Chapter XIII

Decomposition Theorems and Weight

In this chapter we will show every type in a superstable theory can be decomposed in terms of a finite number of regular types. In fact, under a suitable operation we impose a structure on the stationary types which reflects the multiplicative structure of the natural numbers. The weight one types behave as primes in this representation. We first obtain a precise structure theory for finitely generated extensions of an **S**-model. We connect these structural results with the notion of weight. While we would like to develop such a decomposition theorem for models in an arbitrary acceptable class K, we can not do so uniformly. Rather, we first obtain the result for **S**-models in Section 1 then define weight in Section 2. We conclude Section 2 by invoking the notion of weight to prove Lachlan's theorem that a countable superstable theory has either 1 or infinitely many countable models. In Section 3 we show that in an ω-stable theory there are 'enough' **AT**-strongly regular types. With this tool we obtain an extension of the decomposition theorem to all models of an ω-stable theory in Section 4.

Except for a few results at the beginning of Section 2, we assume in this chapter that T is superstable.

1. The Decomposition Theorem For S-Models

In this section we restrict ourselves to the class of **S**-models. We show that each finitely generated **S**-model of a superstable theory has a well defined dimension. We will use this information to decompose all types in a superstable theory as a product of regular types. The results of this section provide one characterization of weight. In Section 4, we consider extending the results of this section to other classes K.

1.1 Definition. Let M be an **S**-model, $A \subseteq M$, and let $R(M, A)$ be the collection of points in M which realize stationary regular types over A. Then $\dim(R(M, A))$ is the cardinality of a maximal independent subset of $R(M, A)$.

Since we are dealing with **S** models in this section we will just refer to regular types (since any regular type over an **S** model is **S**-strongly regular). Note that we do not assert that the dimension defined here determines M. That will be true only in exceedingly special circumstances.

1.2 Exercise. Find **S**-models $M, M' \supseteq N$ which are not isomorphic although $\dim(R(M,N)) = \dim(R(M',N))$.

1.3 Lemma. *Forking is a weakly transitive relation on* $R(M,A)$. *In particular,* $\dim(R(M,A))$ *is well defined.*

Proof. By Theorem XII.3.13, we need only show the first statement. By Lemma XII.3.11, it suffices to define an equivalence relation on $R(M,A)$ such that i) for all $\overline{a}$, forking is fully transitive on $[\overline{a}]$, the equivalence class of $\overline{a}$ and ii) if $\overline{a} \not\downarrow_A I$ then $\overline{a} \not\downarrow_A I \cap [\overline{a}]$. We will show that nonorthogonality of stationary types is such an equivalence relation. To see i), suppose $\overline{c} \cup Z \cup \overline{b} \subseteq [\overline{a}]$, $(\overline{b} \not\downarrow \overline{c} \cup Z; A)$ and $(\overline{c} \not\downarrow Z; A)$. By Lemma XII.3.5 it suffices to show $(\overline{b} \not\downarrow Z; A)$. By regularity, we have $t(\overline{c}; A \cup Z) \perp t(\overline{c}; A)$. Since $\overline{b} \cup \overline{c} \subseteq [\overline{a}]$, $t(\overline{b}; A) \not\perp t(\overline{c}; A)$. Applying the contrapositive of the transitivity of nonorthogonality, we deduce $t(\overline{b}; A) \perp t(\overline{c}; A \cup Z)$. If $(\overline{b} \downarrow A \cup Z; A)$ then $t(\overline{b}; A \cup Z) \parallel t(\overline{b}; A)$. Thus, $t(\overline{b}; A \cup Z) \perp t(\overline{c}; A \cup Z)$. But this contradicts $(\overline{b} \not\downarrow \overline{c} \cup Z; A)$ and thus yields i).

Now suppose $\overline{a} \not\downarrow_A I$. We must show $\overline{a} \not\downarrow_A I_{\overline{a}}$ where $I_{\overline{a}} = I \cap [\overline{a}]$. Let $\hat{I}_{\overline{a}} = I - I_{\overline{a}}$. Assume for contradiction that $\overline{a} \downarrow_A I_{\overline{a}}$. Then certainly $\overline{a} \downarrow_A J_{\overline{a}}$ where $J_{\overline{a}}$ is a maximal independent subset of $I_{\overline{a}}$. Now, $t(\hat{I}_{\overline{a}}; A) \perp t(J_{\overline{a}}; A)$ and $t(\hat{I}_{\overline{a}}; A) \perp t(\overline{a}; A)$ so by the strong triviality of orthogonality (cf. Theorem VI.1.19) $\hat{I}_{\overline{a}} \downarrow_A I_{\overline{a}} \cup \overline{a}$. By monotonicity and transitivity of independence, we deduce $\overline{a} \downarrow_A I_{\overline{a}} \cup \hat{I}_{\overline{a}}$. From this contradiction we have condition ii) and the lemma.

The following result is crucial for our development. This initial version of the *three model theorem* is rather easy to prove for **S**-saturated models. We prove a difficult extension of it to arbitrary models of an ω-stable theory in Theorem 3.3. There are some further variants in the treatment of superstable theories which are not ω-stable [Shelah 198?].

1.4 Theorem. *Let* $N \subseteq M \subset M'$ *and suppose* M *and* N *are* **S**-*saturated. For every* $a \in M' - M$ *such that* $t(a; M)$ *is regular, either* $t(a; M) \dashv N$ *or there is some* $b \in M' - M$ *with* $t(b; M)$ *regular,* $t(b; N) \not\perp t(a; M)$, *and* $b \downarrow_N M$.

Proof. (Fig. 1). Suppose there is $a \in M' - M$ with $t(a; M) \not\dashv N$ and $p = t(a; M)$ regular. Then, by Proposition VI.2.5, for some $A \subset N$ with $|A| < \kappa(T)$, $p \not\dashv A$. Possibly enlarging A inside N, choose a finite set $B \subseteq M - N$ such that p is strongly based on $A \cup B$ and $B \downarrow_A N$. Now by the strong saturation of N choose $B' \subseteq N$, $stp(B'; A) = stp(B; A)$ and $B \downarrow_A B'$. Let p_B denote $p|B$ and $p_{B'}$ the image of p_B under an automorphism fixing A and mapping B to B'. Then $p_{B'}$ is regular and by Corollary VI.2.22 $p_B \not\perp p_{B'}$. Thus, by Theorem XII.4.5 the nonforking extension of $p_{B'}$ to M is realized in M' by some b which satisfies the theorem.

