A}$ when $A = A^-[a]$. The resulting upper bound, $\beth_{m-1}(|\beta + \omega|^{|\omega|})$, on the number of models in power $\aleph_\beta$ is not exact for some theories; it is computed on the worst case assumption that there are infinitely many pairwise orthogonal ENI types over each relevant $A \in \bar{A}$ and that all significant types are normal. A finer analysis must consider the situation where there are only finitely many types over some A and where some of the types may be isolated or abnormal. In Section 2 we computed lower bounds on the number of models if some type p of height r and depth two has R_p, S_p, or some $Q_{p,s}$ infinite. In this section we compute the upper bounds in each case. Moreover, we extend the analysis to deal with abnormal types and with one further problem. Recall that the strategy in Section 2 was to construct models from trees that were $\aleph_1$-ample up to height r and to label the nodes of height r. Ideally, we would just apply this strategy with $r = m - 1$. However, the nodes of height $m - 1$ might create few models while the nodes of height $m - 2$ create many. This problem is illustrated by the following two examples.

5.1 Example. i) The difficulty arises already at depth 2. Let the models of T be a disjoint union of models of the theory T' which has infinitely many disjoint unary predicates and a model of $\mathrm{Th}(Z, S)$ in each predicate and the theory T'' of a single equivalence relation with infinitely many infinite classes. Then in power $\aleph_1$, T has $2^{\aleph_0}$ models given by the depth one theory T', not the $\aleph_0$ models given by the depth two theory T''.

ii) The following, somewhat more complicated example, illustrates the problem at depth 3. Let T be the theory of the disjoint union of the theory T_2 from Examples XVII.3.8 and REF_2, the theory of two refining equivalence relations. Then the depth of T is determined by REF_2 as 3. But T has $|\beta + \omega|^{|\beta+1|^{|\omega|}}$ models of power $\aleph_\beta$ as determined by the depth 2 theory T_2, not the $|\beta + \omega|^{|\beta+1|}$ models of REF_2. In particular, there are $2^{2^{|\omega|}}$ models of power $\aleph_1$, not $2^{|\omega|}$.

Note, however, that while the fragment of the theory that has lower depth may dominate the spectrum function for small values of β, for large β the spectrum function is determined by the types of greater depth. This partially explains why we were able to complete the analysis for large values of $\aleph_\beta$ with less effort in Chapter XVII.

In the light of the example in Section 4 and the analysis in Section 2 we extend the list from Definition 2.9 of kinds of depth two types.

5.2 Definition. i) The depth two type $p \in S(X)$ has kind IV if R_p, S_p, and Q_p, are all finite but some $s \in S_p^e$ is normal.

ii) The depth two type $p \in S(X)$ has kind V if R_p, S_p, and Q_p are all finite and every $s \in S_p^e$ is abnormal.

The next definition and the following two lemmas deal with the problem of abnormal types. Suppose $\bar{A}$ and $\bar{B}$ represent M. By Section XVII.4 we establish a quasiisomorphism between $\bar{A}^1$ and $\bar{B}^1$. In fact, since the depth is finite we can guarantee that this map is an isomorphism of trees. We will label the top nodes of $\bar{A}$ and $\bar{B}$ by full configurations (see the next definition). To see these labels are preserved by the isomorphism of trees, recall that it is induced by nonorthogonality. Thus we would like to show configurations are preserved by nonorthogonality. Lemma 5.5 almost accomplishes that aim and comes close enough to permit us to make the desired calculations. After establishing this preservation result, we finish the computation of the spectrum paying attention to the problem described in Example 5.1.

5.3 Definition. Let $p \in S(X)$ have depth two. Let a realize p. The *full configuration* of a in a model M is

$$\{k_r : k_r = \dim(r, M), r \in R_a\} \cup$$
$$\{l_{\bar{T},s} : l_{\bar{T},s} = |\{b : b \in s(M) \text{ and } \bar{v}_{[b],M} = \bar{T}\}|, s \in S_a\}.$$

Here we need the β-configurations from Definition 3.4 in order to describe models of power $\aleph_\beta$.

5.4 Definition. Let p_1, p_2 be depth 2 types. Then p_1 and p_2 are *similar*, written $p_1 \sim p_2$ if for arbitrary a_1 realizing p_1 and b_1 realizing $s_{a_1} \in S_{a_1}$, there exist a_2 realizing p_2 and b_2 realizing $s_{a_2} \in S_{a_2}$, such that nonorthogonality induces $1-1$ correspondences between R_{a_1} and R_{a_2}, S_{a_1} and S_{a_2}, and between Q_{a_1,b_1} and Q_{a_2,b_2}. For simplicity we denote all these maps as α.

