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The Theory of Pure Operations

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Abstract. This paper continues the study started in [13] where classes of operations were investigated in the partially ordered vector space approach to the theory of statistical physical systems. In this approach the set of states is represented by a norm closed generating cone K in a complete base norm space (V, K, B) and the set of operations is represented by the set $\mathscr{P}$ of positive norm non-increasing linear operators on V. In actual physical experiments it is usually the case that only certain subsets of K are available and it is supposed that the set $\Gamma(K)$ of such subsets is the set of split faces of K. The properties of two important classes of operation are examined. The first class $\boldsymbol{P}$ of strong operations has the property that each member leaves every element of $\Gamma(K)$ invariant and therefore can be measured in every restricted situation. The second class $\boldsymbol{P}_P$ of pure operations has the property above and also sends pure states into pure states. A study is made, in terms of the structure of $\Gamma(K)$, of when such operations are physically relevant. The paper ends with an examination of $\Gamma(K)$, $\boldsymbol{P}$, $\boldsymbol{P}_P$ in the Von Neumann algebra model.

§ 1. Introduction

The need to examine repeated measurements on a statistical physical system led Davies and Lewis [7] to generalise the theory of operations on such a system originally suggested by Haag and Kastler [23]. The central idea is that the set K of states of the system is the important object and that all possible changes in the system can be described by means of operations on K. Whilst K represents all possible states of the system certain well defined subsets of K play important parts in the theory. For details the reader is referred to [12–14]. A strong restriction of the system has as its set H of states a split face of the cone K and H may be regarded as the set of possible states of the system arising from placing the system in some kind of controlled environment. Clearly, the set of operations which leaves H invariant is of some importance. Indeed, if the physical configuration is such that only the particular strong restriction described by H is under consideration, then this set of operations is the only one of importance.

In practice, it is impossible in general to have a completely unrestricted system and in any physical situation the set K is in fact a split face of some larger cone. Fortunately, it is true that in the situation occurring here, the set $\Gamma(H)$ of split faces of the split face H of K is precisely the set of intersections of elements of $\Gamma(K)$ with H. It follows that, even though K is probably only a subset of the set of all states of the system, it is perfectly justifiable to study K in isolation. Such considerations lead to the conclusion that a very important class of operations is that which sends each element H of $\Gamma(K)$ into itself. Such strong operations were mentioned in [13] but none of their properties were discussed. § 3 is devoted to a study of the structure of $\Gamma(K)$ and most of the results given are well known and can be found in the work of Alfsen, Anderson, and Effros [2, 3, 37], Combes and Perdrizet [4, 30], Cunningham [5], Gerzon [19] and Wils [36]. The proofs are given only of those results which do not seem to appear explicitly in the literature. In § 4 the properties of those operations which leave a particular element H in $\Gamma(K)$ invariant are studied whilst in § 5 the set $\boldsymbol{P}$ of strong operations which have this property for every $H \in \Gamma(K)$ is examined. It is in this section that the structure of $\Gamma(K)$ is important. Generally, $\Gamma(K)$ is a complete Boolean algebra and the special case in which $\Gamma(K)$ is atomic is of importance. If $P\Gamma(K)$, the quasi-spectrum, is the set of atoms in $\Gamma(K)$, every j in $\boldsymbol{P}$ has a unique decomposition into operations j_H, H in $P\Gamma(K)$, where j_H is an operation on the system whose set of states is H. A consequence of this result is that if $\Gamma(K)$ is atomic, for each simple observable A, it is possible to find a strong operation j which measures A.

The set of pure states of the system is the set $E(K)$ of points on extreme rays of K. Whilst it is possible that $E(K) = \{0\}$, indeed this occurs in many physical examples, in two important examples, the models of a classical or quantum system with a finite number of degrees of freedom, $E(K)$ is a determining family of states. For each H in $\Gamma(K)$, $E(H) = H \cap E(K)$ and it therefore depends upon the choice of H whether or not pure states are important. It is shown in § 3 that if $E(K) \neq \{0\}$, $P\Gamma(K)$ is non-trivial and indeed that $P\Gamma(K)$ possesses a subset $P_I\Gamma(K)$, the spectrum, such that for $H \in P_I\Gamma(K)$, $E(H) \neq \{0\}$. The join K_I of the elements of $P_I\Gamma(K)$ has the property $E(K_I) = E(K)$ and is the smallest element of $\Gamma(K)$ containing $E(K)$. The systems for which pure states have importance are those for which $K = K_I$. In § 6, the class of strong operations which send $E(K)$ into itself are examined. It is clear that only in the case $K = K_I$ have such pure operations any relevance, since the condition that j sends $E(K)$ into itself makes no extra stipulation on the action of j on states with no component in K_I.

In § 7, the earlier results are applied to the Von Neumann algebra model and using methods of [6, 32, 34, 35] pure operations are identified when the Von Neumann algebra has atomic centre.

The abstract formulation has been studied by Giles [20], Gunson [22], Ludwig [26–28], Mielnik [29] and Pool [38, 39], whilst pure operations in the Von Neumann algebra model have been discussed by Haag and Kastler [23] and Hellwig and Kraus [24, 25]. Results related to those of this paper have also been obtained by Giles and Kummer [21].

§ 2. Preliminaries

For the most part the definitions given in [12–14] will not be repeated. In those papers and in the work of Davies and Lewis [7] and Haag and Kastler [23] can be found the physical motivation for the abstract definitions. For the relevant theory of partially ordered vector spaces see Alfsen [1], Ellis [16, 17], Schaeffer [33] and Størmer [34, 35].

Let (V, K, B) be a complete base norm space having a norm closed cone K with base B. Recall that K is a generating cone for the partially ordered real vector space V and the Minkowski functional $\|.\|_B$ corresponding to the radial, circled convex set $\operatorname{conv}(B \cup -B)$ is the base norm with respect to which K is closed and V is complete. Let (V^*, K^*, e) be the complete Archimedean ordered order unit space dual to (V, K, B). V^* is the Banach space dual of V, K^* is the cone dual to K and e is the strictly positive linear functional on V such that $B = e^{-1}(1) \cap K$. K^* is a weak* closed generating cone for V^* defining an Archimedean ordering on V^* for which e is an order unit. The Minkowski functional $\|.\|_e$ corresponding to the radial, circled convex set $[-e, e] = (-e + K^*) \cap (e - K^*)$ is the order unit norm on V^* and coincides with its norm as a dual space.

Let $\mathfrak{L}(V)$ be the algebra of bounded linear operators on V and let $\mathfrak{L}(V)^+$ be the strongly closed cone in $\mathfrak{L}(V)$ consisting of elements j such that $j(K) \subset K$. Let $\mathscr{P}$ be the subset of $\mathfrak{L}(V)^+$ consisting of elements j such that $\|j\| \leqq 1$. For $j \in \mathfrak{L}(V)$, let j^* be the adjoint mapping on V^*. Then, $j \mapsto j^*$ is an isometric algebraic anti-isomorphism from the algebra $\mathfrak{L}(V)$ onto the algebra $\mathfrak{L}^w(V^*)$ of weak* continuous elements of $\mathfrak{L}(V^*)$ and $j \in \mathfrak{L}(V)^+$ if and only if $j^* \in \mathfrak{L}^w(V^*)^+ = \{q : q \in \mathfrak{L}^w(V^*), q(K^*) \subset K^*\}$. Let $\mathscr{P}^*$ be the image of $\mathscr{P}$ under the mapping $j \mapsto j^*$. For $j \in \mathscr{P}$, $T(j) = j^*(e)$ is defined for $f \in K$ by $T(j)(f) = e(j(f))$ and $j \mapsto T(j)$ is an affine mapping from $\mathscr{P}$ onto the weak* compact convex set $\mathscr{Q} = [0, e] = K^* \cap (e - K^*)$ in V^*.

K is supposed to represent the set of states of a physical system, $\mathscr{P}$ or equivalently $\mathscr{P}^*$ is supposed to represent the set of operations on the system and $\mathscr{Q}$ is supposed to represent the set of simple observables of the system. For $j \in \mathscr{P}$, $f \in K$, $(T(j)(f))/(e(f))$ is the transmission probability of the state f under the operation j designed to measure the simple observable $T(j)$. The set $E(\mathscr{Q})$ of extreme points of $\mathscr{Q}$ is said to be the set of extreme simple observables. In [13] a study was made of some physically interesting subsets of $\mathscr{Q}$ and $\mathscr{P}$ in the general situation. Although it may be necessary to use some of the results of [13] at a later stage, a full list of definitions will not be repeated here.

§ 3. Strong Restrictions

It was remarked in [13] that the set of strong restrictions of the physical system whose set of states is K is in one-one correspondence with two sets, the set $\Gamma(K)$ of split faces of K and the set $\mathscr{P}_{SRW}$ of strongly repeatable reflections of the system. In order to appreciate the physical significance of these two sets it is necessary to repeat some of the general theory of split faces. Most of the results quoted below are well known or are easy consequences of well known results. For details the reader is referred to the work of Alfsen and Anderson [2, 3], Combes and Perdrizet [4, 30], Cunningham [5], Gerzon [19] and Wils [36].

Let U be a real vector space with generating cone U^+. A subset H of U^+ is said to be a face (extremal set, support) if and only if for $a, b \in H$, $\alpha \geqq 0$, $a+b$, $\alpha a \in H$ and if $0 \leqq a \leqq b \in H$, $a \in H$. Let $\Pi(U^+)$ denote the set of faces of U^+. By a slight abuse of notation let $E(U^+) = \{a : a \in U^+, \{\alpha a : \alpha \geqq 0\} \in \Pi(U^+)\}$. $E(U^+)$ is the set of points on extreme rays of U^+. For $H \in \Pi(U^+)$, let $H' = \bigcup\{H_1 : H_1 \in \Pi(U^+), H_1 \cap H = \{0\}\}$ be the complementary set of H. Then H is said to be a split face of U^+ when $H' \in \Pi(U^+)$ and every element $a \in U^+$ has a unique decomposition $a = a_1 + a_2$ with $a_1 \in H$, $a_2 \in H'$. Let $\Gamma(U^+)$ denote the family of split faces of U^+. The following well known result summarises the properties of a single split face.

Proposition 3.1. *The following conditions on $H \in \Pi(U^+)$ are equivalent.*

(i) $H \in \Gamma(U^+)$.

(ii) $H' \in \Gamma(U^+)$, $H'' = H$.

(iii) *There exists $G \in \Pi(U^+)$ such that $G \cap H = \{0\}$ and every $a \in U^+$ has a unique decomposition $a = a_1 + a_2$ where $a_1 \in H$, $a_2 \subset G$.*

(iv) *There exists $G \in \Pi(U^+)$ such that $(G-G) \cap (H-H) = \{0\}$ and $H + G = U^+$.*

In the following, if G, H are subsets of some set F in a real vector space such that every non-zero element a of F has a unique decomposition $a = a_1 + a_2$ where $a_1 \in G$, $a_2 \in H$, the notation $F = G \oplus H$ will be adopted. In particular, for $H \in \Gamma(U^+)$, $U^+ = H \oplus H'$, $U = (H-H) \oplus (H'-H')$.

