ON THE VOLUME IN HOMOGENEOUS SPACES

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Guldin-Pappus's theorem about the volume of a solid of rotation in the euclidean space has been generalized in two ways. G. Koenigs [1] and J. Hadamard [2] proved that the volume generated by a 1-parametric motion of a surface D bounded by a closed curve c is equal to $\sum_i a_i b_i + \sum_{i<j} a_{ij} b_{ij}$, where a_i, $a_{ij} = -a_{ji}$ $(i, j = 1, 2, 3)$ are quantities attached to D with respect to a rectangular coordinate system, while b_j, $b_{ij} = -b_{ji}$ $(i, j = 1, 2, 3)$ are quantities determined by our motion. It is remarkable that a_i, a_{ij} depend only on c and not on D. The theorem was extended to the case of dimension n by G. Guillaumin [3]. Another extension of Guldin-Pappus's theorem was obtained by the author [7] in the following way. The volume V of a solid B in the euclidean space of dimension n is given by $\int v d\sigma$, where v is an $(n-1)$-dimensional volume of a section of B by one of the 1-parametric set of hyperplanes and $d\sigma$ is a component, orthogonal to the hyperplanes, of an arcelement of the locus of the center of gravitation of the section. An analogous result was obtained for the spherical space. In the present paper the author generalizes these results to the case of homogeneous spaces by the method of moving frames of E. Cartan and applies the results to various spaces, and states formulas of the integral geometry in the homogeneous spaces.

1. Preliminaries

1. In the first we quote the matters necessary for our purpose from [4] and [6]. Let G be a group which operates on a space M effectively and transitively. We take an element p_0 of M and a set H of all elements of G which fix p_0. Any element p of M corresponds to a set σH $(\sigma \in G)$ of G/H and M can be identified with G/H in natural way. Now we take a set of elements upon which G operates simply transitively and call each element of the set a frame of M, and to a point p corresponds a set of frames σHR, where R is a

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fixed frame. A relative displacement from a frame TR to SR $(S, T \in G)$ is defined by $T^{-1}S$ and an absolute displacement by ST^{-1}.

2. Now we assume that G is a Lie group of dimension r and H its closed connected subgroup of dimension $n-r$, and consider the homogeneous space G/H which we call M. We denote by S_a an element of G with local parameters $a=(a_1, \ldots, a_r)$. Left invariant differential 1-forms $\omega_p=\omega_p(a, da)$ $(p=1, \ldots, r)$, called relative components, are obtained by taking linear parts in $da=(da_1, \ldots, da_r)$ of the parameters $b=(b_1, \ldots, b_r)$ of $S_b=S_a^{-1}S_{a+da}$, which is a relative displacement from S_aR to $S_{a+da}R$. Relative components $\omega_p=\omega_p(a, da)$ are base of a vector space V dual to the tangent space of G. By taking a suitable linear combination of ω_p with constant coefficients and denoting them ω_p anew we can assume $\omega_i(a, da)=0$ $(i=1, \ldots, n)$ for the elements S_a, S_{a+da} of H. These ω_i's are called principal relative components of M. We can take suitable local parameters $x_1, \ldots, x_n, u_{n+1}, \ldots, u_r$, such that $x=(x_1, \ldots, x_n)$ are considered as local coordinates of M and principal relative components are represented as

$$\omega_i=\sum_j a_{ij}(x_1, \ldots, x_n, u_{n+1}, \ldots, u_r)\,dx_j. \qquad (i, j=1, \ldots, n) \qquad (1.1)$$

Hereafter we take the indices in the following manner:

$$p, q, s, u=1, \ldots, r$$
$$i, j, k=1, \ldots, n \qquad \alpha, \beta, \gamma=n+1, \ldots, r.$$

Then we have

$$d\omega_p=\sum_{qs}\frac{1}{2}c_{qsp}[\omega_q\,\omega_s], \qquad \text{where} \quad c_{qsp}=-c_{sqp},$$

especially,
$$d\omega_i=\sum_{jk}\frac{1}{2}c_{jki}[\omega_j\,\omega_k]+\sum_{\alpha k}c_{\alpha ki}[\omega_\alpha\,\omega_k]. \qquad (1.2)$$

When we take frames S_aS_tR and $S_{a+da}S_{t+dt}R$ instead of S_aR and $S_{a+da}R$,

$$(S_aS_t)^{-1}(S_{a+da}S_{t+dt})=S_t^{-1}(S_a^{-1}S_{a+da})S_t\cdot S_t^{-1}S_{t+dt} \qquad (1.3)$$

and denoting the relative components of $(S_aS_t)^{-1}(S_{a+da}S_{t+dt})$ by π_p we have

$$\pi_p=\sum_q t_{pq}\,\omega_q(a, da)+\omega_p(t, dt), \qquad (1.4)$$

where (t_{pq}) is a matrix representing an element of a linear adjoint group of G corresponding to S_t. We have

$$dt_{pq}=\sum_{us}c_{usp}\,\omega_u(t, dt)\,t_{sq}. \qquad (1.5)$$

If S_t, S_{t+dt} belong to the connected subgroup H of G, we have

$$\omega_i(t, dt) = 0 \tag{1.6}$$

and by (1.4)

$$\pi_i = \sum_j t_{ij}\,\omega_j \tag{1.7}$$

and as a special case of (1.5)

$$dt_{ij} = \sum_{\alpha k} c_{\alpha k i}\,\omega_\alpha(t, dt)\,t_{kj} \tag{1.8}$$

and

$$t_{i\alpha} = 0. \tag{1.9}$$

(t_{ij}) is a matrix representing an element of a linear group of isotropy.