We show that nonorthogonal depth two types are similar and have almost the same configuration.

5.5 Lemma. *Let $\bar{A}$ be an* **AT**-*decomposition of M and suppose $X \in \bar{A}$. Let $p_1, p_2 \in S(X)$ have depth two and Kind IV or V. If $p_1 \not\perp p_2$ then*

i) $p_1 \sim p_2$

ii) *For each a_1 realizing p_1, there is an a_2 realizing p_2 such that $a_1 \not\downarrow_X a_2$ and*

a) *For each* $r \in R_{a_1}$, $\dim(r, M) = \dim(\alpha(r), M) \mod \aleph_0$.
b) *For each* $s \in S_{a_1}$, $\dim(s, M) = \dim(\alpha(s), M) \mod \aleph_0$.
c) *For each* $s \in S_{a_1}$, *each realization* b_1 *of* s, *and each* $q \in Q_{a_1,b_1}$, *there is an integer* k *and a* b_2 *realizing* $\alpha(s)$ *such that*

$$\dim(q, M) = \dim(\alpha(q), M) + k.$$

Proof. Just as we refined our picture of the frame of a model at the beginning of Section 3, we can invoke the tools of Section 1 to clarify the description of a frame at a realization of p_1. Since p_1 has kind IV or V, S_{p_1} is finite. Thus, we can find a type $\tilde{p}_1$ such that if a_1 realizes p_1 there is a finite sequence $\tilde{a}_1$ containing a_1 and realizing $\tilde{p}_1$ such that $\tilde{a}_1$ is atomic over a_1, $a_1 \rhd_X \tilde{a}_1$, and every member of S_{a_1} is strongly based on $\tilde{a}_1 \cup X$.

Since p_1 and p_2 are trivial we can find by Lemma XVI.2.12 a realization $a_2 \in X[a_1]$ of p_2 with $a_1 \not\downarrow_X a_2$. Just as in the preceding paragraph we can choose an $\tilde{a}_2$ on which all members of S_{a_2} are based. By Lemma 1.14 each member of S_{a_1} needs $\tilde{a}_2/X$. Thus, since nonorthogonality is trivial we establish a $1-1$ correspondence between S_{a_1} and S_{a_2}. This correspondence sends s to $\hat{s}$ with $s \not\perp \hat{s}$. Applying Theorem XIV.2.8 we have condition b). To establish the similarity of p_1 and p_2 we note now that since each s_a has depth one, Lemma 1.14 allows us to establish the required $1-1$ correspondence between Q_{a_1,b_1} and Q_{a_2,b_2}. Finally, we establish the correspondence between R_{a_1} and R_{a_2}. For if r is in R_{a_1} and not in R_{a_2} then r is nonorthogonal to some member $\hat{q}$ of Q_{a_2}. But this is impossible. As, $\hat{q}$ is nonorthogonal to some $q \in Q_{a_1}$ and $q \perp r$. This proves i) and a) and b) of ii).

Condition c) follows from the next lemma. We already established the correspondence between a_1 and a_2 and between s and $\alpha(s) = \hat{s}$. Since s and $\hat{s}$ are both strongly based on subsets of $X[a_1]$, we can take them both as types over $X[a_1]$. To simplify our notation and since we don't use any special properties of the construction of $X[a_1]$, we write N for $X[a_1]$. It is important for the following argument to remember that $\overline{v}_{[b]}$ is the collection of $\overline{v}_{b'}$, the vector of dimensions of types based on b', for $b' \in [b]$.

5.6 Lemma. *Let* $s, \hat{s} \in S(N)$ *be nonorthogonal trivial depth one types. There is a* $\overline{k} = \overline{k}_{s,\hat{s}}$ *such that for any* $c \in s(M)$ *there is a* $\hat{c}$ *realizing* $\hat{s}$ *with* $c \not\downarrow_N \hat{c}$ *and* $\overline{v}_{[\hat{c}],M} = \overline{v}_{[c],M} + \overline{k}$.