It follows from Proposition 3.1 that every $H \in \Gamma(U^+)$ determines a unique positive linear mapping p_H on U defined for $a \in U$ by $p_H(a) = a_1$

where $a = a_1 + a_2$ is the unique decomposition of a into $a_1 \in H - H$, $a_2 \in H' - H'$. In this case $0 \leqq p_H(a) \leqq a$, $\forall a \in U^+$ and $p_H^2 = p_H$. p_H is said to be the split projection defined by H. If p is any linear mapping on U such that $0 \leqq p(a) \leqq a$, $\forall a \in U^+$ and $p^2 = p$, it is easy to see that $p(U^+) = H \in \Gamma(U^+)$ and $p_H = p$. The bijection $H \mapsto p_H$ between $\Gamma(U^+)$ and the set of all split projections was characterised in the following manner by Wils [36].

Let $\mathrm{End}(U)$ be the algebra of all endomorphisms of U and let $\mathrm{End}(U)^+ = \{q : q \in \mathrm{End}(U), q(U^+) \subset U^+\}$. If $\mathbf{1}_U$ is the identity operator on U, then $\mathrm{End}(U)^+$ is a cone in $\mathrm{End}(U)$ containing $\mathbf{1}_U$. Let $\mathfrak{Z}(U) = \{q : q \in \mathrm{End}(U), -\alpha \mathbf{1}_U \leqq q \leqq \alpha \mathbf{1}_U$, some $\alpha \geqq 0\}$ be the ideal centre of U. Then $\mathfrak{Z}(U)^+ = \mathfrak{Z}(U) \cap \mathrm{End}(U)^+$ is the smallest face of $\mathrm{End}(U)^+$ containing $\mathbf{1}_U$ and is a generating cone for $\mathfrak{Z}(U)$ with respect to which $\mathbf{1}_U$ is an order unit. Further $\mathfrak{Z}(U)$ is an ordered subalgebra of $\mathrm{End}(U)$ and if $E(\mathfrak{Z}(U)_1^+)$ denotes the set of extreme points of the set $\mathfrak{Z}(U)_1^+ = \{q : q \in \mathrm{End}(U), \mathbf{0} \leqq q \leqq \mathbf{1}_U\}$, Wils proves the following result.

Proposition 3.2. *The mapping $H \mapsto p_H$ is a bijection from $\Gamma(U^+)$ onto $E(\mathfrak{Z}(U)_1^+)$ with inverse $p \mapsto p(U^+)$ and $p_{H'} = \mathbf{1}_U - p_H$.*

Before remarking on the additional lattice structure of $\Gamma(U^+)$ it is convenient to return to examine the complete base norm space (V, K, B). In this case $\Gamma(K)$ is identified with the sets of states of strong restrictions of the system whose set of states is K and $E(\mathfrak{Z}(V)_1^+) = \mathscr{P}_{SRW}$ with the set of strongly repeatable reflections on the system. Further $\mathfrak{Z}(V)_1^+ = \mathscr{P}_W$ is identified with the set of reflections on the system [13]. Whilst it is clear that for $p \in E(\mathfrak{Z}(V)_1^+)$, $p \in \mathfrak{L}(V)$ it is not entirely obvious that $\mathfrak{Z}(V) \subset \mathfrak{L}(V)$.

Proposition 3.3. (i) *$\mathfrak{Z}(V)$ is an ordered subalgebra of $\mathfrak{L}(V)$.*

(ii) *$(\mathfrak{Z}(V), \mathfrak{Z}(V)^+, \mathbf{1}_V)$ is an Archimedean ordered order unit space.*

Proof. All the results other than that which shows that an arbitrary element of $\mathfrak{Z}(V)$ is bounded are elementary consequences of results of [36]. Let $q \in \mathfrak{Z}(V)$ and suppose $-\alpha \mathbf{1}_V \leqq q \leqq \alpha \mathbf{1}_V$. Then, $-\alpha \leqq e(q(f)) \leqq \alpha$, $\forall f \in B$. Therefore, for $f \in B$,

$$\|(\alpha \mathbf{1}_V - q)(f)\|_B = \alpha - e(q(f)) \leqq 2\alpha,$$

and hence $\|q(f)\|_B \leqq 3\alpha$. Let $g \in V$, $\|g\|_B = 1$. Then, for $\varepsilon > 0$, there exist $g_1, g_2 \in B$, $t \in [0, 1]$, such that $g = (1 + \varepsilon)(t g_1 - (1 - t) g_2)$ and hence $\|q(g)\|_B \leqq 3\alpha$. It follows that $q \in \mathfrak{L}(V)$. The above argument is an adaptation of a result of [17].

The main consequence of the continuity of p_H, for $H \in \Gamma(K)$, is that the kernel $H - H$ of $\mathbf{1}_V - p_H$ is norm closed in V and hence, $H = (H - H)$

$\cap K$ is also norm closed in V. Summarising and restating a result proved in [13],

Proposition 3.4. (i) *Every $H \in \Gamma(K)$ is norm closed.*

(ii) *For $H \in \Gamma(K)$, $(H-H, H, H \cap B)$ is a complete base norm space with base norm equal to the norm on $H-H$ when regarded as a subspace of V.*

The image $\mathfrak{Z}^w(V^*)$ of $\mathfrak{Z}(V)$ under the mapping $q \mapsto q^*$ is the set $\{p : p \in \mathfrak{L}^w(V^*),\ -\alpha \mathbf{1}_{V^*} \leqq p \leqq \alpha \mathbf{1}_{V^*}$ some $\alpha \geqq 0\}$ and it is clear that $q \mapsto q^*$ is an order isomorphism between the Archimedean ordered order unit spaces $(\mathfrak{Z}(V), \mathfrak{Z}(V)^+, \mathbf{1}_V)$, $(\mathfrak{Z}^w(V^*)\mathfrak{Z}^w(V^*)^+, \mathbf{1}_{V^*})$ which is at the same time an algebraic isomorphism between ordered algebras. Hence $q \mapsto q^*$ sends $\mathfrak{Z}(V)_1^+$ onto $\mathfrak{Z}^w(V^*)_1^+ = \{p : p \in \mathfrak{L}^w(V^*), \mathbf{0} \leqq p \leqq \mathbf{1}_{V^*}\}$ and $\mathscr{P}_{SRW} = E(\mathfrak{Z}(V)_1^+)$ onto $\mathscr{P}_{SRW}^* = E(\mathfrak{Z}^w(V^*)_1^+) \subset E(\mathfrak{Z}(V^*)_1^+)$. The following result shows that the reverse inclusion holds.

Proposition 3.5. (i) *For each $G \in \Gamma(K^*)$, $G-G$ and G are weak* closed and p_G is weak* continuous.*

(ii) *For $G \in \Gamma(K^*)$, $(G-G, G, p_G(e))$ is a complete Archimedean ordered order unit space with order unit norm equal to the norm on $G-G$ when regarded as a subspace of V^*.*

(iii) $E(\mathfrak{Z}^w(V^*)_1^+) = E(\mathfrak{Z}(V^*)_1^+)$.

Proof. For $G \in \Gamma(K^*)$, in order to show that $G-G$ is weak* closed it suffices to show that the intersection of $G-G$ with all multiples of the unit ball $(-e+K^*) \cap (e-K^*)$ in V^* is weak* closed (see [11], V. 5.7). Let $\lambda \geqq 0$ and let $A \in (-\lambda e + K^*) \cap (\lambda e - K^*) \cap (G-G)$. Then, $p_G(A) = A$, $\lambda e + A$, $\lambda e - A \in K^*$ and by the positivity of p_G, $\lambda p_G(e) + A$, $\lambda p_G(e) - A \in K^*$ which implies that $A \in (-\lambda p_G(e) + K^*) \cap (\lambda p_G(e) - K^*)$. Conversely, for $A \in (-\lambda p_G(e) + K^*) \cap (\lambda p_G(e) - K^*)$, $\lambda p_G(e) + A$, $\lambda p_G(e) - A \in K^*$, $0 \leqq \lambda p_G(e) - A \leqq 2\lambda p_G(e) \in G$ and, since G is a face, $\lambda p_G(e) - A \in G$. Hence, $\lambda p_G(e) - A = p_G(\lambda p_G(e) - A) = \lambda p_G(e) - p_G(A)$ and therefore, $p_G(A) = A$, $A \in G-G$. Further, $-\lambda e \leqq -\lambda p_G(e) \leqq A \leqq \lambda p_G(e) \leqq \lambda e$ and therefore, $A \in (-\lambda e + K^*) \cap (\lambda e - K^*) \cap (G-G)$, which is therefore identical to the weak* closed order interval $(-\lambda p_G(e) + K^*) \cap (\lambda p_G(e) - K^*)$. Hence, $G-G$ is weak* closed which implies that $G = (G-G) \cap K^*$ is weak* closed. Similarly $G'-G'$ is weak* closed and Lemma 4.9 of [30] shows that p_G is weak* continuous. This completes the proof of (i) and (iii) follows immediately.

To prove (ii) notice that for $\lambda \geqq 0$, $(-\lambda p_G(e) + K^*) \cap (\lambda p_G(e) - K^*)$ is contained in $G-G$ and hence is identical to $(-\lambda p_G(e) + G) \cap (\lambda p_G(e) - G)$ the unit ball in $G-G$ with its order unit norm. The two norms on $G-G$ are therefore identical.

The mapping $H \mapsto p_H$ from $\Gamma(K)$ onto $\mathscr{P}_{SRW}$ and $G \mapsto p_G$ from $\Gamma(K^*)$ onto $\mathscr{P}_{SRW}^*$ are bijections and it follows from Proposition 3.5 that $p \mapsto p^*$

is a bijection from $\mathscr{P}_{SRW}$ onto $\mathscr{P}^*_{SRW}$. Hence, it follows that there exists a bijection $H \mapsto H^*$ from $\Gamma(K)$ onto $\Gamma(K^*)$ such that $p_{H^*} = p^*_H$. Notice that,

$$\begin{aligned} H^* &= p_{H^*}(K^*) = p^*_H(K^*) = \ker(\mathbf{1}_{V^*} - p^*_H) \cap K^* \\ &= \ker(\mathbf{1}_V - p_H)^* \cap K^* = (\mathbf{1}_V - p_H)(V)^0 \cap K^* \\ &= (H' - H')^0 \cap K^* = (H')^0 \cap K^* . \end{aligned}$$

Summarising,

Proposition 3.6. *The mapping $H \mapsto H^* = (H')^0 \cap K^*$ is a bijection from $\Gamma(K)$ onto $\Gamma(K^*)$ with inverse $G \mapsto G_* = (G')_0 \cap K$.*

Notice also that the complementations $H \mapsto H'$, $G \mapsto G'$ on $\Gamma(K)$, $\Gamma(K^*)$ respectively correspond to the mappings $p \mapsto \mathbf{1}_V - p$, $q \mapsto \mathbf{1}_{V^*} - q$ in $\mathscr{P}_{SRW}$, $\mathscr{P}^*_{SRW}$ and clearly $(\mathbf{1}_V - p)^* = \mathbf{1}_{V^*} - p^*$.