2. Main theorems

1. An invariant volume can be defined in $M = G/H$ when and only when the linear group of isotropy is unimodular, namely

$$\det(t_{ij}) = 1 \qquad \text{for} \quad S_t \in H, \tag{2.1}$$

in another words,

$$\sum_i c_{\alpha ii} = 0, \tag{2.2}$$

as we see by (1.7) and (1.8) (we have assumed that H is connected). The volume element dV is an exterior product of principal relative components

$$dV = [\omega_1\,\omega_2 \cdots \omega_n] \tag{2.3}$$

with disregard to a constant multiplier.

Now we define an algebraic volume in M. Let E_n be a euclidean space of dimension n and K be a measurable set of E_n. Let a differentiable mapping of K into M be φ, then a mapping φ^* of differential forms in M into forms in K is naturally defined. We define $\int_{\varphi(K)} [\omega_1\,\omega_2 \cdots \omega_n]$ by $\int_K \varphi^*[\omega_1\,\omega_2 \cdots \omega_n]$ and call it an algebraic volume of (φ, K) in M. We define an equivalence relation between (φ, K) and (φ', K') by the existence of a differentiable homeomorphism f such that

$$K' = f(K), \qquad \varphi' = \varphi f^{-1}.$$

An algebraic volume is a quantity defined for the equivalence class.

2. We take cartesian coordinates $x_1, \ldots, x_{n-1}$, u in E_n and consider a domain D on a hyperplane defined by $u = 0$. Let ψ be a $1-1$ mapping of D into M which is univalent on tangent spaces. $\psi(D)$ is then an $n-1$ dimensional submanifold, namely a hypersurface. By a 1-parametric motion represented by

S_t $(\in G,\ t=t(u))$ for an interval I $(0 \leqq u \leqq 1)$ $\psi(D)$ generates a set of M. Putting

$$K = D \times I$$
$$\varphi(p) = S_t(\psi(x)) \text{ for } p = (x_1, \ldots, x_{n-1}, u) \text{ such that}$$
$$x = (x_1, \ldots, x_{n-1}, 0) \in D$$

an algebraic volume can be defined for (φ, K). When we take an $n-1$-parametric set of frames S_aR attached to each point $\psi(x)$ of $\psi(D)$, S_tS_aR is attached to a point $S_t(\psi(x))$. Let the relative components of

$$(S_tS_a)^{-1}(S_{t+dt}S_{a+da}), \qquad S_t^{-1}S_{t+dt}, \qquad S_a^{-1}S_{a+da}$$

be

$$\pi_p, \qquad \tau_p, \qquad \omega_p$$

respectively, then by the relation

$$(S_tS_a)^{-1}(S_{t+dt}S_{a+da}) = S_a^{-1}(S_t^{-1}S_{t+dt})S_a \cdot S_a^{-1}S_{a+da}$$

we have

$$\pi_i = \omega_i + \sum_p a_{ip}\tau_p, \tag{2.4}$$

where (a_{qp}) is an element of a linear adjoint group corresponding to S_a.

As a volume element dV of (φ, K) we have

$$dV = [\pi_1\pi_2 \cdots \pi_n]$$

and by (2.4)

$$[\pi_1\pi_2 \cdots \pi_n] = \sum_{ip} a_{ip}[\omega_1 \cdots \omega_{i-1}\tau_p\omega_{i+1} \cdots \omega_n],$$

because the set of frames S_aR is $n-1$-parametric and S_t is 1-parametric. Putting

$$\Omega_p = \sum_i (-1)^{i-1}a_{ip}[\omega_1 \cdots \hat{\omega}_i \cdots \omega_n], \tag{2.5}$$

where $\hat{\omega}_i$ means that a term ω_i does not appear in the product, we have

$$dV = \sum_p [\tau_p\Omega_p]. \tag{2.6}$$

We call an algebraic volume V of (φ, K) a volume generated by a hypersurface $\psi(D)$ under a 1-parametric motion S_t $(t = t(u),\ 0 \leqq u \leqq 1)$. Putting

$$X_p = (-1)^{n-1}\int_{\psi(D)} \Omega_p, \qquad Y_p = \int_{0 \leqq u \leqq 1} \tau_p \tag{2.7}$$

we get the following theorem by integrating (2.6).

THEOREM 1. *An algebraic volume V generated by a 1-parametric motion of a hypersurface $\psi(D)$ is given by*

$$V=\sum_p X_p Y_p,$$

where X_p's are quantities determined by $\psi(D)$ and Y_p's are those determined by our motion.

This is a generalization of the theorem of Koenigs-Hadamard.