Proof. For ease of reading we find a k_q for a particular $q \in Q_s$. The required $\overline{k}$ is then the finite sequence of the k_q's. For isolated $q \in Q_b$ there is nothing to prove. For each $b \in s(M)$ and each $q_b \in Q_b^e$, we can choose by triviality and Lemma XVI.2.12 a $\hat{b} \in N[b]$ which realizes $\hat{s}(M)$ and as in Proposition 3.5 a $q_{\hat{b}}$ such that $b \not\downarrow_N \hat{b}$ and $q_b \not\perp q_{\hat{b}}$. Let k_q denote $\dim(q_b, M) - \dim(q_{\hat{b}}, M)$. We show the result first for c which depend on b over N and then show that the argument trivially extends to an arbitrary c realizing s. Since $t(\hat{b}; N \cup b)$ is principal, for any $c \in s(M)$ with $c \not\downarrow_N b$, we choose $\hat{c}$ so that $t(b^\frown\hat{b}; N) = t(c^\frown\hat{c}; N)$. Thus, choosing $N[c]$ to contain $\hat{c}$, we have

$$\dim(q_c, N[c]) - \dim(q_{\hat{c}}, N[c]) = \dim(q_b, N[b]) - \dim(q_{\hat{b}}, N[b]).$$

But,

$$\dim(q_c, M) = \dim(q_c, N[c]) + \dim(q_c^{N[c]}, M)$$

and

$$\dim(q_{\hat{c}}, M) = \dim(q_{\hat{c}}, N[c]) + \dim(q_{\hat{c}}^{N[c]}, M).$$

Moreover, by Theorem XIV.2.7, $\dim(q_c^{N[c]}, M) = \dim(q_{\hat{c}}^{N[c]}, M)$. So

$$\dim(q_c, N[c]) - \dim(q_{\hat{c}}, N[c]) = \dim(q_c, M) - \dim(q_{\hat{c}}, M).$$

Similarly,

$$\dim(q_b, N[b]) - \dim(q_{\hat{b}}, N[b]) = \dim(q_b, M) - \dim(q_{\hat{b}}, M) = k_q.$$

This implies $\dim(q_c, M) - \dim(q_{\hat{c}}, M) = k_q$ for any $c^\frown c'$ realizing $t(b^\frown b'; N)$ with $b \not\downarrow_N c$. Thus, $\overline{v}_{\hat{c}} = \overline{v}_c + \overline{k}$. But since c was an arbitrary member of $[b]$, we have $\overline{v}_{[\hat{b}]} = \overline{v}_{[b]} + \overline{k}$.

Now suppose c realizes s and $c \downarrow_N b$. To finish the lemma we must find a $\hat{c}$ with $\overline{v}_{[\hat{c}]} = \overline{v}_{[c]} + \overline{k}$. But this is immediate. For, $N[b]$ and $N[c]$ are isomorphic by an isomorphism which fixes N; taking $\hat{c}$ as the image of $\hat{b}$ under that isomorphism and repeating the previous proof completes the argument.

Let p_1 and p_2 be nonorthogonal and suppose a_1 and a_2 are assigned as in Lemma 5.5. Lemma 5.6 implies that the set of $\bar{T}$ which appear in the full configuration of a_1 (i.e. $l_{\bar{T},s} \neq 0$ (cf. Definition 5.3)) is the translation by a constant sequence $\overline{k}$ of the $\bar{S}$ which appear in the full configuration of a_2. In particular, if $s_1, s_2 \in S(N)$, $s_1 \not\perp s_2$ then s_1 is abnormal iff s_2 is abnormal.

In Section XVII.4 we computed the spectrum of a countable ω-stable T if $\mathrm{dp}(T)$ is infinite and for cardinals $\aleph_\beta$ where β is infinite. We now settle the case for arbitrary uncountable β of T with finite depth m.

To construct models of T, we construct partially labeled trees of height $m-2$ with the top nodes labeled by the full configuration of the type at that node. To demonstrate the lower bounds, we insist when defining the tree that $\mathrm{ht}(A) = k$ if and only if $\mathrm{dp}(t(A; A^-)) = m - k$. Thus, nonorthogonality preserves height. By constructing $\aleph_1$-ample trees, we guarantee as in Section XVII.2 that if $\bar{A}, \bar{B}$ both represent a model M, then for each $A \in \bar{A}$ of height $m-2$, there is an $\hat{A} \in \bar{B}$ also of height $m-2$ such that $A \not\downarrow_{A^-} \hat{A}$. The lower bounds follow since by Lemmas 5.5 and 5.6 the full configuration at A differs from that at $\hat{A}$ by a constant function. We fill out this sketch with