Proposition 3.7. *For each $H \in \Gamma(K)$, $G \in \Gamma(K^*)$, $(H')^* = (H^*)'$, $(G')_* = (G_*)'$.*

From a physical point of view, for $H \in \Gamma(K)$, the dual space of $(H - H, H, H \cap B)$ is of interest and it was shown in Proposition 3.12 of [13] that it may be identified with $(V^*/H^0, K^*/H^0, \phi(e))$ where $\phi: V^* \to V^*/H^0$ is the canonical mapping. Further, H^0 is positively generated. It is easy to see that the mapping $i: V^*/H^0 \to H^* - H^*$ defined by $i(\phi(A)) = p_{H^*}(A)$ is well-defined and is an order isomorphism between the order unit spaces $(V^*/H^0, K^*/H^0, \phi(e))$ and $(H^* - H^*, H^*, p_{H^*}(e))$.

Proposition 3.8. *The complete Archimedean ordered order unit spaces $(V^*/H^0, K^*/H^0, \phi(e))$, $(H^* - H^*, H^*, p_{H^*}(e))$ are each order isomorphic to the dual space of $(H - H, H, H \cap B)$, the order isomorphism between them being continuous for their weak* topologies.*

Notice that this result justifies the notation $H^* = (H')^0 \cap K^*$ since H^* is indeed the cone dual to H.

What has been shown so far is that for each strong restriction of the system there are uniquely defined complete base norm spaces $(H - H, H, H \cap B)$, $(H' - H', H', H' \cap B)$ with duals $(H^* - H^*, H^*, p_{H^*}(e))$, $(H'^* - H'^*, H'^*, p_{H'^*}(e))$ respectively which represent the restricted system and its complement.

Before proceeding to a discussion of pure states of the system, it is useful to examine the lattice properties of the set $\Gamma(U^+)$ of split faces of an arbitrary generating cone U^+ in the real vector space U. The following well known result summarises the situation.

Proposition 3.9. (i) *$\Gamma(U^+)$ is a Boolean algebra under the inclusion ordering and the complementation $H \mapsto H'$. For $H_1, H_2 \in \Gamma(U^+)$, $H_1 \wedge H_2 = H_1 \cap H_2$, $H_1 \vee H_2 = H_1 + H_2$.*

(ii) $E(\mathfrak{Z}(U)_1^+)$ *is a Boolean algebra under the partial ordering inherited from* $\mathfrak{Z}(U)$ *and the complementation* $p \mapsto \mathbf{1}_U - p$. *For* $p_1, p_2 \in E(\mathfrak{Z}(U)_1^+)$, $p_1 \wedge p_2 = p_1 p_2 = p_2 p_1, p_1 \vee p_2 = p_1 + p_2 - p_1 p_2$.

(iii) *The mapping* $H \mapsto p_H$ *from* $\Gamma(U^+)$ *to* $E(\mathfrak{Z}(U)_1^+)$ *is a Boolean algebra isomorphism.*

It is reasonable to ask if there is a special form of Proposition 3.9 when $U = V$, $U^+ = K$. The following result can be extracted easily from a result of Cunningham [5] and a similar result has been proved by Wils [36].

Proposition 3.10. $\Gamma(K)$ *is a complete Boolean algebra under the inclusion ordering and the complementation* $H \mapsto H'$. *For* $\{H_k : k \in \Lambda\} \subset \Gamma(K)$, $\bigwedge_{k\in\Lambda} H_k = \bigcap_{k\in\Lambda} H_k, \bigvee_{k\in\Lambda} H_k = \left(\sum_{k\in\Lambda} H_k\right)^-$.

Some explanation of the notation is necessary here. $\sum_{k\in\Lambda} H_k$ is the set of elements of K consisting of sums $\sum_{k\in\Lambda'} f_k$ where $f_k \in H_k$, $\Lambda' \subset \Lambda$ and Λ' is finite, whilst $\left(\sum_{k\in\Lambda} H_k\right)^-$ denotes the norm closure of $\sum_{k\in\Lambda} H_k$. Compact convex sets with similar properties have been studied by Alfsen and Anderson [2] and by Størmer [35].

It is clear from Proposition 3.9 (iii) that $\mathscr{P}_{SRW}$ is also a complete Boolean algebra and the method of proof of Proposition 3.10 is based on showing this to be true and then using the Boolean algebra isomorphism property. For $\{p_k : k \in \Lambda\} \subset \mathscr{P}_{SRW}$, $f \in K$, the sets $\left\{\left(\bigwedge_{k\in\Lambda'} p_k\right) f : \Lambda' \subset \Lambda, \Lambda' \text{ finite}\right\}$, $\left\{\left(\bigvee_{k\in\Lambda'} p_k\right) f : \Lambda' \subset \Lambda, \Lambda' \text{ finite}\right\}$ are Cauchy nets for the norm topology with norm limits denoted by $\left(\bigwedge_{k\in\Lambda} p_k\right) f$, $\left(\bigvee_{k\in\Lambda} p_k\right) f$ respectively. It follows that $\left(\bigwedge_{k\in\Lambda} p_k\right)$, $\left(\bigvee_{k\in\Lambda} p_k\right)$ are the strong operator limits of the Cauchy nets $\left\{\left(\bigwedge_{k\in\Lambda'} p_k\right) : \Lambda' \subset \Lambda, \Lambda' \text{ finite}\right\}$, $\left\{\left(\bigvee_{k\in\Lambda'} p_k\right) : \Lambda' \subset \Lambda, \Lambda' \text{ finite}\right\}$ respectively. It is clear that for finite subsets Λ' of Λ, $\left(\bigwedge_{k\in\Lambda'} p_k\right)^* = \bigwedge_{k\in\Lambda'} p_k^*$, $\left(\bigvee_{k\in\Lambda'} p_k\right)^* = \bigvee_{k\in\Lambda'} p_k^*$ and it easily follows that for $A \in K^*$, $\left\{\left(\bigwedge_{k\in\Lambda'} p_k^*\right)(A) : \Lambda' \subset \Lambda, \Lambda' \text{ finite}\right\}$, $\left\{\left(\bigvee_{k\in\Lambda'} p_k^*\right)(A) : \Lambda' \subset \Lambda, \Lambda' \text{ finite}\right\}$ are Cauchy nets with weak* limits denoted by $\bigwedge_{k\in\Lambda} p_k^*(A)$, $\bigvee_{k\in\Lambda} p_k^*(A)$ respectively such that $\left(\bigwedge_{k\in\Lambda} p_k\right)^* = \bigwedge_{k\in\Lambda} p_k^*$, $\left(\bigvee_{k\in\Lambda} p_k\right)^* = \bigvee_{k\in\Lambda} p_k^*$.

Proposition 3.11. (i) $\Gamma(K^*)$ *is a complete Boolean algebra under the inclusion ordering and the complementation* $G \mapsto G'$. *For* $\{G_k : k \in \Lambda\} \in \Gamma(K^*)$, $\bigwedge_{k\in\Lambda} G_k = \bigcap_{k\in\Lambda} G_k, \bigvee_{k\in\Lambda} G_k = \left(\sum_{k\in\Lambda} G_k\right)^{-w}$.

(ii) *$H \mapsto H^*$ is a complete Boolean algebra isomorphism from $\Gamma(K)$ onto $\Gamma(K^*)$.*

Some explanation of the notation and terminology is necessary. $\left(\sum_{k \in \Lambda} G_k\right)^{-w}$ is the weak* closure of $\sum_{k \in \Lambda} G_k$ and a complete Boolean algebra isomorphism is an isomorphism which preserves the ordering, complementation, $\bigwedge$ and $\bigvee$.

One interesting consequence is the following. Let $CE(\mathcal{Q})$ be the subset of $E(\mathcal{Q})$ consisting of elements of the form $p_G(e)$ for $G \in \Gamma(K^*)$. $CE(\mathcal{Q})$ is said to be the set of *central extreme simple observables.* Notice that $G = \bigcup_{\lambda \geqq 0} [0, \lambda p_G(e)]$ a result which follows from observations used in the proof of Proposition 3.4. It follows that $p_G(e) = p_F(e)$ if and only if $G = F$ and hence $p \mapsto p(e)$ is a bijection from $\mathcal{P}^*_{SRW}$ onto $CE(\mathcal{Q})$. Further, for $p_1, p_2 \in \mathcal{P}^*_{SRW}$, $p_1 \leqq p_2$ if and only if $p_1(e) \leqq p_2(e)$ and for any $p \in \mathcal{P}^*_{SRW}$, $(\mathbf{1}_V - p)(e) = e - p(e)$.

Proposition 3.12. (i) *$CE(\mathcal{Q})$ is a complete Boolean algebra with respect to the ordering inherited from V^* and the complementation $A \mapsto e - A$. For $\{A_k : k \in \Lambda\} \subset CE(\mathcal{Q})$, $\bigwedge_{k \in \Lambda} A_k = \left(\bigwedge_{k \in \Lambda} p_k\right)(e)$, $\bigvee_{k \in \Lambda} A_k = \left(\bigvee_{k \in \Lambda} p_k\right)(e)$ where $A_k = p_k(e)$.*

(ii) *The five complete Boolean algebras $\Gamma(K)$, $\Gamma(K^*)$, $\mathcal{P}_{SRW}$, $\mathcal{P}^*_{SRW}$, $CE(\mathcal{Q})$ are isomorphic.*

Let $(\operatorname{conv} CE(\mathcal{Q}))^{-w} = F$ and let $C(V^*) = \bigcup_{\lambda \geqq 0} \lambda(F - F)$. Then $C(V^*)$ is said to be the *centre* of (V^*, K^*, e). This problem has been approached from a slightly different angle by Alfsen and Anderson [3] and Wils [36]. A definition which follows naturally is that (V^*, K^*, e) is a *factor* if and only if $CE(\mathcal{Q}) = \{0, e\}$ or equivalently if no restrictions of the physical system exist. Notice that this is really only a statement about B since V^* may be identified with the space $A^b(B)$ of bounded affine functionals on B.

In general it may be necessary from a physical point of view to study a particular strong restriction of the system. It is therefore important to be able to discuss the complete Boolean algebra $\Gamma(H)$ where $H \in \Gamma(K)$.