3. We consider Ω_p in detail. Until now $S_a R$ has been attached to a point of $\psi(D)$, and S_a has been of $n-1$-parameters. Now we consider S_a which is r-parametric, and construct forms

$$\Omega_p=\sum_i(-1)^{i-1}a_{ip}[\omega_1\cdots\hat{\omega}_i\cdots\omega_n], \tag{2.9}$$

where ω_i's are principal relative components of M corresponding to $S_a^{-1}S_{a+da}$ and (a_{qp}) is an element of a linear adjoint group corresponding to S_a. Then we get

THEOREM 2. *Differential forms Ω_p are closed and are forms on M.*

Verification runs as follows by the use of (1.5) (1.2):

$$\begin{aligned}
d\Omega_p=&\sum_i(-1)^{i-1}[da_{ip}\omega_1\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{i>j}(-1)^{i+j}a_{ip}[\omega_1\cdots d\omega_j\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{i<j}(-1)^{i+j-1}a_{ip}[\omega_1\cdots\hat{\omega}_i\cdots d\omega_j\cdots\omega_n]\\
=&\sum_{\alpha ki}(-1)^{i-1}c_{\alpha ki}a_{kp}[\omega_\alpha\omega_1\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{hki}(-1)^{i-1}c_{hki}a_{kp}[\omega_h\omega_1\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{\alpha hi}(-1)^{i-1}c_{h\alpha i}a_{\alpha p}[\omega_h\omega_1\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{\alpha,\,i>j,\,k}(-1)^{i+j}c_{\alpha kj}a_{ip}[\omega_1\cdots\omega_{j-1}\omega_\alpha\omega_k\omega_{j+1}\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{\alpha,\,i<j,\,k}(-1)^{i+j-1}c_{\alpha kj}a_{ip}[\omega_1\cdots\hat{\omega}_i\cdots\omega_{j-1}\omega_\alpha\omega_k\omega_{j+1}\cdots\omega_n]\\
&+\sum_{i>j,\,h,\,k}(-1)^{i+j}\frac{1}{2}c_{hkj}a_{ip}[\omega_1\cdots\omega_{j-1}\omega_h\omega_k\omega_{j+1}\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{i<j,\,h,\,k}(-1)^{i+j-1}\frac{1}{2}c_{hkj}a_{ip}[\omega_1\cdots\hat{\omega}_i\cdots\omega_{j-1}\omega_h\omega_k\omega_{j+1}\cdots\omega_n]\\
=&\sum_{\alpha ij}(-1)^{j-1}c_{\alpha ij}a_{ip}[\omega_\alpha\omega_1\cdots\hat{\omega}_j\cdots\omega_n]\\
&+\sum_{ij}c_{jij}a_{ip}[\omega_1\cdots\omega_n]+\sum_{\alpha i}c_{i\alpha i}a_{\alpha p}[\omega_1\cdots\omega_n]\\
&+\sum_{\alpha,\,i>j}(-1)^{i-1}c_{\alpha jj}a_{ip}[\omega_\alpha\omega_1\cdots\hat{\omega}_i\cdots\omega_n]\\
&+\sum_{\alpha,\,i>j}(-1)^{j}c_{\alpha ij}a_{ip}[\omega_\alpha\omega_1\cdots\hat{\omega}_j\cdots\omega_n]\\
&+\sum_{\alpha,\,i<j}(-1)^{i}c_{\alpha jj}a_{ip}[\omega_\alpha\omega_1\cdots\hat{\omega}_i\cdots\omega_n]
\end{aligned}$$

$$+\sum_{\alpha,\, i<j}(-1)^{j} c_{\alpha ij} a_{ip} [\omega_\alpha \omega_1 \cdots \hat{\omega}_j \cdots \omega_n]$$
$$-\sum_{i>j} c_{jij} a_{ip} [\omega_1 \cdots \omega_n] - \sum_{i<j} c_{jij} a_{ip} [\omega_1 \cdots \omega_n]$$
$$= \sum_{\alpha} (\sum_{j} c_{\alpha jj}) (\sum_{i} (-1)^{i-1} a_{ip} [\omega_\alpha \omega_1 \cdots \hat{\omega}_i \cdots \omega_n] + a_{\alpha p} [\omega_1 \cdots \omega_n]),$$

and so by virtue of (2.2)

$$d\Omega_p = 0. \tag{2.10}$$

When we take local parameters $x_1, \ldots, x_n, u_{n+1}, \ldots, u_r$, for which (1.1) holds good, we have

$$\Omega_p = \sum_i b_{ip}(x, u) [dx_1 \cdots \widehat{dx_i} \cdots dx_n]$$

and by virtue of (2.10) $b_{ip}(x, u)$ do not contain u, which proves Ω_p are forms on $M = G/H$.

Ω_p are forms on M, but they depend on S_a and are not invariant for displacement. They are not intrinsic in this sense. In a simply connected domain in M there exist forms Π_p such that $\Omega_p = d\Pi_p$. If the domain D has regular boundary C and $\psi(D)$ is in the domain above stated, we get

$$X_p = (-1)^{n-1} \int_{\psi(D)} \Omega_p = \int_{\psi(D)} d\Pi_p = \int_{\psi(C)} \Pi_p \tag{2.11}$$

and X_p depend on (ψ, C) and not on (ψ, D).