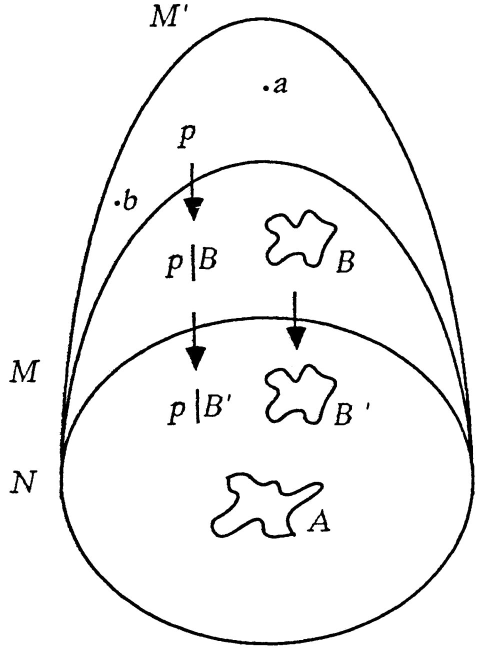

Fig. 1. The three model lemma (Strongly saturated version)

The last result would be more satisfying if it asserted either $t(M'; M) \dashv N$ or there is a $b \in M' - M$ with $b \downarrow_N M$ and $t(b; M)$ is regular. We obtain this stronger result in Lemma 4.3.

The following proof makes essential use of the fact that **S** is a powerful isolation relation. In Section 4 we will obtain a weaker version of this theorem for the class of **AT**-models.

1.5 Theorem (The Decomposition Theorem). *Let M be an* **S***-saturated model of a superstable theory. For any finite sequence $\bar{c}$, if X is a basis for $R(M[\bar{c}], M)$ then $M[\bar{c}] \approx M[X]$.*

Proof. (Fig. 2). Let X be a basis for $R(M[\bar{c}], M)$. By Exercise X.1.21 $\bar{c}$ depends over M on each $x \in X$. So by Theorems VI.1.19 and II.2.16, X is finite. We show first that $X \rhd_M X^\frown\bar{c}$. Let $M_0 = M[X]$. We will construct a sequence of models $\langle M_i : i \leq \beta\rangle$ for some β and elements $\langle a_i : i < \beta\rangle$ such that $\bar{c} \in M_\beta$ and if $\bar{b} \in M_i$ for some $i < \beta$, then $X \rhd_M X^\frown\bar{b}$. Suppose we have constucted the sequence up to M_n and A_n. If $\bar{c} \notin M_n$, choose $a_{n+1} \in M_n[\bar{c}] - M_n$ to realize a regular type over M_n. Let $M_{n+1} = M_n[a_{n+1}]$. Then $t(\bar{c}; M_{n+1})$ forks over M_n for each n, so for some finite β, $\bar{c} \in M_\beta$. The maximality of X implies that there is no $b \in M - M_n$ such that $t(b; M_n)$ is regular and $b \downarrow_M M_n$. From Theorem 1.4, we deduce $t(a_{n+1}; M_n) \dashv M$. Thus for each n, $M_n \rhd_M M_{n+1}$. By the transitivity of dominance (Lemma

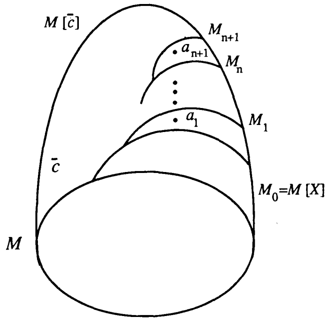

Fig. 2. The decomposition lemma

VI.3.10), $M_0 \rhd_M M_n$, for each n. But $X \rhd_M M_0$. So $X \rhd_M M_n$ for each n. In particular, $X \rhd_M X^\frown \bar{c}$. From Theorem X.2.3, we deduce there is a copy, M^0, of $M[X]$ which includes $\bar{c}$. From Theorem IX.4.1 and the fact (Theorem X.3.1) that all prime models are strictly prime, we deduce $M[X]$ is **S**-prime over $M \cup \bar{c}$ as required.

To see that the assumption that $\bar{c}$ is finite (or at least countable) is essential for Theorem 1.5, consider the theory of two refining equivalence relations. Let $M \models T$ and let C be an uncountable collection of points which are all in the same new E_1 class but are pairwise E_2 inequivalent. Now a subset of $M[C]$ which is maximal with respect to independence over M contains only a single element d and $M[d]$ contains only countably many E_2 classes. Thus $M[C]$ is not prime over a basis.

1.6 Exercise. Show that if X and Y are finite bases for $R(M,N)$ then $N[X] \approx N[Y] \approx M$.

1.7 Historical Notes. This section gives a more conceptual version of the argument in Theorem V.3.8 of [Shelah 1978].

2. *Weight*

We begin this section by defining weight and proving the rudimentary properties of weight which hold in any stable theory. Then we show that if T is superstable the weight of a type, p, can be thought of as the number of 'prime' factors of p under a suitable decomposition into regular types.

Intuitively, the weight of a type p is the maximal number of independent points that can be found such that a given realization of p depends upon each of them. A precise definition is somewhat more complicated. We have to take a supremum over various possibilites twice. In i) of the following definition we assign a 'preweight' to each type q by finding the maximal cardinality of an independent set C such that each element of C depends over dom q on some realization of q. In ii) we guarantee that the weight of a type p is a parallelism invariant by taking the supremum of the preweights of the nonforking extensions of p. Note that each q has a preweight since no C satisfying the conditions in ii) can have cardinality more than $\lambda \cdot \kappa(T)$ when q is a λ-type.

2.1 Definition. (Fig. 3).

i) Let $q \in S(B)$. The *pre-weight* of q ($\mathrm{pwt}(q)$) is the supremum of the cardinalities of sets C such that C is an independent set of sequences over B and for some $\overline{a}$ realizing q, $\overline{a} \not\downarrow_B \overline{c}$ for each $\overline{c} \in C$.

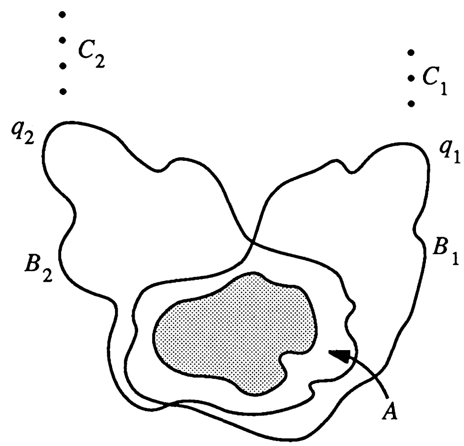

Fig. 3. $\mathrm{wt}(p) = \mathrm{pwt}(q_2) = 4$

ii) Let $p \in S(A)$. The *weight* of p ($\mathrm{wt}(p)$) is the supremum over all $B \supseteq A$ and all nonforking extensions q of p to $S(B)$ of $\mathrm{pwt}(q)$.

To shorten notation we often write $\mathrm{wt}(A; B)$ for $\mathrm{wt}(t(A; B))$. There is no assumption in this definition that p is the type of a finite sequence. However, the proofs in this section are given for types of finite sequences. To extend to λ-types one must replace $\kappa(T)$ by $\kappa(T) + \lambda$.

It is clear from the definition that nonforking extensions can not increase weight. The following result then follows from Corollary II.2.11.

2.2 Exercise. Let p' be a nonforking extension of p; show $\mathrm{wt}(p) = \mathrm{wt}(p')$.

The following exercise shows that forking extensions can increase weight.