5.7 Theorem. *Let T be an ω-stable countable theory with NDOP and suppose $\mathrm{dp}(T) = m < \omega$. Consider those $p \in S(N)$, for some model N of T, which have height $m-2$.*

The spectrum function of T is determined by the following cases.

i) If some such p has kind I then

$$I^*(\aleph_\beta, AT) = \min(2^{\aleph_\beta}, \beth_{m-1}(|\beta + 1|^{|\omega|})).$$

ii) If no p of height $m-2$ has kind I but there is one with kind II, kind III, or kind IV then

$$I^*(\aleph_\beta, AT) = \min(2^{\aleph_\beta}, \beth_{m-1}(|\beta+\omega|)).$$

iii) If every type of height $m-2$ has kind V then

$$I^*(\aleph_\beta, AT) = \min(2^{\aleph_\beta}, \beth_{m-2}(|\beta+\omega|^{|\beta+1|})).$$

Proof. Let M be a model of T with power $\aleph_\beta$. Let $(\bar{A}, <)$ be a representation of M and let $\bar{A}^l$ be the $\kappa(\beta)$-labeled tree obtained by restricting $(\bar{A}, <)$ to the nodes of height at most $m-2$ and labeling each node $A^-[a]$ of height $m-2$ by the full configuration of a in M. Thus, $\kappa(\beta)$ denotes the number of possible labels for the representing tree of a model with power $\aleph_\beta$; later, we calculate the value of $\kappa(\beta)$ in each case. Ideally, we would now construct a $1-1$ correspondence between the set of such labeled trees and the models of power $\aleph_\beta$. We can not quite do this. But, we have a map σ (for surjective) from the class, $T_{\beta,\kappa}$, of such trees onto the class of models of power $\aleph_\beta$. To obtain $\sigma(\bar{A}^l)$, choose points to witness the configurations described by the labels and take the model prime over the resulting tree. Thus, $T_{\beta,\kappa}$ is an upper bound.

We will show, by a separate argument in each case, that $T'_{\beta,\kappa}$, the number of κ-labeled $\aleph_1$-ample trees of height $m-2$ provides a lower bound on the number of models of power at most $\aleph_\beta$. For, if the $\aleph_1$-ample trees $\bar{A}^l$ and $\bar{B}^l$ both represent a model M, Theorem XVII.4.7 implies the unlabeled trees are isomorphic. By an ad hoc argument in each case, we will show that this isomorphism preserves labels.

Writing κ for $\kappa(\beta)$, Lemma XVII.3.17, yields the upper bound:

$$|T_{\beta,\kappa}| = P_{\beta,m-3,\kappa} = \beth_{m-3}(|\beta+\omega|^\kappa).$$

By Theorem 2.8, we have the lower bound:

$$|T'_{\beta,\kappa}| = \beth_{m-2}(\kappa).$$

In every case

$$\kappa(\beta) \geq |\beta+\omega|^{|\beta+1|}. \qquad (*)$$

For, there is at least one type q of height m which needs a type s. We obtain $|\beta+\omega|^{|\beta+1|}$ labels for realizations of p which are preserved by the isomorphism of $\bar{A}^l$ and $\bar{B}^l$. Namely for each f mapping $|\beta+1|$ into the set of cardinals less than or equal $\aleph_\beta$ and each $\gamma < \beta+1$ let

$$|\{b \in M_f : \dim(q_b) = \aleph_\gamma\}| = f(\gamma).$$

Now $(*)$ implies that the upper and lower bounds yield the same function of $\kappa(\beta)$, namely, $\beth_{m-2}(\kappa(\beta))$. To complete the proof we just need to compute $\kappa(\beta)$ in each case and verify that $T'_{\beta,\kappa}$ is indeed the lower bound.

Case i). As noted in Exercise 2.12, the lower bound from Theorem 2.11 equals the upper bound from Theorem XVII.3.19 so we have nothing to prove.