4. If we take $S_c(\psi(D))$ in stead of $\psi(D)$ with S_c constant in G, frames $\bar{S}_a R = S_c S_a R$ are attached to it, and if we take $\bar{S}_t$ such that

$$\bar{S}_t = S_t S_c^{-1},$$

we have

$$\bar{S}_t \bar{S}_a = S_t S_a,$$

and the motion of $\psi(D)$ by S_t is the same with that of $S_c(\psi(D))$ by $\bar{S}_t$. Owing to the relation

$$\bar{S}_a^{-1} (S_b^{-1} S_{b+db}) \bar{S}_a = S_a^{-1} S_c^{-1} (S_b^{-1} S_{b+db}) S_c S_a$$

an element of a linear adjoint group corresponding to $\bar{S}_a$ is a product of those of S_a and S_t, and

$$\bar{a}_{sp} = \sum_q a_{sq} c_{qp}, \tag{2.12}$$

where $(\bar{a}_{sp})$, (a_{sq}), (c_{qp}) correspond to $\bar{S}_a$, S_a, S_c respectively. By the relation $\bar{S}_a^{-1} \bar{S}_{a+da} = S_a^{-1} S_{a+da}$ we have $\bar{\omega}_i = \omega_i$, and putting

$$\bar{\Omega}_p = \sum_i (-1)^{i-1} \bar{a}_{ip} [\bar{\omega}_1 \cdots \hat{\bar{\omega}}_i \cdots \bar{\omega}_n]$$

we get
$$\bar{\Omega}_p = \sum_q \Omega_q c_{qp} \tag{2.13}$$

and also for
$$X_p = (-1)^{n-1} \int \Omega_p, \qquad \bar{X}_p = (-1)^{n-1} \int \bar{\Omega}_p,$$
$$\bar{X}_p = \sum_q X_q c_{qp}. \tag{2.14}$$

By the use of (2.14) we can nomalize X_p, namely by taking S_c suitably we reduce X_p's to a system which is as simple as possible. Examples will be given later.

5. Next we turn to another generalization of Guldin-Pappus's theorem. We assume $M = G/H$ has an $n-1$-dimensional submanifold L which is transformed into itself transitively by a certain connected closed subgroup of G. We call L a W-hypersurface. L is derived from a solution of a set of completely integrable system

$$\rho_v = 0 \qquad (v = 1, 2, \ldots, k) \tag{2.15}$$

by E. Cartan's theory, where ρ_v's are linearly independent linear combinations of relative components with constant coefficients. As L is of dimension $n-1$, principal relative components ω_i are not independent along L, and so by taking a suitable linear combination of them and suitable linear combinations of relative components ω_p, both of constant coefficients, we can assume anew

$$\omega_1 = 0, \qquad \omega_\lambda = 0 \qquad (\lambda = n+1, \ldots, n+k-1) \tag{2.16}$$

instead of (2.15). As this system is completely integrable we have

$$c_{ij1} = 0 \qquad c_{i\sigma 1} = 0 \qquad c_{\sigma\rho 1} = 0$$
$$c_{ij\lambda} = 0 \qquad c_{i\sigma\lambda} = 0 \qquad c_{\sigma\rho\lambda} = 0,$$

where
$$i, j = 2, \ldots, n$$
$$\lambda = n+1, \ldots, n+k-1 \qquad \sigma, \rho = n+k, 1, \ldots, r. \tag{2.17}$$

We take up a solution of (2.16) which contains an identity of G and denote each element of the solution by S_a which generates a subgroup J of G. Then a set of frames $S_a R$ is attached to all points of a submanifold of M which we can assume to be L. If J is connected we have by virtue of (1.9)

$$a_{1\rho} = 0, \qquad a_{\lambda\rho} = 0, \qquad a_{1i} = 0, \qquad a_{\lambda i} = 0 \tag{2.18}$$

with the indices as in (2.17), where (a_{pq}) is an element of a linear adjoint group corresponding to $S_a \in J$. The forms (2.9) reduce to

$$\Omega_p = a_{1p}[\omega_2 \cdots \omega_n]$$

on the manifold L and we get

$$\Omega_i = 0, \qquad \Omega_\rho = 0 \tag{2.19}$$
$$(i, j = 2, \ldots, n, \qquad \rho = n+k, \ldots, r)$$

by virtue of (2.18).

We consider a euclidean space of dimension n with coordinates $x_1, \ldots, x_{n-1}, u$ and a 1-parametric set of domains $D(u)$ $(0 \leqq u \leqq 1)$ contained in a domain D on the hyperplane $u = 0$. Let ψ be a differentiable mapping which maps D on the W-hypersurface L, and S_t $(t = t(u),\ 0 \leqq u \leqq 1)$ be a 1-parametric motion on M. Then $S_t(\psi(D(u))$ generates a part of M and a volume element dV is given by

$$dV = \sum_p [\tau_p \Omega_p] \tag{2.20}$$

by taking an $n-1$-parametric set of frames $S_a R$ along L. If K is a set of points $p = (x_1, \ldots, x_{n-1}, u)$ such that $x = (x_1, \ldots, x_{n-1}, 0) \in D(u)$ and φ is defined by $\varphi(p) = S_t(\psi(x))$, an algebraic volume of (φ, K) is given by

$$V = \sum_p \int X_p \tau_p, \tag{2.21}$$

when we put

$$X_p = X_p(u) = (-1)^{n-1} \int_{\psi(D(u))} \Omega_p. \tag{2.22}$$

By virtue of (2.19) we have $X_i = 0$, $X_\rho = 0$ and so

$$V = \int X_1 \tau_1 + \sum_\lambda \int X_\lambda \tau_\lambda \qquad (\lambda = n+1, \ldots, n+k-1). \tag{2.23}$$

Thus we get

THEOREM 3. *An algebraic volume generated by a* 1*-parametric set of W-hypersurfaces is given by* (2.23).

X_p's are quantities depending on $(\psi, C(u))$, where $C(u)$ is a boundary of $D(u)$.