2.3 Exercise. Let T be the theory of two refining equivalence relations with infinite classes. Let $p \in S^2(\emptyset)$ be generated by $E_1(x, y) \wedge \neg E_2(x, y)$. Show that p has weight one and, in fact, is regular but that the extension p' of p generated by $E_1(x, a)$ has weight two.

2.4 Exercise. Show that if T is the theory of two crosscutting equivalence relations with all classes (and all intersections of classes from the two equivalence relations) infinite then the weight of the unique 1-type over the empty set is two.

2.5 Exercise. Let T be the theory of an infinite set, its three element subsets and a binary relation which holds between a three element set and each of its elements. Show the weight of a three element set is three.

2.6 Exercise. Show that it is necessary to pass through the notion of preweight by exhibiting a type with differing preweights depending on the choice of B. (Hint: Consider the theory of three crosscutting equivalence relations.)

2.7 Proposition. *Let $p \in S(A)$ and let M be a strongly $\kappa_r(T)$-saturated model containing A. There is a nonforking extension p' of p to $S(M)$ with* $\mathrm{wt}(p) = \mathrm{pwt}(p')$.

Proof. Consider any $B \supseteq A$, $\bar{b}$ realizing p, and $C = \langle \bar{c}_i : i < \alpha \rangle$ with $\alpha < \kappa(T)$ such that $\bar{b} \downarrow_A B$, for each i, $\bar{b} \not\downarrow_B \bar{c}_i$, and the $\bar{c}_i$ are independent over B. Using the local character of forking, there is a set $D \subseteq B$ with $|D| < \kappa_r(T)$ which satisfies the same conditions. Now choose $A_0 \subseteq A$ such that $\bar{b} \cup D \cup C \downarrow_{A_0} A$. By the strong $\kappa_r(T)$-saturation of M, we can choose $D' \subseteq M$, so that $t(D'; A_0 \cup \bar{b} \cup C) = t(D; A_0 \cup \bar{b} \cup C)$. Now choose $C' \cup \bar{b}'$ so that $t(C' \cup \bar{b}'; A_0 \cup D') = t(C \cup \bar{b}; A_0 \cup D')$, $\bar{b}'$ realizes $stp(\bar{b}; A_0)$, and $C' \cup \bar{b}' \downarrow_{A_0} M$. By Corollary II.2.10, we see that the required p' is $t(\bar{b}'; M)$.

Remember that if $\kappa(T)$ is regular and in particular if T is superstable then strong $\kappa_r(T)$-saturation is just **S**-saturation.

2.8 Definition. Let $W(M, A)$ denote the set of $\overline{m} \in M$ such that $t(\overline{m}; A)$ has weight one.

Note that if $p \in S(A)$, the assertion that p has weight one implies that for any $\bar{a}$ realizing p and any sequences $\bar{b}$ and $\bar{c}$, if $(\bar{a} \not\downarrow \bar{b}; A)$ and $(\bar{a} \not\downarrow \bar{c}; A)$ then $(\bar{b} \not\downarrow \bar{c}; A)$. That is, we can pivot on $\bar{a}$. The weight one types play a very important role in the theory. They are a slight generalization of the regular types.

2.9 Theorem. *Supppose $A \subseteq M$ and $Z \subseteq W(M, A)$. Suppose $Y \subseteq M - A$ is independent over A and f mapping Y into Z satisfies $y \not\downarrow_A f(y)$. Then,*

i) *f is* $1 - 1$.
ii) *$f(Y)$ is independent over A.*

Proof. i) is obvious from the definition of a weight one type. For ii) we can assume by the finite character of forking that Y is finite. We show by induction on $|B|$ for $B \subseteq Y$, that $f(B) \cup (Y - B)$ is independent over A. When $B = Y$ we have the theorem. If $B = \emptyset$ there is nothing to show. Suppose $Y = Y_0 \cup B \cup y$ where we have by induction that $Y_0 \cup y \cup f(B)$ is independent. Then by hypothesis $y \not\downarrow_A f(y)$. So, if $f(y) \not\downarrow_A (Y_0 \cup f(B))$, since $f(y)$ has weight one, $y \not\downarrow_A (Y_0 \cup f(B))$ contrary to the induction hypothesis.

The argument for Theorem 2.9 can be rephrased as follows. Suppose p and q are complete weight one types over A which are weakly orthogonal. Let $R(a, b)$ hold if a realizes p, b realizes q and $a \not\downarrow_A b$. Now for every a

realizing p there is an a' realizing q with $R(a, a')$ and vice versa. Moreover, if $a_1, \ldots, a_n$ realize p, $a'_1, \ldots, a'_n$ realize q, and for each $i < n$, $R(a_i, a'_i)$ holds then exactly the same dependence relations hold among the $a_1, \ldots, a_n$ and the $a'_1, \ldots, a'_n$. This shows that weak non-orthogonality preserves any property of a type defined solely in terms of the properties of forking on realizations of the type. (Compare Chapter XVI.2 and [Baldwin 1984].)

For the next few results we assume T is superstable and tie together the definition of weight with the decomposition of **S**-saturated models in Section 1.

2.10 Theorem. *For any $A \subseteq M$, $R(M, A) \subseteq W(M, A)$. That is, every regular type has weight one.*

Proof. Suppose $t(\overline{m}; A)$ is regular, $\overline{b} \not\downarrow_A \overline{m}$ and $\overline{c} \not\downarrow_A \overline{m}$. By the definition of regularity, $t(\overline{m}; A \cup \overline{c}) \perp t(\overline{m}; A)$. From Exercise XII.4.7 we can conclude $t(\overline{b}; A) \triangleright^e t(\overline{m}; A)$. Choose an **S**-model N with $\overline{m}^\frown \overline{b}^\frown \overline{c} \downarrow_A N$. Then $\overline{b} \triangleright_N \overline{m}$. Now if $\overline{c} \downarrow_A \overline{b}$, we deduce in turn $\overline{c} \downarrow_N \overline{b}$, $\overline{c} \downarrow_N \overline{m}$, and, $\overline{c} \downarrow_A \overline{m}$ by the definition of domination and Corollary II.2.10. Thus, $\overline{c} \not\downarrow_A \overline{b}$ as required.

Suppose that $\overline{a}$ realizing $p \in S(M)$ and Y with $|Y| = \mu$ witness that $\mathrm{pwt}(p) = \mu$. Then, just rewording the definition, $t(Y; M) \triangleright_M p$ and no subset of Y dominates p. With this observation we can link the weight of a type with the regular decomposition of a model prime over a realization of the type.