Proposition 3.13. (i) *For $H \in \Gamma(K)$, $G \in \Gamma(H)$ if and only if $G \in \Gamma(K)$. $G \subset H$.*

(ii) *For $H \in \Gamma(K)$, $G \in \Gamma(K)$ if and only if $G \cap H \in \Gamma(H)$, $G \cap H' \in \Gamma(H')$.*

Proof. Let p^H_G be the element of $E(\mathfrak{Z}(H-H)^+_1) = \mathcal{P}^H_{SRW}$ corresponding to $G \in \Gamma(H)$ and let $p = p^H_G p_H$. Then p is clearly linear, $0 \leqq p \leqq \mathbf{1}_V$ and $p^2 = p^H_G p_H p^H_G p_H = p^H_G p^H_G p_H = p$ and hence $p \in \mathcal{P}_{SRW}$, $p(K) = G \in \Gamma(K)$. For $G \in \Gamma(K)$, $G \subset H$ implies $p_G \leqq p_H$ and hence $p_G \in \mathcal{P}^H_{SRW}$, $G \in \Gamma(H)$. That completes the proof of (i).

For $H, G \in \Gamma(K)$, $G \cap H$, $G \cap H' \in \Gamma(K)$ and by (i), $G \cap H \in \Gamma(H)$, $G \cap H' \in \Gamma(H')$. Conversely, for $H \in \Gamma(K)$ if $G \cap H \in \Gamma(H)$, $G \cap H' \in \Gamma(H')$, then, $(G \cap H) \cap (G \cap H') = \{0\}$, and hence $G = (G \cap H) \oplus (G \cap H') \in \Gamma(K)$.

Whilst the subject of a particular strong restriction of the system is under consideration the following result concerning simple observables is appropriate.

Proposition 3.14. *Let $H \in \Gamma(K)$ and let $\mathcal{Q}^H$ be the set of simple observables of the corresponding restricted system. Then,*

(i) $\mathcal{Q}^H = \mathcal{Q} \cap H^*$,
(ii) $E(\mathcal{Q}^H) = E(\mathcal{Q}) \cap \mathcal{Q}^H$,
(iii) $CE(\mathcal{Q}^H) = CE(\mathcal{Q}) \cap \mathcal{Q}^H$,
(iv) $\mathcal{Q} = \mathcal{Q}^H \oplus \mathcal{Q}^{H'}$,
(v) $E(\mathcal{Q}) = E(\mathcal{Q}^H) \oplus E(\mathcal{Q}^{H'})$,
(vi) $CE(\mathcal{Q}) = CE(\mathcal{Q}^H) \oplus CE(\mathcal{Q}^{H'})$.

Proof. The proof of (i) follows from Proposition 3.5. If $A \in E(\mathcal{Q}^H)$, $A = tA_1 + (1-t)A_2$, $A_1, A_2 \in \mathcal{Q}$, $t \in (0, 1)$, then

$$A = p_{H^*}(A) = t p_{H^*}(A_1) + (1-t)\, p_{H^*}(A_2),\ p_{H^*}(A_1),\ p_{H^*}(A_2) \in \mathcal{Q}^H$$

and it follows that $A = p_{H^*}(A_1) = p_{H^*}(A_2)$. Hence,

$$t(\mathbf{1}_{V^*} - p_{H^*})(A_1) + (1-t)(\mathbf{1}_{V^*} - p_{H^*})(A_2) = 0$$

and $(\mathbf{1}_{V^*} - p_{H^*})(A_i) = 0$, $i = 1, 2$, $A = A_1 = A_2$ and $A \in E(\mathcal{Q})$. Conversely, if $A \in E(\mathcal{Q}) \cap \mathcal{Q}^H$, clearly $A \in E(\mathcal{Q}^H)$.

For $A \in CE(\mathcal{Q}^H)$, there exists $G \in \Gamma(H)$ such that $A = p_{G^*}(e)$ and by Proposition 3.13 (i), $G \in \Gamma(K)$, $p_{G^*} \in \mathcal{P}^*_{SRW}$, $p_{G^*}(e) \in CE(\mathcal{Q})$. Conversely, for $A \in CE(\mathcal{Q}) \cap \mathcal{Q}^H$, there exists $G \in \Gamma(K)$ such that $A = p_{G^*}(e)$ and $0 \leqq p_{G^*}(e) \leqq p_{H^*}(e)$ and by Proposition 3.12, $G \subset H$. But $G, H \in \Gamma(K)$ and it follows from Proposition 3.13 (i) that $G \in \Gamma(H)$, $p_{G^*}(e) \in CE(\mathcal{Q}^H)$.

The proof of (iv) is trivial using the unique decomposition of elements of K^* into elements of H^*, $H^{*\prime}$, defined by

$$A = p_{H^*}(A) + p_{H^{*\prime}}(A)\,.$$

If $p_{H^*}(A) \in E(\mathcal{Q}^H)$, $p_{H'^*} \in E(\mathcal{Q}^{H'})$ and $A = tA_1 + (1-t)A_2$, $A_i \in \mathcal{Q}$, $i = 1, 2$, $t \in (0, 1)$, then $p_{H^*}(A) = t p_{H^*}(A_1) + (1-t)\, p_{H^*}(A_2)$ and $p_{H^*}(A) = p_{H^*}(A_i)$, $i = 1, 2$. Similarly $p_{H^{*\prime}}(A) = p_{H^{*\prime}}(A_i)$, $i = 1, 2$, and adding gives $A = A_i$, $i = 1, 2$, $A \in E(\mathcal{Q})$. Conversely, let $A \in E(\mathcal{Q})$ and suppose $p_{H^*}(A) = tA_1 + (1-t)A_2$, $A_i \in \mathcal{Q}^H$, $i = 1, 2$, $t \in (0, 1)$. Then,

$$A = p_{H^*}(A) + p_{H^{*\prime}}(A) = t(A_1 + p_{H^{*\prime}}(A)) + (1-t)(A_2 + p_{H^{*\prime}}(A))$$

and $0 \leqq A_i + p_{H^{*\prime}}(A) \leqq p_{H^*}(e) + p_{H^{*\prime}}(e) = e$, $i = 1, 2$. Therefore, $A = A_i + p_{H^{*\prime}}(A)$, $p_{H^*}(A) = A_i$, $i = 1, 2$ and $p_{H^*}(A) \in E(\mathcal{Q}^H)$. Similarly $p_{H^{*\prime}}(A) \in E(\mathcal{Q}^{H'})$.

Suppose now that $p_{H^*}(A) \in CE(\mathscr{Q}^H)$, $p_{H^{*\prime}}(A) \in CE(\mathscr{Q}^{H'})$. Then, there exist $G_1 \in \Gamma(H) \subset \Gamma(K)$, $G_2 \in \Gamma(H') \subset \Gamma(K)$ such that $p_{H^*}(A) = p_{G_1^*}(e)$, $p_{H^{*\prime}}(A) = p_{G_2^*}(e)$. Hence, $A = (p_{G_1^*} + p_{G_2^*})(e)$ and since $G_1 \subset H$, $G_2 \subset H'$, $G_1 \cap G_2 = \{0\}$ and $p_{G_1^*} + p_{G_2^*} = p_{G_1^* \oplus G_2^*}$. Therefore, $A = (p_{G_1^* \oplus G_2^*})(e) \in CE(\mathscr{Q})$. Conversely, if $A \in CE(\mathscr{Q})$, $A = p_{G^*}(e)$ some $G \in \Gamma(K)$ and hence $p_{H^*}(A) = p_{H^* \cap G^*}(e) = p_{(H \cap G)^*}(e)$. But $H \cap G \in \Gamma(H)$ and $p_{H^*}(A) \in CE(\mathscr{Q}^H)$. Similarly $p_{H^{*\prime}}(A) \in CE(\mathscr{Q}^{H'})$.

The first step towards examining pure states is to study the atoms of the complete Boolean algebra $\Gamma(K)$. $H \in \Gamma(K)$ is said to be the set of states of a *primary strong restriction* if and only if $\Gamma(H) = \{\{0\}, H\}$. The following result is then immediate from Propositions 3.13–14.

Proposition 3.15. *The following conditions on $H \in \Gamma(K)$ are equivalent.*

(i) *H is the set of states of a primary strong restriction.*
(ii) *$G \in \Gamma(K)$, $G \subset H$ implies $G = \{0\}$ or $G = H$.*
(iii) $CE(\mathscr{Q}^H) = \{0, p_{H^*}(e)\}$.
(iv) *$(H^* - H^*, H^*, p_{H^*}(e))$ is a factor.*
(v) $C(H^* - H^*) = \{\lambda p_{H^*}(e) : \lambda \in R\}$.

Let $P\Gamma(K)$ denote the set of elements of $\Gamma(K)$ satisfying the conditions of Proposition 3.15. $P\Gamma(K)$ is said to be the *quasi-spectrum* of K.

Proposition 3.16. *For $G, H \in P\Gamma(K)$, either $G \cap H = \{0\}$ or $G = H$.*

A question which might arise is how strong restrictions are defined physically. There is no single answer to the question but it is known that one possible manner is the specification of some equilibrium state which produces an environment in which the statistical system evolves. This is equivalent to the specification of a particular element or set of elements of K. For $G \subset K$, there exists a smallest element H_G of $\Gamma(K)$ containing G and this is interpreted as the set of states of the strong restriction defined by G. In particular, for $f \in K$, H_f denotes the smallest element of $\Gamma(K)$ containing f. Notice that this definition differs in general from that of Wils [36]. $f, g \in K$ are said to be *quasi-equivalent* if $H_f = H_g$ and $f \in K$ is said to be *primary* if $H_f \in P\Gamma(K)$.

The first important result about pure states is the following.

Proposition 3.17. *If $f \in E(K)$, f is primary.*

Proof. If $f = 0$, $H_f = \{0\} \in P\Gamma(K)$ and therefore suppose $f \neq 0$, $G \in \Gamma(K)$, $G \neq \{0\}$, $G \subset H_f$. If $f \in G$, $H_f \subset G$ which implies that $H_f = G$ and f is primary. Therefore suppose $f \notin G$ in which case $f = p_G(f) + (\mathbf{1}_V - p_G)(f)$ and since $f \in E(K)$, $p_G(f) = \alpha f$, $(\mathbf{1}_V - p_G)(f) = (1-\alpha) f$. Applying p_G gives $p_G(f) = 0$ or $\alpha = 1$. If $\alpha = 1$, $f = p_G(f)$ which implies that $f \in G$ giving a contradiction. Therefore, $p_G(f) = 0$ and $f \in G'$. Hence,

$G \subset H_f \subset G'$ and since $G \cap G' = \{0\}$, $G = \{0\}$ giving a further contradiction and completing the proof.

It is important to notice that the existence of non-zero pure states is completely immaterial in the general theory. It is only in some physical examples that pure states seem to be important.

A subset Q of $E(K)$ is said to be a *sector* if and only if $Q \neq \{0\}$ and there exists an element $H \in P\Gamma(K)$ such that $H \cap E(K) = Q$. Let $\hat{K}$ denote the set of sectors of $E(K)$ and let $P_I\Gamma(K)$ denote the subset of $P\Gamma(K)$ consisting of elements H such that $H \cap E(K) \neq \{0\}$. The following result summarises the properties of sectors.