We can simplify the formula (2.23) in the following manner. If we take $\bar{S}_a$, and $\bar{S}_t$ such that

$$\bar{S}_a = S_c S_a \qquad \bar{S}_t = S_t S_c^{-1}$$

where S_c $(c = c(u))$ belong to the subgroup J, we have

$$\bar{X}_1 = X_1 c_{11} + \sum_\lambda X_\lambda c_{\lambda 1}, \qquad \bar{X}_\lambda = X_1 c_{1\lambda} + \sum_\mu X_\mu c_{\mu\lambda}. \tag{2.24}$$
$$(\lambda,\ \mu = n+1, \ldots, \qquad n+k-1)$$

By taking $c = c(u)$ suitably we can simplify X_p. Especially if the linear group defined by linear transformations (2.24), namely a linear group of isotropy on a homogeneous space G/J, is transitive, we can reduce in such a way that $\bar{X}_\lambda = 0$ hold good, and we get from (2.23)

$$V = \int X_1 \tau_1. \tag{2.25}$$

Examples will be given in the following section.

3. Applications

We take in the euclidean space of dimension n rectangular coordinates $x = (x_1, \ldots, x_n)$. A displacement from a point $\xi = (\xi_1, \ldots, \xi_n)$ to $\xi' = (\xi'_1, \ldots, \xi'_n)$ is given by

$$\xi \to \xi' : \xi'_i = \sum_j p_{ji} \xi_j + x_i \qquad (i,\ j = 1, \ldots, n), \tag{3.1}$$

where (p_{ij}) is a proper orthogomal matrix. Let the fundamental frame R be given by a set of vectors

$$A^\circ = (0, \ldots, 0), \qquad e_1^\circ = (1, 0, \ldots, 0), \ldots, e_n^\circ = (0, \ldots, 0, 1) \tag{3.2}$$

and a frame derived from (3.2) by the displacement (3.1) be ·

$$A = (x_1, \ldots, x_n), \qquad e_1 = (p_{11}, \ldots, p_{1n}), \ldots, e_n = (p_{n1}, \ldots, p_{nn}).$$

When we put

$$R^\circ = \begin{pmatrix} A^\circ \\ e_1^\circ \\ \vdots \\ e_n^\circ \end{pmatrix}, \qquad R = \begin{pmatrix} A \\ e_1 \\ \vdots \\ e_n \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & x_1 & \cdots & x_n \\ 0 & p_{11} & \cdots & p_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & p_{n1} & \cdots & p_{nn} \end{pmatrix}, \tag{3.3}$$

we get

$$R = SR^\circ. \tag{3.4}$$

We take another displacement represented by

$$\xi' \to \xi'' : \xi''_i = \sum_j q_{ji} \xi'_j + y_i \tag{3.5}$$

and denote a corresponding frame transformation by

$$R = TR^{\circ}.$$

To the displacement $\xi \to \xi''$, which is a composition of (3.1) and (3.5), corresponds a frame transformation from R° to R such that

$$R = STR^{\circ}.$$

Relative displacement from SR to TR is given by TS^{-1} and by virtue of

$$(S + dS)S^{-1} = E + dSS^{-1}$$

relative components ω_i, $\omega_{ij} = -\omega_{ji}$ $(i, j = 1, \ldots, n)$ are coefficients of dSS^{-1}, namely

$$dSS^{-1} = \begin{pmatrix} 0 & \omega_1 & \cdots & \omega_n \\ 0 & \omega_{11} & \cdots & \omega_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix}. \tag{3.7}$$

This is usually represented as

$$dA = \sum_i \omega_i e_i, \qquad de_i = \sum_j \omega_{ij} e_j \qquad (i, j = 1, \ldots, n).$$

Next we take a fixed displacement represented by a matrix C. Relative components of the displacement from CSR to $C(S + dS)R$ are given by the coefficients of a matrix

$$d(CS)(CS)^{-1} = C(dSS^{-1})C^{-1}.$$

We denote as

$$C = \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ 0 & c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & c_{n1} & \cdots & c_{nn} \end{pmatrix}, \qquad d(CS)(CS)^{-1} = \begin{pmatrix} 0 & \bar{\omega}_1 & \cdots & \bar{\omega}_n \\ 0 & \bar{\omega}_{11} & \cdots & \bar{\omega}_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \bar{\omega}_{n1} & \cdots & \bar{\omega}_{nn} \end{pmatrix}$$

with a proper orthogonal matrix (c_{ij}), and we get

$$\bar{\omega}_i = \sum_k c_{ik}(\omega_k + \sum_j c_j \omega_{jk}) = \sum_k c_{ik}\omega_k + \sum_{j<k} (c_j c_{ik} - c_k c_{ij})\,\omega_{jk} \tag{3.8}$$

$$\bar{\omega}_{ij} = \sum_{kh} c_{ik} c_{jh} \omega_{kh} = \sum_{k<h} (c_{ik} c_{jh} - c_{ih} c_{jk})\,\omega_{kh}. \tag{3.9}$$

This is a transformation of a linear adjoint group corresponding to C.