2.11 Theorem. *Let T be superstable.*

i) *Suppose M is **S**-saturated and $p \in S(M)$; then* $\mathrm{wt}(p) = \mathrm{pwt}(p, M) = \dim(R(M[p], M))$.
ii) *Thus, for any p, $\mathrm{wt}(p) = n$ if and only if there is an **S**-saturated M containing* $\mathrm{dom}\, p$ *and a nonforking extension p' of p to $S(M)$ such that* $\dim(R(M[p'], M)) = n$.
iii) $\dim(W(M, A)) = \dim(R(M, A))$. *Thus,* $\dim(W(M, A))$ *is well defined.*

Proof. i) The first equality holds by Propostion 2.7. For the second, let Y be an independent set which witnesses the preweight of p'. Since $t(Y; M) \triangleright_M p$, applying Theorem X.2.5 p is realized by some $\overline{a} \in M[Y]$. But, $M[\overline{a}]$ must be $M[Y]$. Otherwise a proper subset of Y would dominate $\overline{a}$. Map Y into $R(M[p], M)$ choosing $f(y) \in M[y] - M$ to realize a regular type over M. By FI_1 for each y, $y \not\downarrow_M f(y)$. By Theorem 2.9, we deduce f is 1-1 and $f(Y)$ is independent. Thus $|Y| \leq \dim(R(M[p], M))$. But the other inequality is obvious by Theorem 2.10 so we have i).

Now ii) and iii) follow easily from Theorem 2.9 and i).

Now, we calculate the relation between the weight of a sequence and the weight of its components.

2.12 Theorem. i) *For any $\overline{a}$, $\overline{b}$, and A:*

$$\mathrm{wt}(\overline{a}^\frown \overline{b}; A) \leq \mathrm{wt}(\overline{a}; A) + \mathrm{wt}(\overline{b}; A \cup \overline{a}).$$

ii) *If* $\bar{a} \downarrow_A \bar{b}$,

$$\mathrm{wt}(\bar{a}^\frown \bar{b}; A) = \mathrm{wt}(\bar{a}; A) + \mathrm{wt}(\bar{b}; A).$$

Proof. Without loss of generality, we replace A by an **S**-saturated model, M. For i), we show:

$$\mathrm{wt}(\bar{a}^\frown \bar{b}; M) \leq \mathrm{wt}(\bar{a}; M) + \mathrm{wt}(\bar{b}; M \cup \bar{a}).$$

For this, suppose I is an independent set which witnesses the weight of $t(\bar{a}^\frown \bar{b}; M)$. Let I_0 be a maximal subset of I such that $I_0 \cup \bar{a}$ is independent. It is now straightforward to check a) for each $\bar{i} \in I_0$, $\bar{i} \not\downarrow_{M \cup \bar{a}} \bar{b}$ and so $|I_0| \leq \mathrm{wt}(\bar{b}; M \cup \bar{a})$ and b) if $\bar{i} \in I - I_0$ then $\bar{i} \not\downarrow_M \bar{a}$ so $|I - I_0| \leq \mathrm{wt}(\bar{a}; M)$. This establishes the first claim.

For ii), suppose C is a basis for $R(M[\bar{a}], M)$ and D is a basis for $R(M[\bar{b}], M)$. Now $C \downarrow_M D$ and for each $e \in C \cup D$, $\bar{a}^\frown \bar{b} \not\downarrow_M e$. Thus $\mathrm{wt}(\bar{a}^\frown \bar{b}; M) \geq \mathrm{wt}(\bar{a}; M) + \mathrm{wt}(\bar{b}; M)$. Since $\mathrm{wt}(\bar{b}; M \cup \bar{a}) = \mathrm{wt}(\bar{b}; M)$, we finish.

We can now read off a number of the characteristics of weight. First we make precise the decomposition of each type into regular types.

2.13 Definition. Let $\langle r_i : i < \alpha \rangle$ be a sequence of stationary types over a set A. Then the *product* of the r_i, denoted $\otimes r_i$, is the unique type of an independent sequence of tuples realizing the r_i. The r_i need not be distinct.

Note that if $\lg(r_i) = m_i$ then $\lg(\otimes r_i) = \sum_i m_i$.

2.14 Exercise. Show $\otimes r_i \perp \otimes r'_j$ if and only if $r_i \perp r'_j$ for each i and j.

2.15 Theorem. *Let T be superstable and q a stationary type. There exist regular types r_i, $i < \mathrm{wt}(q)$, such that $q \mathrel{\underset{\sim}{\mapsto}}_S \otimes r_i$. That is, $q \sqsupseteq^e \otimes r_i$.*

Proof. Choose an **S**-saturated M, $q' \in S(M)$ with $q' \parallel q$, and let the r_i be a list with appropriate repetitions of the types realized in a basis for $R(M[q'], M)$. We get the second statement because $\mathrel{\underset{\sim}{\mapsto}}_S$ is the same as $\sqsupseteq^e$.

This theorem shows that the quotient of the $\mapsto_S$ order on the class of stationary types in a countable superstable theory by the congruence relation $\mathrel{\underset{\sim}{\mapsto}}_S$ yields a structure similar to the natural numbers under the partial order of divisibility. The equivalence classes of the regular types correspond to the prime numbers.

The following exercises record some of the essential properties of weight. Exercise 2.18 is especially important. All of these exercises follow easily from the definitions.

2.16 Exercise. Show that if $A = \{\bar{a}_i : i < \mu\}$ then for any B,

$$\mathrm{wt}(A; B) \leq \sum_{i<\mu} \mathrm{wt}(\bar{a}_i; A_i \cup B).$$

2.17 Exercise. Show that $A \subseteq C$ implies $\mathrm{wt}(A; B) \leq \mathrm{wt}(C; B)$.

2.18 Exercise. Show that if $\bar{a} \rhd_M \bar{b}$ then $\mathrm{wt}(\bar{a}; M) \geq \mathrm{wt}(\bar{b}; M)$.

2.19 Exercise. Conclude from the previous exercise that if $\bar{a} \rhd_M \bar{b}$ and $\mathrm{wt}(\bar{a}; M) = 1$ then $\mathrm{wt}(\bar{b}; M) = 1$.

2.20 Exercise. Show that if p is K-strongly regular then $\mathrm{wt}(M[p]; M)$ is one.

2.21 Exercise. For any M and $\bar{a}$, $\mathrm{wt}(\bar{a}; M) = \mathrm{wt}(M[\bar{a}]; M)$.

The weight one types play a central role in the classification theory. Modulo $\sqsubseteq^e$ they are equivalent to regular types and thus share most of the important properties of regular types. The following results summarize some of their most important properties.

2.22 Theorem. *Suppose T is superstable. Let $p \in S(A)$. The following are equivalent.*

i) *p has weight 1.*
ii) *Suppose $\bar{a}$ realizes p, $\bar{a} \downarrow_A B$, $\bar{b} \not\downarrow_B \bar{a}$ and $\bar{c} \not\downarrow_B \bar{a}$. Then $\bar{b} \not\downarrow_B \bar{c}$.*
iii) *Let M be $\mathbf{S}$-saturated and $p' \in S(M)$ a nonforking extension of p. Then p' is $\mathbf{S}$-minimal.*
iv) *There is a regular type q with $p \sqsubseteq^e q$.*

Proof. ii) is just a restatement of the definition of weight one. If ii) holds and $\bar{a}$ realizes p' then for any $\bar{b}$ such that $\bar{a} \not\downarrow_M \bar{b}$, $\bar{b} \rhd \bar{a}$. Thus p is realized in $M[\bar{a}]$ and p is $\mathbf{S}$-minimal. (Indeed, we have shown that p' has weight one implies that if $p' \not\perp q$ then $q \mapsto_S p'$.)