Proposition 3.18. (i) *$H \mapsto H \cap E(K)$ is a bijection from $P_I\Gamma(K)$ onto $\hat{K}$.*

(ii) *For $Q_1, Q_2 \in \hat{K}$, either $Q_1 = Q_2$ or $Q_1 \cap Q_2 = \{0\}$.*

(iii) *For $f, g \in E(K), f, g \neq 0$, f and g are quasi-equivalent if and only if $Q_f = Q_g$ where $Q_f = H_f \cap E(K)$.*

(iv) $E(K) = \bigcup \{Q : Q \in \hat{K}\}$.

(v) *$K_I = \bigvee \{H : H \in P_I\Gamma(K)\}$ is the smallest element of $\Gamma(K)$ containing $E(K)$ and $E(K_I) = E(K)$.*

(vi) $P\Gamma(K_I) = P_I\Gamma(K)$.

Proof. For each $H \in P_I\Gamma(K), H \cap E(K) = E(H) \in \hat{K}$ and hence $H \mapsto H \cap E(K)$ maps $P_I\Gamma(K)$ onto $\hat{K}$. Suppose $H_1, H_2 \in P_I\Gamma(K)$, $H_1 \cap E(K) = H_2 \cap E(K)$. Then, either $H_1 = H_2$ or $H_1 \cap H_2 = \{0\}$ which implies that $H_1 \cap E(K) = H_2 \cap E(K) = \{0\}$ giving a contradiction. That completes the proof of (i) and (ii) follows from Proposition 3.16.

Let $f, g \in E(K)$ be quasi-equivalent. Then, $H_f = H_g \in P_I\Gamma(K)$ and from (i) it follows that $Q_f = Q_g$. Conversely, if $Q_f = Q_g$ it follows from (i) that $H_f = H_g$ and f, g are quasi-equivalent.

If $f \in E(K)$, then $f \in Q_f \in \hat{K}$. Conversely if $f \in Q$ some $Q \in \hat{K}$, then $f \in E(K)$ and hence (iv) holds.

If $H_{E(K)}$ is the smallest element of $\Gamma(K)$ containing $E(K)$, it is clear that since $E(K) \subset K_I$, $H_{E(K)} \subset K_I$. Let $H \in P_I\Gamma(K)$, $H \cap E(K) = Q \subset E(K)$. Then $H_Q = H$ and since $Q \subset E(K), H = H_Q \subset H_{E(K)}$. This holds for all $H \in P_I\Gamma(K)$ and hence $K_I \subset H_{E(K)}$. Further, $E(K_I) = K_I \cap E(K) = E(K)$ and the proof of (v) is complete.

Let $H \in P_I\Gamma(K)$. Then $H \in \Gamma(K_I)$ and clearly H is minimal and therefore $P_I\Gamma(K) \subset P\Gamma(K_I)$. Conversely, let $H \in \Gamma(K_I)$ and suppose $H \cap E(K) = \{0\}$. It follows that $E(K) \subset H' \cap K_I$ and by (v), $K_I \subset H' \cap K_I$ which implies that $K_I \subset H'$ giving a contradiction unless $H = \{0\}$. Therefore, for $H \in P\Gamma(K_I), H \neq \{0\}, H \cap E(K) \neq \{0\}$ and $H \in P_I\Gamma(K)$. This completes the proof of (vi).

$P_I\Gamma(K)$ or $\hat{K}$ is said to be the *spectrum* of K and K_I is said to be the set of states of the *pure restriction* of the system.

The final result in this section generalises Lemma 4.3 of [32] and depends upon a lemma due to Gerzon, (i) of the following.

Proposition 3.19. (i) *For* $H \in \Gamma(K)$,

$$H' = \{g : g \in K, \|(f/e(f)) - (g/e(g))\|_B = 2, \forall f \in H, f \neq 0\} \cup \{0\}.$$

(ii) *For* $f, g \in E(K), f, g \neq 0$, *if* $\|(f/e(f)) - (g/e(g))\|_B < 2$, $Q_f = Q_g$.

Proof. Let $f \in H, g \in H', f, g \neq 0$ and let $f_1, f_2 \in K$ satisfy $f_1 - f_2 = (f/e(f)) - (g/e(g))$. Then, it follows that

$$p_H(f_1) - p_H(f_2) = (f/e(f)),\ p_{H'}(f_1) - p_{H'}(f_2) = -(g/e(g)),$$

and by the definition of $\|.\|_B$ that

$$e(p_H(f_1)) + e(p_H(f_2)) \geqq 1,\ e(p_{H'}(f_1)) + e(p_{H'}(f_2)) \geqq 1.$$

Therefore, $e(f_1) + e(f_2) \geqq 2$ and

$$\begin{aligned}\|(f/e(f)) - (g/e(g))\|_B = \inf\{e(f_1) + e(f_2) : f_1, f_2 \in K, f_1 - f_2 \\ = (f/e(f)) - (g/e(g))\} \\ \geqq 2.\end{aligned}$$

But since $\|f/e(f)\|_B = \|g/e(g)\|_B = 1$, it follows that

$$\|(f/e(f)) - (g/e(g))\|_B \leqq 2$$

and therefore $\|(f/e(f)) - (g/e(g))\|_B = 2$.

Conversely, let $g \in K, g \neq 0$ and $\|(f/e(f)) - (g/e(g))\|_B = 2, \forall f \in H$, $f \neq 0$. If $p_H(g) = 0, g \in H'$ and the result is proved and therefore suppose $p_H(g) \neq 0$. If $p_{H'}(g) = 0, g \in H$ and this is clearly contradictory. Therefore, suppose $p_{H'}(g) \neq 0$ and then,

$$\begin{aligned}2e(g)\, e(p_H(g)) &= \|e(g)\, p_H(g) - e(p_H(g))\, g\|_B \\ &= \|e(p_{H'}(g))\, p_H(g) - e(p_H(g))\, p_{H'}(g)\|_B \\ &= 2e(p_{H'}(g))\, e(p_H(g))\end{aligned}$$

from the first part of the proof. Since $e(p_H(g)) \neq 0$, it follows that $e(g) = e(p_{H'}(g)), e(p_H(g)) = 0$ giving a contradiction. This completes the proof of (i).

To prove (ii), let $f, g \in E(K), f, g \neq 0$ and let

$$\|(f/e(f)) - (g/e(g))\|_B < 2.$$

Then, by (i), $g \notin H'_f$ and hence $g = p_{H_f}(g) + p_{H'_f}(g), p_{H'_f}(g) \neq g$. But since $g \in E(K), p_{H_f}(g) = \alpha g, p_{H'_f}(g) = (1-\alpha)\, g, \alpha \neq 0$ and again using (i), $p_{H_f}(g) = g, g \in H_f, H_g = H_f, Q_g = Q_f$.

§ 4. Restricted Operations

The main concern of this section is to consider the action of operations on a particular strong restriction of a system. Most of the results are direct consequences of § 3 of [13] and many easy proofs are omitted. Throughout the section $H \in \Gamma(K)$ is the set of states of a fixed strong restriction of the system whose set of states is K. H automatically defines subsystems corresponding to the base norm spaces $(H-H, H, H \cap B)$, $(H'-H', H', H' \cap B)$ respectively. Already in Proposition 3.14 the properties of the sets $\mathcal{Q}^H, \mathcal{Q}^{H'}$ of simple observables in the restricted system and its complement have been examined. Here, this study is extended to one of classes of operations of the restricted system and its complement and their relation to corresponding classes of operations for the whole system. From a physical point of view it would be expected that there exists a set of operations on the whole system whose elements decompose into operations on the restricted system and its complement with similar properties.

Before commencing a systematic discussion of such operations some notation is established. In [13] many subsets of K, the set of states, $\mathcal{P}$ the set of operations and $\mathcal{Q}$ the set of simple observables were defined. The convention which will be adopted here is that the corresponding sets for the restricted system with set of states H, will be denoted by the same symbol with the superscript H. For example $K^H = H$, $\mathcal{P}^H$, $\mathcal{Q}^H$ are the sets of states, operations, simple observables respectively of the restricted system. For simplification, in this section p, p' are written for p_H, $p_{H'}$, respectively.

Since $V = (H-H) \oplus (H'-H')$ it follows that $j \in \mathfrak{L}(V)$ has the following decomposition

$$j = pjp + p'jp + pjp' + p'jp'$$

into elements $pjp \in \mathfrak{L}(H-H)$, $p'jp \in \mathfrak{L}(H-H, H'-H')$, $pjp' \in \mathfrak{L}(H'-H', H-H)$, $p'jp' \in \mathfrak{L}(H'-H')$, and $j \in \mathfrak{L}(V)^+$ if and only if each of its four constituents are positive. If $j \in \mathcal{P}$, it is clear that $pjp \in \mathcal{P}^H$, $p'jp' \in \mathcal{P}^{H'}$ and if the notion of strong restriction is to be physically meaningful the elements of $\mathcal{P}$ which are of interest are those for which $j(H) \subset H, j(H') \subset H'$. From above, it is clear that these conditions are satisfied if and only if $pj = jp$. Denote by $\boldsymbol{P}^H$ the set of all operations j such that $pj = jp$. The properties of $\boldsymbol{P}^H$ are summarised below.

Proposition 4.1. (i) *$\boldsymbol{P}^H$ is a uniformly closed face of $\mathcal{P}$ containing* $\mathbf{0}, \mathbf{1}_V$.

(ii) $\boldsymbol{P}^H = \{j : j \in \mathcal{P}, j(H) \subset H, j(H') \subset H'\}$.

(iii) *For* $j_1, j_2 \in \boldsymbol{P}^H, j_1 j_2 \in \boldsymbol{P}^H$.

(iv) $j \mapsto (pjp, p'jp')$ *is an affine isomorphism from* $\boldsymbol{P}^H$ *onto* $\mathcal{P}^H \times \mathcal{P}^{H'}$ *(with the linear structure inherited from* $\mathfrak{L}(H-H) \times \mathfrak{L}(H'-H')$*) which preserves multiplication.*

(v) *For* $j \in \boldsymbol{P}^H$, $T(j) = T^H(pjp) + T^{H'}(p'jp')$, $p^*(T(j)) = T^H(pjp)$, $p'^*(T(j)) = T^{H'}(p'jp')$.

(vi) *For* $j \in \boldsymbol{P}^H$, $T(j) \in E(\mathcal{Q})$ *if and only if* $T^H(pjp) \in E(\mathcal{Q}^H)$, $T^{H'}(p'jp') \in E(\mathcal{Q}^{H'})$.

(vii) *For* $j \in \boldsymbol{P}^H$, $T(j) \in CE(\mathcal{Q})$ *if and only if* $T^H(pjp) \in CE(\mathcal{Q}^H)$, $T^{H'}(p'jp') \in CE(\mathcal{Q}^{H'})$.

(viii) $j \mapsto T(j)$ *is an affine mapping from* $\boldsymbol{P}^H$ *onto* $\mathcal{Q}$.

(ix) *For* $A \in \mathcal{Q}$, $T^{-1}(A) \cap \boldsymbol{P}^H$ *is a uniformly closed convex subset of* $\boldsymbol{P}^H$ *and is a face of* $\boldsymbol{P}^H$ *if and only if* $p^*(A) \in E(\mathcal{Q}^H)$, $p'^*(A) \in E(\mathcal{Q}^{H'})$.