Let S be as in (3.3) and T be a matrix representing a 1-parametric motion and put

$$d(ST)(ST)^{-1}=\begin{pmatrix} 0 & \pi_1 & \cdots & \pi_n \\ 0 & \pi_{11} & \cdots & \pi_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \pi_{n1} & \cdots & \pi_{nn}\end{pmatrix}, \qquad dTT^{-1}=\begin{pmatrix} 0 & \tau_1 & \cdots & \tau_n \\ 0 & \tau_{11} & \cdots & \tau_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \tau_{n1} & \cdots & \tau_{nn}\end{pmatrix} \tag{3.10}$$

$$\pi_{ij}=-\pi_{ji} \qquad \tau_{ij}=-\tau_{ji}.$$

By the relation

$$d(ST)(ST)^{-1}=dSS^{-1}+S(dTT^{-1})S^{-1}$$

we have

$$\pi_i=\omega_i+\sum_j p_{ij}(\tau_j+\sum_k x_k\tau_{kj}) \tag{3.11}$$

$$\pi_{ij}=\omega_{ij}+\sum_{kh} p_{ik}p_{jh}\tau_{kh}. \tag{3.12}$$

(a) Euclidean spaces with points as its elements

In this case a group G in the general theory is a group of displacement and H is a group $SO(n)$. Principal relative components of our spaces $M=G/H$ are $\omega_1, \ldots, \omega_n$. We take a hypersurface F to whose points we attach frames SR of $n-1$-parameters. As a volume element of a part generated by a 1-parametric motion T of the hypersurface we have by the use of (3.11)

$$dV=[\pi_1 \cdots \pi_n].$$

Putting
$$\Omega_i=\sum_j(-1)^{j-1}p_{ji}[\omega_1 \cdots \hat{\omega}_j \cdots \omega_n] \tag{3.13}$$

$$\Omega_{ij}=x_i\Omega_j-x_j\Omega_i \tag{3.14}$$

we have
$$dV=\sum_i[\tau_i\Omega_i]+\sum_{i<j}[\tau_{ij}\Omega_{ij}]. \tag{3.15}$$

By the relation $\omega_i=\sum_j p_{ij}dx_j$ (3.13) reduces to

$$\Omega_i=(-1)^{i-1}[dx_1 \cdots \widehat{dx_i} \cdots dx_n]. \tag{3.16}$$

When we put

$$\Pi_i=\frac{1}{n-1}\sum_{j(\neq i)}(-1)^{i+j}x_j[dx_1 \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots dx_n] \tag{3.17}$$

$$\Pi_{ij}=\frac{1}{2}(x_i^2+x_j^2)[dx_1 \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots dx_n]$$

we have
$$\Omega_i=d\Pi_i, \qquad \Omega_{ij}=d\Pi_{ij} \tag{3.18}$$

and Ω_i, Ω_{ij} are forms on M, though they depend on a coordinate system.

If we take a domain D on the hypersurface, an algebraic volume V generated by a 1-parametric motion T of D is

$$V=\sum_i X_iY_i+\sum_{i<j}X_{ij}Y_{ij} \tag{3.19}$$

where we have put $X_i = (-1)^{n-1}\int_D \Omega_i$, $X_{ij} = (-1)^{n-1}\int_D \Omega_{ij}$, $Y_i = \int \tau_i$, $Y_{ij} = \int \tau_{ij}$. X_i, X_{ij} depend on the boundary of D and not on D itself. This is the theorem of Koenigs-Hadamard-Guillaumin.

If we take frames $\bar{S}R = SCR$ and a motion $\bar{T} = C^{-1}T$ with a constant matrix C, we get the same generation of D and we have

$$V = \sum_i X_i Y_i + \sum_{i<j} X_{ij} Y_{ij} \tag{3.21}$$

and by (2.13), (3.8), (3.9) relations between X_i, X_{ij} and $\bar{X}_i$, $\bar{X}_{ij}$ are

$$\bar{X}_i = \sum_j X_j c_{ji} \tag{3.22}$$

$$\bar{X}_{ij} = \sum_k X_k (c_i c_{kj} - c_j c_{ki}) + \sum_{k<h} X_{kh} (c_{ki} c_{hj} - c_{hi} c_{kj}). \tag{3.23}$$

By taking (c_{ij}) suitably, we can assume

$$\bar{X}_2 = 0, \ldots, \bar{X}_n = 0.$$

For $c_{ij} = \delta_{ij}$ and $X_2 = 0, \ldots, X_n = 0$ we have

$$\bar{X}_{i1} = c_i X_1 + X_{i1} \qquad (i \neq 1). \tag{3.24}$$

Hence by taking c_i suitably we can assume $\bar{X}_{i1} = 0$ in case $X_1 \neq 0$. If frames are so chosen from first, we have

$$V = X_1 Y_1 + \sum_{i<j} X_{ij} Y_{ij} \qquad (i, j = 2, \ldots, n) \tag{3.25}$$

and by taking (c_{ij}) suitably we can simplify in such a way that X_{ij} $(i<j)$ except X_{23}, X_{45}, . . . vanish. If $X_1 = 0$, we have

$$V = \sum_{i<j} X_{ij} Y_{ij} \tag{3.26}$$

where X_{ij} $(i<j)$ except X_{12}, X_{34}, . . . vanish.