Finally, suppose p' is $\mathbf{S}$-minimal. Let $\bar{a}$ realize p'. We want to show $\mathrm{wt}(p) = \mathrm{wt}(p') = 1$. If not, there exist $\mathrm{wt}(p') = n > 1$ independent elements in $M[\bar{a}] - M$. Choose any two, say $\bar{b}$ and $\bar{c}$. Since p' is $\mathbf{S}$-minimal there is an imbedding of $M[p']$ into $M[\bar{b}]$ and thus there are n independent elements contained in $M[p']$. But they are independent from $\bar{c}$ and all depend on $\bar{a}$, contradicting that $\mathrm{wt}(p') = n$. The equivalence of i) and iv) is immediate from Theorems 2.10 and 2.15.

2.23 Exercise. Show that if p has weight 1 then forking is mildly transitive on $p(\mathcal{M})$.

The $\mathbf{S}$ in part iii) of Theorem 2.10 cannot be replaced by an arbitrary acceptable K. Note that we have characterized $\mathrm{wt}(p) = 1$ in terms of dependence on two sets which need not realize p. Thus weight one is somewhat stronger than the assertion that forking is mildly transitive on $p(\mathcal{M})$ but, in view of Example XII.3.10, not as strong as asserting that forking is fully transitive on $p(\mathcal{M})$.

The following remarks are obvious from the assertion that each weight one type is $\sqsubseteq^e$ equivalent to a regular type.

2.24 Corollary. *Suppose T is superstable.*

i) *Nonorthogonality is transitive on weight one types.*
ii) *If q has weight one, $p \not\perp q$, and $r \not\perp q$ then $p \not\perp r$.*
iii) *If $p \not\perp A$, $q \not\perp p$ and p has weight one then $q \not\perp A$.*

2.25 Exercise. Show using Example XII.1.11 ii) that a type may have weight 1 without being **AT**-minimal.

2.26 Theorem. *If $A \subseteq M$ and $p \in S(A)$ has weight 1 then* $\dim(p, M)$ *is well defined.*

Proof. Suppose E and F are maximal independent subsets of $p(M)$ with $|E| = \mu < |F| = \mu'$. By Theorem 2.12, $\mathrm{wt}(E; A) = \mu$. But F is an independent set and each element of F depends on E. This contradicts the definition of weight one and yields the theorem.

We conclude this section by using weight to give a quick proof of an important result of Lachlan.

2.27 Theorem. *If T is a countable superstable theory then T has either* 1 *or infinitely many countable models.*

Proof. If T has fewer than $2^{\aleph_0}$ countable models then T is small and so admits prime models over finite sets. If T is not $\aleph_0$-categorical there is a finite set A with a nonprincipal type $p \in S(A)$. Let $E = \langle \bar{e}_i : i < \omega \rangle$ be an infinite set of indiscernibles based on p. Define models $\langle M_i : i < \omega \rangle$ so that M_i is prime over $A \cup E_i$. Suppose that k is the weight of p. By Theorem 2.12 and induction $\mathrm{wt}(E_i; A) = ik$. Note that for any $f \in p(M_i)$, $t(f; A \cup E_i)$ is isolated while $t(f; A) = p$ is not, so $f \not\downarrow_A E_i$. But then by the definition of weight, any independent (over A) set of realizations of p in M has cardinality at most ik. Thus, the models $\langle M_{ik} : i < \omega \rangle$ are pairwise nonisomorphic and we finish.

2.28 Exercise. Show that if any of $R_M(p), U(p), R_C(p)$ equals 1 then p is regular and thus has weight one.

The following exercises are taken from [Lascar 1984]. They follow easily from the U-rank inequalities discussed in Chapter VII.2.

2.29 Exercise. Suppose $p, q \in S(A)$, $U(p) \geq \omega^\alpha$ and $U(q) < \omega^\alpha$. Show $p \perp q$.

2.30 Exercise. If $U(p) = \omega^\alpha$ then p is regular.

From Theorem 2.22 iv) and Exercise 2.30 it is immediate that if $p \sqsubseteq q$ and $U(q) = \omega^\alpha$ then p has weight one. Lascar [Lascar 1984] establishes the converse, provided we work in T^{eq}. This proviso is essential; it is easy to find T and regular p with $U(p) = 2$ if we ignore T^{eq}.

2.31 Historical Notes. Shelah introduced the concept of weight in Section V.3 of [Shelah 1978]. We follow the somewhat simpler definition of [Makkai 1984]. The description of the relation of weight to regular types returns to Shelah's exposition. The product of types notation was introduced by Lascar in [Lascar 1976].

Theorem 2.27 has a long history. The first step in this direction is the proof by Baldwin and Lachlan [Baldwin & Lachlan 1971] that the conclusion holds for countable theories which are $\aleph_1$ categorical. This proof used many special properties of $\aleph_1$-categorical theories and yielded the additional infomation that all countable models of such a theory are homogeneous. Then Lachlan [Lachlan 1973] proved Theorem 2.27 by a complicated argument using rank. Lascar [Lascar 1976] simplified the proof by the use of U rank. The proof here is just a translation of his. Finally, Pillay [Pillay 1983] has given an even simpler proof and extended the result to what he calls the class of *normal theories.*

3. Ubiquity of Regular Types

This section is devoted to showing there are enough K-strongly regular types so that if a type $p \not\dashv M$ for $M \in K$ then $p \not\perp q$ for some K-strongly regular $q \in S(M)$. In the next section we will use this result to prove a decomposition theorem for finitely generated members of K. Throughout this section, we assume that T is superstable.

It is easy to deduce the following result from Theorems 2.15 and VI.1.19.

3.1 Proposition. *If $p \not\dashv M$ and M is* **S**-*saturated then for some regular $r \in S(M)$, $p \not\perp r$.*

Proof. For some $q \in S(M)$, $p \not\perp q$. Decompose q as $\otimes r_i$ with each r_i regular. We must have $p \not\perp r_i$ for some i.

It is much more difficult to show for an arbitrary acceptable class K, e.g **AT** in an ω-stable theory, that $p \not\dashv M$ implies $p \not\perp r$ for some K-strongly regular r. We require one easy lemma before the main assault.

3.2 Lemma. *If N is a model and M is* **S**-*saturated, $r \in S(M)$ is regular and $r \not\dashv N$ then for every $p \in S(M)$, if p is realized in $M[r]$ then $p \not\dashv N$.*

Proof. There is a $q \in S(N)$ with $r \not\perp q$. Since p is realized in $M[r]$, $p \not\perp r$. Thus, by Corollary XII.4.9 $p \not\perp q$, a fortiori, $p \not\dashv N$.

This next proof requires both the use of strong **I**-saturation as in Section 1 of this chapter and the consideration of indiscernibles to study orthogonality as in the last part of Section VI.1. The main thrust of this result is a *three model theorem.* Let $N \subseteq M \subseteq M_1$. Unless $t(M_1; M) \dashv N$, we want to find a $p \in S(M)$ which is realized in M_1, does not fork over N, and is K-strongly regular for an appropriate K. There are several variants on theorems with this general form. In the case at hand we make minimal assumptions on N and allow M to carry the burden. This is appropriate as the interesting application of this result is to arbitrary models N of an ω-stable theory.