Case ii). If p has kind II or kind III we obtain the following upper bound on $\kappa(\beta)$. Since R_p is countable the number of labels contributed by R_p is at most $|\beta+\omega|^\omega$ and the number contributed by each s is at most $|\beta+\omega|^{(|\beta+\omega|^k)}$ (which is raised to the ω again in case III since there are infinitely many s). In either case $I^*(\aleph_\beta, AT) \leq \beth_{m-3}(|\beta+\omega|^{\kappa(\beta)}) \leq \beth_{m-1}(|\beta+\omega|)$. For kind II this is the lower bound computed in Theorem 2.11 ii) and we finish. However, if p has kind III the computation in Section 2 provides only a lower bound of

$$\beth_{m-2}(|\beta+1|^{|\omega|}).$$

But, we saw just before beginning the case analysis that there is always a lower bound of

$$\beth_{m-2}(|\beta+\omega|^{|\beta+1|}).$$

So there are $\beth_{m-2}(\sup(|\beta+1|^{|\omega|}, |\beta+\omega|^{|\beta+1|})) = \beth_{m-1}(|\beta+\omega|)$ models of power $\aleph_\beta$ as required.

If R_p, S_p and each $Q_{p,s}$ are finite $\kappa(\beta)$ is bounded by the product $|\beta+\omega|^k$, contributed by R_p, times $|\beta+\omega|^{(|\beta+\omega|^k)}$, contributed by each s, which again yields: $I^*(\aleph_\beta, AT) \leq \beth_{m-3}(|\beta+\omega|^{\kappa(\beta)}) \leq \beth_{m-1}(|\beta+\omega|)$. We note now that in the case of kind IV this upper bound is attained while we show below that for kind V it is not. If some p with depth $m-2$ has kind IV, then some $s \in S_p^e$ is a normal type. Thus there are $|\beta+\omega|$ distinct configurations for realizations of s. We can distinguish a realization a of p by the set of s-configurations which occur β times in the full configuration of a. From Lemma 5.6 we see that if $\{\bar{T}_i : i < |\beta+\omega|\}$ appears in a label for M then the set of partial configurations associated with $\alpha(s)$ in another representation of M differs from the set of $\bar{T}_i$ by some constant $\bar{k}$. Thus, there are $2^{|\beta+\omega|}$ possible labels for the depth $m-2$ types and clearly, $2^{|\beta+\omega|}$ of these differ by nonconstant functions so there are $\beth_{m-2}(2^{|\beta+\omega|}) = \beth_{m-1}(|\beta+\omega|)$ models as required.

iii) Since for each p, S_p is finite, we label each realization of p by the number of realizations b of each $s \in S_p$ that have a given configuration for Q_b. Since each member of S_p^e is abnormal, there are only $|\beta+1|$ possible configurations so for some finite k , there are $|\beta+\omega|^{|\beta+k|}$ labels. This upper bound on $\kappa(\beta)$ is the same as our general lower bound so in this case:

$$I^*(\aleph_\beta, AT) = \beth_{m-2}(|\beta+\omega|^{|\beta+1|}).$$

There are more kinds than there are distinct spectrum functions. This illustrates the disadvantage of coding our structure results as simple enumeration problems. The following exercise shows that this kind of coding can reflect some further structure.

5.8 Exercise. Let T be an ω-stable theory of depth m. Show that if every type of height $m-2$ is of kind IV then $I(\aleph_\beta, S) = \beth_{m-2}(|\beta+\omega|^{|\beta+1|})$ but if T has a type of depth $m-2$ with kind II, then $I(\aleph_\beta, S) = \beth_{m-1}(|\beta+\omega|)$.

5.9 Historical Notes. The conjectures considered in this chapter were solved by Shelah in [Shelah, Harrington, & Makkai 1984] and [Shelah 198?] (cf. [Harrington & Makkai 1985]). Neither of the 'main gap' papers deal explicitly with the finite depth case. The first detailed discussion of this case appears in Saffe [Saffe 1983]. Our proof here combines the techniques of [Harrington & Makkai 1985] with the analysis by Bouscaren [Bouscaren 1984] of depth 2 theories. This proof and exposition rely on many conversations with Harrington, Buechler, and Saffe. We have answered a question of Saffe [Saffe 1983] by providing a single classification which explains the behavior of the spectrum function both at $\aleph_0$ and in uncountable cardinals.

The proof here still falls somewhat short of the assignment of invariants discussed in Section I.1. That is, rather than assigning to each model a tree of ordinals, we have only assigned a set of trees and then computed the spectrum by a fluke of cardinal arithmetic. The only complete solution of that problem so far requires the additional assumption that T is $\aleph_0$-categorical. Buechler accomplishes this in [Buechler 1985b] by interweaving the geometrical ideas of Zilber and [Cherlin, Harrington, & Lachlan 1985] with Shelah's analysis.