Next we turn to an application of the Theorem 3. We take a hyperplane E_{n-1} and denote by J the set of all displacements which fix E_{n-1}. If we take a fundamental frame $(A^\circ, e_1^\circ, \ldots, e_n^\circ)$ with A°, $e_2^\circ, \ldots, e_n^\circ$ on E_{n-1}, we have for a displacement S of J represented as in (3.3)

$$x_1 = 0, \quad p_{11} = 1, \quad p_{12} = 0, \ldots, p_{1n} = 0. \tag{3.27}$$

Hence $\omega_1 = 0$ and by (3.13), (3.14)

$$\Omega_i = p_{1i}[\omega_2 \cdots \omega_n], \qquad \Omega_{ij} = x_i \Omega_j - x_j \Omega_i$$

and so

$$\Omega_1 = [\omega_2 \ldots \omega_n], \qquad \Omega_i = 0 \qquad (i \geqq 2)$$
$$\Omega_{i1} = x_i \Omega_1, \qquad \Omega_{ij} = 0 \qquad (j > i \geqq 2).$$

For a domain $D(u)$ $(0 \leqq u \leqq 1)$ on E_{n-1} we take a suitable frame SCR $(C \in J)$, then we have by (3.24) $\overline{X}_{i1} = 0$. Thus a volume generated by $D(u)$ under a motion $T = T(u)$ $(0 \leqq u \leqq 1)$ for a suitably chosen frame on $D(u)$ is given by

$$V = \int X_1 \tau_1. \tag{3.29}$$

$X_1 = \int_{D(u)} [\omega_2 \cdots \omega_n]$ is an $n-1$-dimensional volume and τ_1 is a component, orthogonal to the hyperplane on which $D(u)$ under T lies, of an arcelement of the locus of center of gravitation of $D(u)$ (c.f. [7] p. 113).

We give another application of the Theorem 3. Let K_{n-1} be a hypersphere of radius r and J be the set of all displacements which fix K_{n-1}. We take an origin A° of the fundamental frame at the center of K_{n-1} and $(A, e_1, \ldots, e_n)$ with A on K_{n-1}, and e_1 on an outer normal at A. Then we have $x_i = rp_{1i}$ in (3.3). By the relation $d(A - re_1) = \sum_i \omega_i e_i - r\sum_i \omega_{1i} e_i = 0$ we have along K_{n-1}

$$\omega_1 = 0, \qquad \omega_i - r\omega_{1i} = 0 \qquad (i = 2, \ldots, n). \tag{3.30}$$

Hence by (3.13), (3.14)

$$\Omega_i = p_{1i}[\omega_2 \cdots \omega_n]$$
$$\Omega_{ij} = x_i \Omega_j - x_j \Omega_i = r(p_{1i}\Omega_j - p_{1j}\Omega_i) = 0.$$

We take a domain $D(u)$ on K_{n-1} and an algebraic volume gemerated by $D(u)$ under a motion $T = T(u)$ $(0 \leqq u \leqq 1)$ of K_{n-1} is

$$V = \sum_i \int X_i \tau_i, \quad \text{where} \quad X_i = (-1)^{n-1} \int_{D(u)} \Omega_i. \tag{3.31}$$

If v is an $n-1$-dimensional volume of $D(u)$ and $k = (-1)^{n-1} r v^{-1}$, $(kX_1, \ldots, kX_n)$ are coordinates of a center of gravitation of $D(u)$. By taking on each $D(u)$ a suitable frame we can reduce X_i to such ones that $X_2 = 0, \ldots, X_n = 0$. Hence

$$V = \int X_1 \tau_1 \tag{3.32}$$

Thus we get (3.32), where V is an algebraic volume generated by a set of domains of $D(u)$ on a hypersphere K_{n-1} of radius r under a motion $T = T(u)$ $(0 \leqq u \leqq 1)$ and $r^{-1} v X_1$ is a distance from the center of K_{n-1} to a center of gravitation $g(u)$ of $D(u)$, and τ_1 is an orthogonal component of an arcelement of the locus of center of the hypersphere K_{n-1} to the direction from the center of K_{n-1} to $g(u)$. According to the general theory a formula

$$V=\int X_1\tau_1+\sum_i\int(X_i-rX_{1i})(\tau_i-r\tau_{1i})$$

should appear in the first on account of (3.30). A modification has been made by taking an origin A° at the center of K_{n-1} from first.

(b) Euclidean space with hyperplanes as its elements

In this case we take frames $(A, e_1, \ldots, e_n)$ with $A, e_2, \ldots, e_n$ on a hyperplane. Then principal relative components are $\omega_1, \omega_{12}, \ldots, \omega_{1n}$. We take an $n-1$-parametric set D of hyperplanes to which we attach an $n-1$-parametric set of frames SR and a 1-parametric motion $T=T(u)$ $(0\leqq u\leqq 1)$. Then an element dV of an algebraic volume generated by D under a motion T is given by

$$dV=[\pi_1\pi_{12}\cdots\pi_{1n}],$$

where π_1, π_{1i} are given by (3.10). We put

$$p_i=p_{1i},\qquad h=(A, e_1),\qquad d\sigma=[\omega_{12}\cdots\omega_{1n}],$$

$$\Lambda_i=\sum_k(-1)^{k-1}p_{ki}[\omega_{12}\cdots\hat{\omega}_{1k}\cdots\omega_{1n}] \tag{3.33}$$

and also

$$\Omega_i=p_i\,d\sigma \tag{3.34}$$

$$\Omega_{ij}=x_i\Omega_j-x_j\Omega_i+p_i[\omega_1\Lambda_j]-p_j[\omega_1\Lambda_i]. \tag{3.35}$$