3.3 Theorem (The three model theorem). *Let K admit regular types. Suppose $N \subseteq M \subseteq M_1$ and M is* **S**-*saturated. If for each $a \in M_1 - M$, $t(a; M) \not\dashv N$ then there is a $b \in M_1 - M$ such that $b \downarrow_N M$ and $t(b; M)$ is K-strongly regular.*

Proof. (Fig. 4). Since K admits regular types (Definition XII.2.1) we can choose $p \in S(M)$ and an **I**-formula p_0 over N such that p_0 weakly isolates $p|N$ in (M, M_1). By hypothesis, $p \not\perp N$. To complete the proof of the theorem, we establish the following claim.

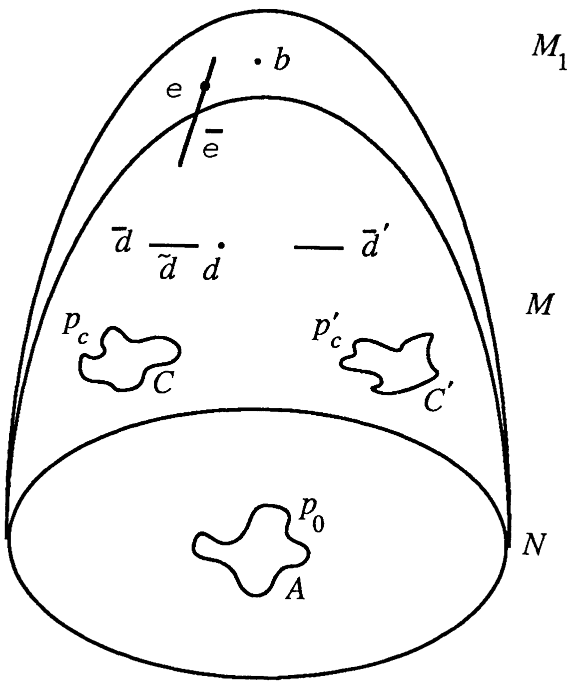

Fig. 4. The three model lemma (ω-stable version)

CLAIM. p does not fork over N.

Let $b \in M_1 - M$ realize p and choose $A \subseteq N$ and $C \subseteq M$ such that $|A| < \kappa(T)$, $|C| < \kappa(T)$, p_0 is over A and

$$b \underset{A \cup C}{\downarrow} M \text{ and } p_C = t(b; A \cup C) \text{ is stationary} \tag{1}$$

$$C \underset{A}{\downarrow} N \tag{2}$$

$$p_C \not\perp A\,. \tag{3}$$

We would be done if $b \downarrow_A A \cup C$, so we can assume for contradiction that $b \not\downarrow_A C \cup A$. We now invoke the test in Corollary VI.2.22 for $p_C \not\perp A$.

Choose $C' \subseteq M$ such that:

$$C' \underset{A}{\downarrow} C \tag{4}$$

$$stp(C; A) = stp(C'; A). \tag{5}$$

Let $p_{C'}$ denote the image of p_C under an automorphism which fixes A and maps C to C'. By Corollary VI.2.22, $p_C \not\perp p_{C'}$ so by Corollary VI.2.18 there is an n such that $p_C^n \not\perp^w p_{C'}^n$ and in particular $p_C^n \not\perp^a p_{C'}^n$. Let $\overline{d} = \tilde{d}^\frown d \in M^n$ with $\lg(\tilde{d}) = n-1$ realize p_C^n and let $\overline{d}' \in M$ realize $p_{C'}^n$. We can demand:

$$\overline{d} \underset{A \cup C}{\downarrow} A \cup C \cup C' \tag{6}$$

$$\overline{d}' \underset{A \cup C}{\downarrow} A \cup C \cup C' \tag{7}$$

$$\overline{d} \underset{A \cup C \cup C'}{\not\downarrow} \overline{d}' \tag{8}$$

and

$$C \cup \overline{d} \underset{N}{\downarrow} C'. \tag{9}$$

We will apply Axiom X.1.31 as in the proof of Theorem X.1.38. Since $(b \not\downarrow C \cup A; A)$ there is a formula $\psi(x, \overline{y}) \in F(A)$ and a sequence $\overline{c} \subset C$ such that:

$$\models \psi(b, \overline{c}) \text{ and if } \models \psi(b^*, \overline{c}^*) \text{ then } b^* \not\downarrow_A \overline{c}^* \text{ for any } \overline{c}^* \text{ and any } b^* \text{ realizing } t(b; A). \tag{$*$}$$

Without loss of generality we can take $\overline{c}$ as an enumeration of C. Similarly, (8) implies there is a formula $\theta(\overline{x}, \overline{y}; \overline{x}', \overline{y}') \in F(A)$ and $\overline{c}'$, (which without loss of generality we again take as an enumeration of C') such that:

$$\models \theta(\overline{d}, \overline{c}; \overline{d}', \overline{c}') \text{ and for any } \overline{e}, \overline{f} \text{ if } \models \theta(\overline{d}, \overline{c}; \overline{e}, \overline{f}) \text{ then } \overline{d} \not\downarrow_{A \cup \overline{c}} \overline{e}^\frown \overline{f}. \tag{$**$}$$

Let

$$r(\overline{d}, \overline{c}; \overline{x}, \overline{c}') = \bigcup_{i<n} p_0(x_i) \cup \{\theta(\overline{d}, \overline{c}, \overline{x}, \overline{c}')\} \cup \bigcup_{i<n} \{\psi(x_i; \overline{c}')\}.$$

Then $r(\overline{d}, \overline{c}; \overline{x}, \overline{c}')$ is consistent since it is satisfied by $\overline{d}'$. Note that in the important **AT** case, r is a single formula. Applying (9) and the argument for Corollary X.1.13, we can choose a $\overline{c}'' \in N$ such that $r(\overline{d}, \overline{c}; \overline{x}, \overline{c}'')$ is consistent. By (1) and the definition of p_C^n,

$$t(b; \tilde{d}^\frown \overline{c}^\frown \overline{c}' \cup A) = t(d; \tilde{d}^\frown \overline{c}^\frown \overline{c}' \cup A).$$

Thus, $r(\tilde{d}, b, \overline{c}; \overline{x}, \overline{c}'')$ is consistent. Let $\overline{e} \in M_1$ realize this last type. In particular, $\models \theta(\tilde{d}, b, \overline{c}, \overline{e}, \overline{c}'')$. We now show $\overline{e} \cap (M_1 - M) \neq \emptyset$. Suppose to the contrary, that $\overline{e} \subseteq M$. From ($**$) we have $(\tilde{d}^\frown b \not\downarrow_{A \cup C} \overline{e}^\frown \overline{c}'')$. By monotonicity, this implies $b \not\downarrow_{A \cup C} \overline{e} \cup C'$ which contradicts (1). This yields $\overline{e} \cap (M_1 - M) \neq \emptyset$ as required.