Proof. Let $\mathcal{P}(H) = \{j : j \in \mathcal{P}, j(H) \subset H\}$. Then, $j \in \mathcal{P}(H)$ if and only if $pjp = jp$ from which it follows that $\mathcal{P}(H)$ is convex and uniformly closed. Let $j \in \mathcal{P}(H), j = tj_1 + (1-t)j_2, j_1, j_2 \in \mathcal{P}$, $t \in (0,1), f \in H$. Then, $tj_1(f) + (1-t)j_2(f) = j(f) \in H$ and since H is a face $j_1(f), j_2(f) \in H$ and j_1, $j_2 \in \mathcal{P}(H)$ showing that $\mathcal{P}(H)$ is a face of $\mathcal{P}$. Similar remarks apply to $\mathcal{P}(H')$ and since $\boldsymbol{P}^H = \mathcal{P}(H) \cap \mathcal{P}(H')$, (i) holds. Notice that (ii) and (iii) are immediate consequences of the definition.

$j \mapsto (pjp, p'jp')$ is clearly an affine mapping from $\boldsymbol{P}^H$ into $\mathcal{P}^H \times \mathcal{P}^{H'}$ and it follows from the decomposition of any element of $\mathcal{P}$, that the mapping is an isomorphism with inverse $(j, j') \mapsto jp + j'p'$. Further, for $j_1, j_2 \in \boldsymbol{P}^H$, $pj_1j_2p = pj_1ppj_2p$, $p'j_1j_2p' = p'j_1p'p'j_2p'$ and (iv) holds.

For $j \in \boldsymbol{P}^H$, $j = pjp + p'jp'$ and therefore,

$$\begin{aligned} T(j) = j^*(e) &= p^*j^*p^*(e) + p'^*j^*p'^*(e) \\ &= (pjp)^*(p^*(e)) + (p'jp')^*(p'^*(e)) \\ &= T^H(pjp) + T^{H'}(p'jp'). \end{aligned}$$

In addition, $T(j)$ has the unique decomposition

$$T(j) = p^*(T(j)) + p'^*(T(j))$$

where $p^*(T(j)) \in \mathcal{Q}^H$, $p'^*(T(j)) \in \mathcal{Q}^{H'}$. It follows that $p^*(T(j)) = p^*j^*p^*(e) = T^H(pjp)$, $p'^*(T(j)) = T^{H'}(p'jp')$. This shows that (v) holds and (vi), (vii) follow from Proposition 3.14.

Proposition 3.2 of [13] shows that $j \mapsto T(j)$ is an affine mapping from $\mathcal{P}$ onto $\mathcal{Q}$ and it therefore only remains to show that $j \mapsto T(j)$ restricted to $\boldsymbol{P}^H$ maps onto $\mathcal{Q}$. For $A \in \mathcal{Q}$, $p^*(A) \in \mathcal{Q}^H$, $p'^*(A) \in \mathcal{Q}^{H'}$ and hence by Proposition 3.2 of [13], there exist $j \in \mathcal{P}^H$, $j' \in \mathcal{P}^{H'}$ such that $T^H(j) = p^*(A)$, $T^{H'}(j') = p'^*(A)$. Therefore, $jp + j'p' \in \boldsymbol{P}^H$ and $T(jp + j'p') = T^H(j) + T^{H'}(j') = A$ by (v) and hence (viii) holds.

(ix) is a consequence of Proposition 3.2 (iii) of [13] and (i), (vi) above.

Recall that, for $A \in \mathcal{Q}$, $K_A = \{A\}_0 \cap K$, $H_A = \{e - A\}_0 \cap K$. For $j \in \boldsymbol{P}^H$ let $\boldsymbol{T}_j^H = T^{-1}(e - T(j)) \cap \boldsymbol{P}^H = \mathcal{T}_j \cap \boldsymbol{P}^H$ be the set of operations in $\boldsymbol{P}^H$ complementary to j. (The convention that bold face letters with superscript H represent the intersection with $\boldsymbol{P}^H$ of subsets of $\mathscr{P}$ represented by the same script symbol is adopted throughout.) The next result describes a little more about the properties of simple observables.

Proposition 4.2. (i) *For* $A \in \mathscr{Q}$, $K_A = K^H_{p^*(A)} \oplus K^{H'}_{p'^*(A)}$.

(ii) *For* $j \in \boldsymbol{P}^H$, $\boldsymbol{T}^H_j$ *is a uniformly closed convex subset of* $\boldsymbol{P}^H$ *and is a face of* $\boldsymbol{P}^H$ *if and only if* $T^H(pjp) \in E(\mathscr{Q}^H)$, $T^{H'}(p'jp') \in E(\mathscr{Q}^{H'})$.

(iii) *For* $j \in \boldsymbol{P}^H, j' \in \boldsymbol{T}^H_j$,

$$T^H((pjp)(pj'p)) = T^H(pj'p) - T^H((pj'p)^2),$$
$$T^{H'}((p'j'p')(p'jp')) = T^{H'}(p'jp') - T^{H'}((p'jp')^2).$$

Proof. For $f \in K_A$, $f = p(f) + p'(f)$, $p(f) \in H$, $p'(f) \in H'$ and $0 \leqq A(p(f)) \leqq A(f) = 0$ and $p(f) \in \{p^*(A)\}_0 \cap H = K^H_{p^*(A)}$. Similarly $p'(f) \in K^{H'}_{p'^*(A)}$. Conversely, if $f \in K$, $p(f) \in K^H_{p^*(A)}$, $p'(f) \in K^{H'}_{p'^*(A)}$, $A(f) = (p^*(A))(p(f)) + (p'^*(A))(p'(f)) = 0$ and $f \in K_A$. The proof of (i) is now complete.

$\boldsymbol{T}^H_j = \mathscr{T}_j \cap \boldsymbol{P}^H$ and, by Proposition 3.3 (iv) of [13], is a uniformly closed convex subset of $\boldsymbol{P}^H$. Proposition 4.1 (ix) shows that $\boldsymbol{T}^H_j$ is a face of $\boldsymbol{P}^H$ if and only if $p^*(e-A) \in E(\mathscr{Q}^H)$, $p'^*(e-A) \in E(\mathscr{Q}^{H'})$ and hence if and only if $p^*(A) \in E(\mathscr{Q}^H)$, $p'^*(A) \in E(\mathscr{Q}^{H'})$ where $A = T(j)$. (ii) now follows as a result of Proposition 4.1 (v) and (iii) is an immediate consequence of Proposition 3.3 (v) of [13].

Recall that $\mathscr{P}_T = \mathscr{T}_0$ is the set of transmissions of the system and let $\boldsymbol{P}^H_T = \mathscr{P}_T \cap \boldsymbol{P}^H$. It follows from Proposition 4.1 (v) that for $j \in \boldsymbol{P}^H_T$,

$$e = T^H(pjp) + T^{H'}(p'jp') \leqq p^*(e) + p'^*(e) = e$$

and hence $T^H(pjp) = p^*(e)$, $T^{H'}(p'jp') = p'^*(e)$. The main result concerning transmissions is therefore the following.

Proposition 4.3. $j \in \boldsymbol{P}^H_T$ *if and only if* $pjp \in \boldsymbol{P}^H_T$, $p'jp' \in \boldsymbol{P}^{H'}_T$.

A result analogous to that of Proposition 3.4 of [13] can clearly be reproduced by examining constituents of elements of $\boldsymbol{P}^H_T$.

Recall that for $j \in \boldsymbol{P}^H$, $\mathscr{L}_j = \{j' : j' \in \mathscr{P}, T(j'j) = 0\}$, $\mathscr{R}_j = \{j' : j' \in \mathscr{P}, T(jj') = 0\}$, $\mathscr{U}_j = \mathscr{L}_j \cap \mathscr{R}_j$ and therefore, $\boldsymbol{L}^H_j = \mathscr{L}_j \cap \boldsymbol{P}^H$, $\boldsymbol{R}^H_j = \mathscr{R}_j \cap \boldsymbol{P}^H$, $\boldsymbol{U}^H_j = \mathscr{U}_j \cap \boldsymbol{P}^H$. It is clear that results similar to those of Proposition 3.5 of [13] may be applied to the present situation but they will not be reproduced here.

The set $\boldsymbol{P}^H_R$ of operations $j \in \boldsymbol{P}^H$ such that $\boldsymbol{T}^H_j \cap \boldsymbol{U}^H_j \neq \emptyset$ forms an important class of operations, namely the *repeatable* operations in $\boldsymbol{P}^H$. Notice that $j \in \boldsymbol{P}^H_R$ implies that there exists $j' \in \boldsymbol{T}^H_j$ such that $T(j) = T(j^2)$, $T(j') = T(j'^2)$ and therefore from Proposition 4.1 (v),

$$T^H(pjp) = T^H((pjp)^2),\ T^H(pj'p) = T^H((pj'p)^2),$$

$$T^{H'}(p'jp') = T^{H'}((p'jp')^2),\ T^{H'}(p'j'p') = T^{H'}((p'j'p')^2).$$

Using Proposition 3.5 (iv) of [13] it follows that $pjp \in \mathscr{P}_R^H, p'jp' \in \mathscr{P}_R^{H'}$. By reversing the above argument it follows that for $j \in \boldsymbol{P}^H$, $j \in \boldsymbol{P}_R^H$ if $pjp \in \mathscr{P}_R^H, p'jp' \in \mathscr{P}_R^{H'}$. Summarising,

Proposition 4.4. (i) *For $j \in \boldsymbol{P}^H$, $\boldsymbol{L}_j^H$, $\boldsymbol{R}_j^H$, $\boldsymbol{U}_j^H$ are uniformly closed faces of $\boldsymbol{P}^H$.*

(ii) *For $j \in \boldsymbol{P}^H, j \in \boldsymbol{P}_R^H$ if and only if $pjp \in \mathscr{P}_R^H, p'jp' \in \mathscr{P}_R^{H'}$.*

In [13] the notions of (a) and (c)-repeatability were introduced. Recall that for $j \in \mathscr{P}_R, j$ is said to be (a)-repeatable providing that there exists $j' \in \mathscr{T}_j \cap \mathscr{U}_j$ such that,

(a) For $j_1 \in \mathscr{P}$, $T(j_1 j) = T(j)$, $T(j_1 j') = 0$ implies $T(j_1) = T(j)$.

Suppose $j, j' \in \boldsymbol{P}_R^H$ satisfy (a). Then it easily follows that $pjp, p'jp'$ are (a)-repeatable operations in $\mathscr{P}^H, \mathscr{P}^{H'}$ with complementary operations $pj'p, p'j'p'$ respectively. However, the converse assertion is not necessarily true. This suggests that in the context of restricted operations the definition of (a)-repeatability is not what is required. Indeed, it is clear that only operations in $\boldsymbol{P}^H$ should be considered. Guided by such considerations an element $j \in \boldsymbol{P}_R^H$ is said to be *$(a)^H$-repeatable* if there exists $j' \in \boldsymbol{P}_R^H$ satisfying

$(a)^H$ For $j_1 \in \boldsymbol{P}^H$, $T(j_1 j) = T(j)$, $T(j_1 j') = 0$ implies $T(j_1) = T(j)$.