Then we get by (3.11) (3.12)

$$dV=\sum_i[\tau_i\Omega_i]+\sum_{i<j}[\tau_{ij}\Omega_{ij}]. \tag{3.36}$$

$d\sigma$ is an $n-1$-dimensional volume element of spherical representation of normals of hyperplanes belonging to D, and $h=(A, e_1)$ is a distance from the origin of a fundamental frame to a hyperplane. By the general theory Ω_i, Ω_{ij} are closed forms, which can also be verified directly. As for Ω_{ij} we have

$$\Omega_{ij}=[dh,\ p_i\Lambda_j-p_j\Lambda_i] \tag{3.37}$$

because of the relations

$$dh=d(A, e_1)=(dA, e_1)+(A, de_1)=\omega_1+\sum_{ij}\omega_{1i}p_{ij}x_j,$$

and Ω_{ij} are closed forms on account of the formula

$$\Lambda_i = (-1)^i \sum_j (-1)^{j-1} p_j [dp_1 \cdots \widehat{dp_i} \cdots \widehat{dp_j} \cdots dp_n]. \tag{3.38}$$

An algebraic volume V generated by a domain D of hyperplanes under a 1-parametric motion T is

$$V = \sum_i X_i Y_i + \sum_{i<j} X_{ij} Y_{ij}, \tag{3.39}$$

where X_i, X_{ij}, Y_i, Y_{ij} are as in (3.20) and a simplification of (3.39) are quite the same as in (3.25), (3.26). We take a set of all hyperplanes through one point and a rotation group $SO(n)$ around the point. Then for frames SR ($S \in SO(n)$) with an origin A at the point we have

$$x_1 = 0, \ldots, x_n = 0, \qquad \omega_1 = 0, \cdots, \omega_n = 0$$

and for these frames Ω_{ij} in (3.35) vanish and $X_{ij} = 0$. Hence an algebraic volume V of a set $D(u)$ of hyperplanes passing through $A = A(u)$ $(0 \leqq u \leqq 1)$ is given by

$$V = \sum_i \int X_i \tau_i, \qquad \text{where} \qquad X_i = (-1)^{n-1} \int_{D(u)} \Omega_i.$$

We take a spherical representation of normals of hyperplanes, and the center of gravitation of the representation of $D(u)$ has coordinates $(kX_1, \ldots, kX_n)$, v being an $n-1$-dimensional volume of the spherical representation, and $k = (-1)^{n-1} v^{-1}$. By a assuitable choice of C for which $c_i = 0$ we have

$$\overline{X}_i = \sum_j X_j c_{ji}$$

by (3.22) and by a suitable choice of (c_{ij}) we get a formula

$$V = \int X_1 \tau_1, \tag{3.40}$$

where $v^{-1} X_1$ is a distance from $A(u)$ to a center of gravitation $g(u)$ of $D(u)$ and τ_1 is a component of an arcelement of the locus of a center $A(u)$ to the direction from $A(u)$ to $g(u)$.

(c) Euclidean space with straight lines as its elements

We take an origin A and e_1 of a frame on a straight line. Then principal relative components of the space with straight lines as its elements are

$$\omega_2, \ldots, \omega_n, \qquad \omega_{12}, \ldots, \omega_{1n}.$$

For an element dV of an algebraic volume generated by a $2n-3$-parametric set D of straight lines under a 1-parametric motion we have

$$dV=[\pi_2\cdots\pi_n,\ \pi_{12}\cdots\pi_{1n}]$$

in the notation as in (3.10). We put

$$p_i=p_{1i},\qquad dS=[\omega_2\cdots\omega_n],\qquad d\sigma=[\omega_{12}\cdots\omega_{1n}],$$

$$M_i=\sum_i(-1)^{j-1}p_{ji}[\omega_2\cdots\hat{\omega}_j\cdots\omega_n] \tag{3.41}$$

$$\Lambda_i=\sum_j(-1)^{j-1}p_{ji}[\omega_{12}\cdots\hat{\omega}_{1j}\cdots\omega_{1n}] \tag{3.42}$$

and also

$$\Omega_i=[M_i\,d\sigma] \tag{3.43}$$

$$\Omega_{ij}=x_i\Omega_j-x_j\Omega_i+(-1)^n[dS,\ p_i\Lambda_j-p_j\Lambda_i]. \tag{3.44}$$

These are closed forms by the general theory although they are also verified by

$$M_i=(-1)^i\sum_j(-1)^{j-1}p_j[dx_1\cdots\widehat{dx_i}\cdots\widehat{dx_j}\cdots dx_n]$$

and (3.38). An algebraic volume generated by a set of straight lines under a 1-parametric motion is given by (3.19).

If we consider a set of all straight lines which are perpendicular to a fixed direction which we take as e_n, we have

$$p_{1n}=0,\ \ldots,\ p_{n-1n}=0,\qquad p_{nn}=1,\qquad \omega_{1n}=0,\ \ldots,\ \omega_{n-1n}=0$$

and so by virtue of (3.43) (3.44) we get

$$\Omega_i=0,\qquad \Omega_{ij}=0\qquad (i<j<n).$$

We consider a $2n-3$-parametric set $D(u)$ of straight lines which are perpendicular to a fixed direction each. An algebraic volume V generated by a set $D(u)$ $(0\leqq u\leqq 1)$ is given by

$$V=\sum_i\int X_{in}\tau_{in}. \tag{3.45}$$

A transformation C which fixes a direction e_n satisfies the relation $c_{1n}=0,\ \ldots,$ $c_{n-1n}=0$, $c_{nn}=1$ and by (3.23) we can reduce $X_{1n},\ \ldots,\ X_{n-1n}$ to $X_{in}=0$ $(i\geqq 2)$, and we get a formula

$$V=\int X_{1n}\tau_{1n}. \tag{3.46}$$

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