Choose e from $\overline{e}$ with $e \in M_1 - M$. Then e realizes p_0 so $t(e; A) = t(b; A)$. But then since $\models \psi(e, \overline{c}''),$ ($*$) yields $(e \not\downarrow_A \overline{c}''$, contrary to the weak isolation and we deduce the claim.

3.4 Corollary. *Let K admit regular types. If $N \in K$ and $q \not\perp N$ then there is a K-strongly regular type $r \in S(N)$ such that $q \not\perp r$.*

Proof. Without loss of generality, $q \in S(M)$ for a model $M \supseteq N$ which is **S**-saturated, **I**-saturated, and $|N|^+$-saturated, since we can take the nonforking extension of q to such a model. We can decompose q as a product of regular types r_i. Then one of the $r_i \not\perp N$ and so by Lemma 3.2 any type $p' \in S(M)$ which is realized in $M[r_i]$ is not orthogonal to N. By Theorem 3.3, there is a K-strongly regular $r \in S(N)$ whose nonforking extension to M is realized in $M[r_i]$. So $r \not\perp q$ and we finish.

3.5 Exercise (Saffe). Show that the assumption that N is a model is essential for Theorem 3.3 by considering the theory of two crosscutting equivalence relations.

Now we can obtain a factorization of any type into a product of K-strongly regular types whenever K admits regular types.

3.6 Theorem. *If T is superstable and K admits regular types then for any $N \in K$ and for any type $p \in S(N)$ there exists a family of K-strongly regular types $r_i \in S(N)$ for $i < n$ such that $p \sqsupseteq^e \otimes r_i$.*

Proof. By Theorem 2.15 $p \sqsupseteq^e \otimes r'_i$ for a sequence of r'_i each of which is regular. Now if $p \in S(N)$, each r'_i is not orthogonal to N. Thus, for each r'_i there is a strongly regular $r_i \in S(N)$ with $r'_i \not\perp r_i$. Then $\otimes r_i \sqsupseteq^e \otimes r'_i$ so $p \sqsupseteq^e \otimes r_i$.

Steinhorn has pointed out that Example XII.1.11 ii) shows that even for ω-stable theories we cannot strengthen Theorem 3.6 by replacing $\sqsupseteq^e$ by $\succsim_{\mathbf{AT}}$. We proved in Chapter VI that if M is **S**-saturated then for complete types over M weak orthogonality implies orthogonality. In Theorem XII.4.10 we removed the restriction on M but required both types to be **AT**-strongly regular and stationary. Now we remove the restriction that the types be strongly regular.

3.7 Corollary. *If T is superstable and K admits stationary regular types then for each $M \in K$, if $p, q \in S(M)$ and $p \perp^w q$ then $p \perp q$.*

Proof. By Theorem 3.6 we can let $q \sqsupseteq^e \otimes r_i$ where each r_i is K-strongly regular. It suffices to show $p \perp r_i$ for each i. But if $p \not\perp r_i$ then $p \not\perp^w r_i$ by Theorem XII.4.10.

The hypotheses of Corollary 3.7 express the properties required to prove the result. Nevertheless, they are unnatural. If K is **S** then the result is easy (Theorem VI.1.40). If K is **AT** then the only class of theories known to obey the hypotheses are the ω-stable theories. Thus, we have only a weak generalization of the following theorem of Lascar.

3.8 Corollary. *If T is ω-stable, $M \models T$, $p, q \in S(M)$ and $p \perp^w q$ then $p \perp q$.*

3.9 Exercise. Show the Example VI.1.33.i) is a small superstable theory so the assertion, 'For types over models, $\perp^w$ implies $\perp$' can not be extended from ω-stable to small superstable theories.

Two elements of a partial order are called *disjoint* if they have no common lower bound. We can now give a syntactic characterization of two types, p and q, being disjoint in the $\mapsto_K$-order. Namely, $p \perp q$. This observation follows quickly from the next theorem. It is unlikely to be an accident that Birkhoff's classic lattice theory text [Birkhoff 1949] uses $\perp$ to denote disjointness.

3.10 Theorem. *If the acceptable class K admits stationary regular types, $M \in K$, and $p \in S(M)$ is K-strongly regular then for any $q \in S(M)$ if $q \not\perp p$ then $q \mapsto_K p$.*

Proof. By Corollary 3.4 $q \sqsupseteq^e \otimes r_i$ where each $r_i \in S(M)$ is K-strongly regular. Thus $q \not\perp p$ implies $\otimes r_i \not\perp p$ implies for some i, $r_i \not\perp p$. By Theorem XII.4.10 $r_i \mapsto_K p$ and so $q \mapsto_K p$.

This theorem again improves Theorem XII.4.10 by removing the hypothesis that both of the types are K-strongly regular.

Now to see the equivalence of disjointness and orthogonality, let $M \in K$ and $p, q \in S(M)$. If p and q have a common lower bound, they have one which is regular. But then, by Corollary XII.4.9, $p \not\perp q$. Conversely, if $p \not\perp q$ then by Theorem 2.15 there exist regular factors r_1 of p and r_2 of q such that $p \not\perp r_1$, $r_1 \not\perp r_2$, and $q \not\perp r_2$. By Theorem 3.10, p and q are not disjoint. Of course, for $K = S$, we could prove this result already in Section XII.4.

There were actually three stages in the proof of Theorems 3.6, 3.10 and their assorted corollaries. We first have the result if M is an **S**-model and the types are arbitrary. Then we allow the model to be arbitrary but demand regularity of the types. Finally, we remove the regularity demand. This progression is obscured, if Theorem 3.10 itself is looked at, by the fact that the type p in Theorem 3.10 must be regular in all incarnations of that theorem.

Using Proposition 3.1 we can show that a weight one type which is not orthogonal to a set B shares many of the properties of a type actually over B. For example, we have the following generalization of Theorem VI.2.21.

3.11 Theorem. *Suppose* $\mathrm{wt}(p) = 1$, *$p \dashv A$, and $p \not\dashv B$. Then if $B \downarrow_A C$, $p \dashv C$.*

Proof. Without loss of generality, replace B by an **S**-model M. By Proposition 3.1, there is a regular $r \in S(M)$ with $p \not\perp r$. By Corollary 2.24, $p \dashv A$ implies $r \dashv A$. By Theorem VI.2.21, $r \dashv C$ whence by Corollary 2.24 again, now using that $\mathrm{wt}(p) = 1$, $p \dashv C$.

3.12 Exercise. Suppose T is stationary and K admits stationary regular types. Show that for any $M \in K$ each of the following properties implies its successor on the list: $p \mapsto_K q$, $p \rhd q$, $p \not\perp^a q$, $p \not\perp q$.

Consider the three implications in the preceding exercise. Corollary 3.7 shows the last can be reversed. We should not expect the second to be reversed for arbitrary p and q. For, we might factor have p as $r_1 \otimes r_2$ and q as $r_2 \otimes r_3$ where r_1, r_2, and r_3 are pairwise orthogonal regular types. The first implication is an equivalence for $K = \mathbf{S}$ (Corollary X.2.5); it cannot be reversed for $K = \mathbf{AT}$ ((Exercise X.2.6). Theorem 3.1 shows that all four conditions are equivalent if both p and q are K-strongly regular.