It is clear that the following result holds.

Proposition 4.5. *$j \in \boldsymbol{P}^H$ is $(a)^H$-repeatable with complementary operation j' if and only if $pjp, p'jp'$ are (a)-repeatable in $\mathscr{P}^H, \mathscr{P}^{H'}$ with complementary operations $pj'p, p'j'p'$ respectively.*

Notice that although $(a)^H$-repeatability is a weaker condition than (a)-repeatability for elements of $\boldsymbol{P}_R^H$ it is strong enough to ensure that $T(j) \in E(\mathscr{Q})$. This follows immediately from Proposition 4.1 (v) and Proposition 3.14 since $T^H(pjp) \in E(\mathscr{Q}^H)$, $T^{H'}(p'jp') \in E(\mathscr{Q}^H)$.

Proposition 4.6. *If $j \in \boldsymbol{P}^H$ is $(a)^H$-repeatable, $T(j) \in E(\mathscr{Q})$.*

The position regarding (c)-repeatability is much more simple. Recall that $j \in \mathscr{P}_R$ is said to be (c)-repeatable providing that there exists $j' \in \mathscr{T}_j \cap \mathscr{U}_j$ such that,

(c) For $A \in \mathscr{Q}$, $A(j(f)) = A(j'(f)) = 0$, $\forall f \in K$ implies $A = 0$.

It follows that there exists a natural definition for (c)-repeatability of elements $j \in \boldsymbol{P}_R^H$. Such an element is said to be *(c)-repeatable* if there exists $j' \in \boldsymbol{T}_j^H \cap \boldsymbol{U}_j^H$ such that (c) above holds. A simple argument then shows that the following result holds.

Proposition 4.7. *$j \in \boldsymbol{P}^H$ is (c)-repeatable with complementary operation j' if and only if $pjp, p'jp'$ are (c)-repeatable in $\mathscr{P}^H, \mathscr{P}^{H'}$ with complementary operations $pj'p, p'j'p'$ respectively.*

Recall that, for $j \in \boldsymbol{P}^H$, $G_j = \{f : f \in K, j(f) = f\}$ is the set of unchanged states of the system and that $\boldsymbol{P}_F^H = \{j : j \in \boldsymbol{P}^H, G_j = H_{T(j)}\}$ is the set of filterings in $\boldsymbol{P}^H$. The result below whose proof is trivial summarises the properties of G_j, $\boldsymbol{P}_F^H$.

Proposition 4.8. (i) *For* $j \in \boldsymbol{P}^H$, $G_j = G^H_{pjp} \oplus G^{H'}_{p'jp'}$.
(ii) *For* $j \in \boldsymbol{P}^H, j \in \boldsymbol{P}_F^H$ *if and only if* $pjp \in \mathscr{P}_F^H$, $p'jp' \in \mathscr{P}_F^{H'}$.

An $(a)^H$ or (c)-repeatable operation $j \in \boldsymbol{P}^H$ with complementary operation j' is said to be *strongly* $(a)^H$ *or* (c)*-repeatable* respectively provided that $j, j' \in \boldsymbol{P}_F^H$. Let $\boldsymbol{P}^H_{SRa}$, $\boldsymbol{P}^H_{SRc}$ denote the sets of strongly $(a)^H$ and (c)-repeatable operations in $\boldsymbol{P}^H$ respectively. Then, combining Propositions 4.5, 4.7 and 4.8 (iii) with Proposition 3.9 (iv) of [13] gives the following.

Proposition 4.9. *For* $j \in \boldsymbol{P}^H, j \in \boldsymbol{P}^H_{SRa}$ *(or* $\boldsymbol{P}^H_{SRc}$*) if and only if* $pjp \in \mathscr{P}^H_{SRa}$ *(or* $\mathscr{P}^H_{SRc}$*),* $p'jp' \in \mathscr{P}^{H'}_{SRa}$ *(or* $\mathscr{P}^{H'}_{SRc}$*).*

Recall that the set $\mathscr{P}_W$ of reflections is the order unit interval in $\mathfrak{Z}(V)$ and clearly $j \in \mathscr{P}_W \cap \boldsymbol{P}^H = \boldsymbol{P}^H_W$ if and only if $0 \leqq j \leqq \mathbf{1}_V$, $pj = jp$. However, since $\mathfrak{Z}(V)$ is commutative this latter condition is redundant and hence $\mathscr{P}_W = \boldsymbol{P}^H_W$. Further, for $j \in \mathscr{P}_W$, it is clear that $pjp \in \mathscr{P}^H_W$, $p'jp' \in \mathscr{P}^{H'}_W$. Summarising,

Proposition 4.10. (i) $\mathscr{P}_W \subset \boldsymbol{P}^H$.
(ii) *For* $j \in \mathscr{P}$, $j \in \mathscr{P}_W$ *if and only if* $pjp \in \mathscr{P}^H_W$, $p'jp' \in \mathscr{P}^{H'}_W$.

The final class of operations which could be examined in the present context is the set $\mathscr{P}_{SRW}$ of strongly repeatable reflections. However, as was seen in § 3, such an examination is equivalent to one of $\Gamma(K)$ and Proposition 3.13 (ii) describes the situation arising in this case.

§ 5. Strong Operations

This section is concerned with the set of operations which preserve all strong restrictions of the system. This set $\boldsymbol{P} = \bigcap \{\boldsymbol{P}^H : H \in \Gamma(K)\}$ is said to be the set of *strong operations* on the system and, being the intersection of uniformly closed faces of $\mathscr{P}$ is itself a uniformly closed face. It follows from Proposition 4.10 (i) that $\mathscr{P}_W \subset \boldsymbol{P}$ and in particular $\mathscr{P}_{SRW} \subset \boldsymbol{P}$. The question arises if it is possible for the trivial case $\boldsymbol{P} = \{\lambda \mathbf{1}_V : 0 \leqq \lambda \leqq 1\}$ to arise. If this is the case it follows that $\mathscr{P}_{SRW} = \{0, \mathbf{1}_V\}$ or what is equivalent $\Gamma(K) = \{\{0\}, K\}$. But then it follows that $\mathscr{P} = \boldsymbol{P}$ and therefore V is of dimension unity. Therefore provided that V is of dimension greater than unity the trivial case does not arise.

The question of whether $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathscr{Q}$ is clearly of some importance since, if this is not the case, then, simple observables exist which cannot be measured by means of strong operations. Initially it will be assumed that $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$ and an example will be considered at the end of the section when this certainly is the case.

The first result summarises the properties of $\boldsymbol{P}$ which arise naturally from the corresponding results for $\boldsymbol{P}^H$.

Proposition 5.1. (i) *$\boldsymbol{P}$ is a uniformly closed face of $\mathcal{P}$ and $\boldsymbol{P} = \{\lambda 1_V : 0 \leqq \lambda \leqq 1\}$ if and only if V is of dimension unity.*

Suppose that $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$. Then,

(ii) *For $j \in \boldsymbol{P}$, $T(j) = T^H(p_H j p_H) + T^{H'}(p_{H'} j p_{H'})$, $\forall H \in \Gamma(K)$.*

(iii) *For $j \in \boldsymbol{P}$, $T(j) \in E(\mathcal{Q})$ if and only if $T^H(p_H j p_H) \in E(\mathcal{Q}^H)$, $\forall H \in \Gamma(K)$.*

(iv) *For $j \in \boldsymbol{P}$, $T(j) \in CE(\mathcal{Q})$ if and only if $T^H(p_H j p_H) \in CE(\mathcal{Q}^H)$, $\forall H \in \Gamma(K)$.*

(v) *For $A \in \mathcal{Q}$, $T^{-1}(A) \cap \boldsymbol{P}$ is a uniformly closed convex subset of $\boldsymbol{P}$ and is a face of $\boldsymbol{P}$ if and only if $p_H^*(A) \in E(\mathcal{Q}^H)$, $\forall H \in \Gamma(K)$.*

In what follows it will be supposed that the system satisfies the condition that $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$. Therefore, for $j \in \boldsymbol{P}$, $\boldsymbol{T}_j = \bigcap \{\boldsymbol{T}_j^H : H \in \Gamma(K)\}$ is non-empty. Let $\boldsymbol{L}_j = \bigcap \{\boldsymbol{L}_j^H : H \in \Gamma(K)\}$. $\boldsymbol{R}_j = \bigcap \{\boldsymbol{R}_j^H : H \in \Gamma(K)\}$, $\boldsymbol{U}_j = \boldsymbol{L}_j \cap \boldsymbol{R}_j$. Then $j \subset \boldsymbol{P}$ is said to be *repeatable* if and only if $\boldsymbol{U}_j \cap \boldsymbol{T}_j \neq \emptyset$ and $\boldsymbol{P}_R$ will denote the set of repeatable elements of $\boldsymbol{P}$. The following result is a consequence of Proposition 4.4.

Proposition 5.2. *If $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$ and $j \in \boldsymbol{P}$,*

(i) *$\boldsymbol{L}_j$, $\boldsymbol{R}_j$, $\boldsymbol{U}_j$ are uniformly closed faces of $\boldsymbol{P}$.*

(ii) *$j \in \boldsymbol{P}_R$ if and only if $p_H j p_H \in \boldsymbol{P}_R^H$, $\forall H \in \Gamma(K)$.*

Under the conditions of Proposition 5.2, $j \in \boldsymbol{P}_R$ is said to be (a)-*repeatable* if there exists $j' \in \boldsymbol{U}_j \cap \boldsymbol{T}_j$ such that,

(a) For $j_1 \in \boldsymbol{P}$, $T(j_1 j) = T(j)$, $T(j_1 j') = 0$ implies $T(j_1) = T(j)$.

Similarly j is said to be (c)-*repeatable* if there exists $j' \in \boldsymbol{U}_j \cap \boldsymbol{T}_j$ such that,

(c) For $A \in \mathcal{Q}$, $A(j(f)) = A(j'(f)) = 0$, $\forall f \in K$ implies $A = 0$.

Then, the following result is immediate from Propositions 4.5–4.7.

Proposition 5.3. (i) *If $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$ then, $j \in \boldsymbol{P}$ is (a) (or (c))-repeatable with complementary operation j' if and only if for each $H \in \Gamma(K)$, $p_H j p_H$ is (a) (or (c))-repeatable in $\boldsymbol{P}^H$ with complementary operation $p_H j' p_H$.*

(ii) *If $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$ and if $j \in \boldsymbol{P}$ is (a)-repeatable, $T(j) \in E(\mathcal{Q})$.*

Let $\boldsymbol{P}_F = \bigcap \{\boldsymbol{P}_F^H : H \in \Gamma(K)\}$. Then, it clearly follows from Proposition 4.8 (ii) that $j \in \boldsymbol{P}_F$ if and only if $p_H j p_H \in \boldsymbol{P}_F^H$, $\forall H \in \Gamma(K)$. Let $\boldsymbol{P}_{SRa}$ (or $\boldsymbol{P}_{SRc}$) denote the set of (a) (or (c))-repeatable operations $j \in \boldsymbol{P}$ such that both j and its complementary operation j' lie in $\boldsymbol{P}_F$. Then, it is immediate from Proposition 4.9 that the following result holds.