3.13 Historical Notes. The main result of this section, Theorem 3.3, first occurs in [Shelah 1982a]. Our proof is derived from comments of Buechler. Corollary 3.7 is due to Lascar.

4. *Linear Decomposition of Finitely Generated* AT-*Models*

In Section 1 we showed that in the category of **S**-models of a superstable theory any finitely generated **S**-model $N = M[\bar{c}]$ could be represented as $M[X]$ where X is a set of independent realizations of regular types over M. The cardinality of X is an invariant of the pair (N, M). In fact, it is the weight of $t(\bar{c}; M)$. We would like to provide a similar decomposition for finitely generated models (in the class **AT**) when T is ω-stable. The following examples show that an exact analog is impossible.

4.1 Example. i) Consider again Example XII.1.11. If we let M be a model of T and as before choose $a^\frown b \notin M$, not on the same Z-component but in the same equivalence class which does not intersect M, then it is impossible to choose an independent set X realizing **AT**-strongly regular types over M with $M[a, b] \approx M[X]$. Thus the best description of a 'basis' for $M[a, b]$ over M is to take the pair $\langle a, b\rangle$ as the basis and note that $t(b; M[a]) \dashv M$.

The next example shows the sensitivity of these notions to the proper choice of $M[a]$.

ii) Let T be the theory of a single unary function f satisfying axioms which assert $f^3(x) = f^2(x)$ but there are infinitely many elements with neither $f(x) = x$ nor $f^2(x) = f(x)$. Thus any countable model of T can be broken up into components where each component is a collection of sequences of integers of length at most two with $f(s) = s^-$ if $\lg(s) > 0$ and $f(s) = s$ if $\lg(s) = 0$. Now let M be a countable model of T and choose $a \in \mathcal{M}$ such that $f(a) \notin M$ but $f^2(a) \in M$. Choose $b \in \mathcal{M}$ such that $f(a) = f(b)$. Consider the model $M[a, b]$. Now $a \rhd_M M[a, b]$ so it is impossible to find a pair of elements in $M[a, b]$ which are independent over M. But there are copies of $M[a]$ with $b \notin M[a]$.

The first of these examples is fairly representative of the situation. The notion of a canonical resolution for N over M induces an invariant which characterizes the pair (M, N) up to isomorphism over M.

4.2 Definition. i) A *K-regular resolution* of $N = M[\bar{c}]$ is a sequence of models $M_0 = M, M_1, \ldots M_k$ such that each M_{i+1} is K-prime over a basis X_i for $R(N, M_i)$ and $M_k \approx M[\bar{c}]$.

ii) A *canonical K-regular resolution* of $M[\bar{c}]$ over M is a K-resolution of $M[\bar{c}]$ over M with minimal length.

We will apply the following lemma which is the promised strengthening of Lemma 1.4. Note that this result relies on the complicated Lemma 3.3.

4.3 Lemma. *Let $N \subseteq M \subset M'$ be models of the countable ω-stable theory T. Either $t(M';M) \dashv N$ or for some $a \in M' - M$, $a \downarrow_N M$, and $t(a;N)$ is strongly regular.*

Proof. Suppose $t(M';M) \not\dashv N$, by monotonicity for some $\bar{a} \in M' - M$, $p = t(\bar{a};M) \not\dashv N$. By Theorem 3.4 there is a K-strongly regular $p' \in S(N)$ with $p \not\perp p'$. By Theorem XII.4.5, the nonforking extension of p' to $S(M)$ is realized in $M[\bar{a}] \subseteq M'$ which yields the result.

4.4 Lemma. *Let $N \subseteq M \subset M'$ be models of the countable ω-stable theory T. If N is finitely generated over M there is a finite canonical* **AT**-*regular resolution of N over M.*

Proof. We need only show there is a resolution as any shortest such will be canonical. Construct a sequence of submodels of $N = M[\bar{c}]$ according to the definition of a resolution. We can find such a sequence by Theorem 4.3. We need only show that the construction stops after finitely many steps. It suffices to show that for each i, $(\bar{c} \not\downarrow X_i; M_i)$, since such an infinite sequence would contradict the definition of $\kappa(T)$. So suppose for contradiction that $\bar{c} \downarrow_{M_i} X_i$. In particular, $\bar{c} \notin M_{i+1}$. Choose a strongly regular type q in $S(M_{i+1})$ realized by $b \in M_{i+1}[\bar{c}] - M_{i+1}$. By the maximality of X_i, we see $b \not\downarrow_M M_{i+1}$. But $\bar{c} \not\downarrow_{M_i} M_{i+1}$ so $\bar{c} \downarrow_{M_{i+1}} \bar{b}$ by the strong regularity of q. This contradicts the choice of $b \in M_{i+1}[\bar{c}]$.

Note that in forming a canonical resolution the choice of points or even of strongly regular types over M_i realized in X_i is not completely determined. But both the cardinality of X_i and the isomorphism type of M_{i+1} over M_i is fixed. Thus we justify the term 'canonical resolution'.

This uniqueness claim depends essentially upon the restriction in the definition of canonical resolution that k be minimal with $M_k \approx M[\bar{a}]$. For example, let T be the theory of an equivalence relation with infinitely many classes which are all infinite. If $M \models T$ and the equivalence class of a does not intersect M, then for any b which is equivalent to a, it is possible to choose $M[b] \subseteq M[a]$ which is isomorphic to but not equal to $M[a]$. In fact, we can choose arbitrarily long **AT**-resolutions of $M[a]$ over M.

4.5 Theorem. *Let $N \subseteq M \subset M'$ be models of a countable ω-stable theory T. If N is a finitely generated over M, there is finite sequence of models $M_0 = M, M_1, \ldots, M_k = N$ such that each M_i is determined up to isomorphism over M_{i-1}.*

Proof. Choose $\langle M_i : i \leq k\rangle$ inductively to satisfy the definition of an **AT**-resolution. Stop when some $M_i \approx N$. The uniqueness follows by induction from the following exercise.

4.6 Exercise. If X and Y are bases for $R(N, M)$ then $M[X] \approx M[Y]$. (Hint: Note that forking defines a $1-1$ correspondence between X and Y. Apply Corollary XII.1.15.)

4.7 Definition. Let $M_0 = M, M_1, \ldots, M_k = N$ be the canonical resolution of N over M. The *length* of this resolution is the sum of the dimensions of $R(M_{i+1}, M_i)$ for $i < k$.

Of course, both the integer k and the length of the resolution are invariants of the pair (M, N).

The following important reformulation of Lemma 4.3 provides a useful weakening of Theorem 1.5 for the class of **AT**-models of an ω-stable theory.

4.8 Exercise. If T is ω-stable, $M \subseteq N$, and $M, N \models T$ then $R(N, M) \triangleright_M N$.

4.9 Historical Notes. The results here refine the decompositions of models of ω-stable theories in [Lascar 1984] and [Makkai 1984].