Proposition 5.4. *For $j \in \boldsymbol{P}$, $j \in \boldsymbol{P}_{SRa}$ (or $\boldsymbol{P}_{SRc}$) with complementary operation j' if and only if for each $H \in \Gamma(K)$, $p_H j p_H \in \mathcal{P}_{SRa}$ (or $\mathcal{P}_{SRc}$) with complementary operation $p_H j' p_H$.*

The final result is immediate from Proposition 4.10.

Proposition 5.5. $\mathscr{P}_W \subset \boldsymbol{P}$.

The Boolean algebra $\Gamma(K)$ is said to be atomic provided that there exists a family $\{H_i : i \in \Lambda\} \subset \Gamma(K)$ such that for $i \neq i'$, $H_i \wedge H_{i'} = \{0\}$, $K = \bigvee_{i \in \Lambda} H_i$ and every $H \in \Gamma(K)$ is of the form $\bigvee_{i \in \Lambda'} H_i, \Lambda' \subset \Lambda$. Notice that it follows immediately that $P\Gamma(K) = \{H_i : i \in \Lambda\}$ and this is clearly a general result in Boolean algebra theory. The following result stems from the particular properties of $\Gamma(K)$.

Proposition 5.6. *$\Gamma(K)$ is atomic if and only if $K = \bigvee \{H : H \in P\Gamma(K)\}$.*

Proof. If $\Gamma(K)$ is atomic then it follows from the remark above that $K = \bigvee \{H : H \in P\Gamma(K)\}$. Conversely, suppose that $K = \bigvee \{H : H \in P\Gamma(K)\}$ and then it remains to prove that for $G \in \Gamma(K)$, $G = \bigvee_{i \in \Lambda} H_i, H_i \in P\Gamma(K)$. Let $G_1 = \bigvee \{H : H \in P\Gamma(K), H \subset G\}$. Then, it follows that $G_1 \subset G$. However, for $f \in G$, there exists a Cauchy net $\{f_\Lambda\}$ where $\{H_i : i \in \Lambda\}$ ranges over all finite subsets of $P\Gamma(K)$ such that $f_\Lambda = \sum_{i \in \Lambda} g_i, g_i \in H_i$ and $\{f_\Lambda\}$ has the limit f in the norm topology. Since p_G is continuous for the norm topology and $p_G(f) = f$ it is clear that $\{f_\Lambda\}$ can be supposed to be such that $p_G(f_\Lambda) = f_\Lambda, f_\Lambda \in G$ and hence $g_i \in G, \forall i \in \Lambda$. It follows that $f_\Lambda \in G_1$ and since G_1 is norm closed $f \in G_1$, $G \subset G_1$ and the proof is complete.

The importance of Proposition 5.6 lies in the following results.

Proposition 5.7. *Suppose that $\Gamma(K)$ is atomic. Then, each $j \in \boldsymbol{P}$ is the limit in the strong operator topology of the Cauchy net whose elements are of the form $\sum_{i \in \Lambda} p_{H_i} j p_{H_i}$ where $\{H_i : i \in \Lambda\}$ ranges over all finite subsets of $P\Gamma(K)$. If, for each $H \in P\Gamma(K), j_H \in \mathscr{P}^H$, there exists uniquely $j \in \boldsymbol{P}$ such that $p_H j p_H = j_H$.*

Proof. Let $j \in \boldsymbol{P}, f \in K$ and then it easily follows since the norm is additive on K that the set of elements of the form $f_\Lambda = \sum_{i \in \Lambda} (p_{H_i} j p_{H_i})(f)$ where $\{H_i : i \in \Lambda\}$ ranges over all finite subsets of $P\Gamma(K)$ is a Cauchy net with limit $j_1(f)$ in the norm topology. Simple limit arguments show that $j_1 \in \mathscr{P}$. Moreover, for $H \in P\Gamma(K)$,

$$(j_1 p_H)(f) = \lim \sum_{i \in \Lambda} (p_{H_i} j p_{H_i} p_H)(f) = (j p_H)(f)$$

from which it follows that $j_1 \in \boldsymbol{P}$. For $f \in K$, the set of elements of the form $g_\Lambda = \sum_{i \in \Lambda} p_{H_i}(f)$ forms a Cauchy net with limit f in the norm topology and clearly from above $j_1(g_\Lambda) = j(g_\Lambda)$. The continuity of j and j_1 for the norm topology then imply that $j_1 = j$.

Suppose that for each $H \in P\Gamma(K), j_H \in \mathscr{P}^H$. Then, for $f \in K$, the set of elements of the form $f_\Lambda = \sum_{i \in \Lambda} (j_{H_i} p_{H_i})(f)$ where $\{H_i : i \in \Lambda\}$ ranges over all finite subsets of $P\Gamma(K)$ is a Cauchy net with limit $j(f)$ in the norm topology. Simple limit arguments show that $j \in \mathscr{P}$ and clearly $p_H j p_H = j p_H = j_H p_H, \forall H \in P\Gamma(K)$. It follows that $j \in \boldsymbol{P}$. To prove uniqueness, notice that if $j_1, j_2 \in \boldsymbol{P}, p_H j_1 p_H = p_H j_2 p_H, \forall H \in P\Gamma(K)$, it follows from the first part that $j_1 = j_2$.

The next result is a corollary of Proposition 5.7 and shows that the assumption that was made earlier that $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathscr{Q}$ is true in the atomic case.

Proposition 5.8. *If $\Gamma(K)$ is atomic $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathscr{Q}$.*

Proof. Let $A \in \mathscr{Q}$, and for each $H \in P\Gamma(K)$, let $j_H \in \mathscr{P}^H$ satisfy $T^H(j_H) = p_{H^*}(A)$. Such elements exist by Proposition 3.2 of [13]. From Proposition 5.7 it follows that there exists $j \in \boldsymbol{P}$ such that j is the limit in the strong operator topology of the Cauchy net whose elements are of the form $\sum_{i \in \Lambda} j_{H_i} p_{H_i}$ as $\{H_i : i \in \Lambda\}$ ranges over finite subsets of $P\Gamma(K)$. Therefore,

$$\begin{aligned} T(j)(f) = e(j(f)) &= \lim \sum_{i \in \Lambda} e(j_{H_i} p_{H_i}(f)) \\ &= \lim \sum_{i \in \Lambda} (p_{H_i^*}(A))(p_{H_i}(f)) \\ &= \lim \sum_{i \in \Lambda} A(p_{H_i}(f)) \\ &= A(f) \end{aligned}$$

and $T(j) = A$. This completes the proof.

The final result of this section whose proof follows from Proposition 3.18 shows that the results above hold in one important special case.

Proposition 5.9. *If $K_I = K$, then $\Gamma(K)$ is atomic.*

Since for $H \in P\Gamma(K)$, the set of strong operations on the system with set of states H is $\mathscr{P}^H$, Proposition 5.7 shows how in the atomic case elements of $\boldsymbol{P}$ may be decomposed into operations on non-decomposable systems. Proposition 5.8 shows that under the same conditions all simple observables may be measured by strong operations.

§ 6. Pure Operations

It is mainly when a consideration of pure states of the system is required that a study of the situation in the case $\Gamma(K)$ atomic is of importance. This can easily be seen from Proposition 5.9.

If K represents the set of states of any physical system an element $j \in \boldsymbol{P}$ is said to be a *pure operation* when $j(E(K)) \subset E(K)$. The set of pure operations will be denoted by $\boldsymbol{P}_P$. Notice that the condition $j(E(K)) \subset E(K)$ produces no extra condition on elements of K'_I and therefore, in discussing pure operations there is no loss of generality in supposing that $K = K_I$. This condition will be assumed for the whole of this section. Notice that it follows from Proposition 5.8 that $j \mapsto T(j)$ maps $\boldsymbol{P}$ onto $\mathcal{Q}$.

Elements of $\boldsymbol{P}_P$ possess certain trivial properties among them being that for each $Q \subset \hat{K}, j \in \boldsymbol{P}_P, j(Q) \subset Q$. However the most important property of pure operations stems from Proposition 5.9.

Proposition 6.1. *For $j \in \boldsymbol{P}$, $j \in \boldsymbol{P}_P$ if and only if for each $H \in P\Gamma(K) = P_I\Gamma(K)$, $p_H j p_H \in \boldsymbol{P}_P^H$ the set of pure operations on the system with set of states H.*

Proof. It follows from Proposition 3.18 that, for $H \in P\Gamma(K)$, $E(H) = Q = H \cap E(K)$ and since $j(H) \subset H, j(E(K)) \subset E(K)$ it follows that $p_H j p_H \in \boldsymbol{P}_P^H$. Conversely, let $j \in \boldsymbol{P}$ satisfy $j(E(H)) \subset E(H)$, $\forall H \in P\Gamma(K)$. Each $f \in E(K)$ lies in a unique sector $Q = H \cap E(K)$ for some $H \in P\Gamma(K)$ and hence $j(f) \in E(H) \subset E(K)$. This completes the proof.

The main consequence of this result is that in order to make a complete study of pure operations it suffices to examine the case in which $K_I = K, \Gamma(K) = \{\{0\}, K\}$. This fact will be used when the algebraic models are studied below.

§ 7. The Von Neumann Algebra Model

For details of the theory of C^*-algebras and Von Neumann algebras the reader is referred to Dixmier [9, 10], Effros [15], and Prosser [31].

Let $\mathfrak{B}$ be a Von Neumann algebra acting on a Hilbert space X and let 1_X, the identity operator on X be also the identity in $\mathfrak{B}$. Let $\mathfrak{B}_*$ be the pre-dual of $\mathfrak{B}$, $\mathfrak{B}_*^h$ the self-adjoint part of $\mathfrak{B}_*$ and $\mathfrak{B}_*^+$ the positive part of $\mathfrak{B}_*^h$. If $\mathfrak{X} = \{f : f \in \mathfrak{B}_*^+, 1_X(f) = 1\}$, $(\mathfrak{B}_*^h, \mathfrak{B}_*^+, \mathfrak{X})$ is a complete base norm space with norm closed cone $\mathfrak{B}_*^+$ and dual $(\mathfrak{B}^h, \mathfrak{B}^+, 1_X)$ where $\mathfrak{B}^h$, $\mathfrak{B}^+$ are the self-adjoint and positive parts of $\mathfrak{B}$ respectively. The order unit norm in $\mathfrak{B}^h$ coincides with the operator norm and the base norm in $\mathfrak{B}_*^h$ coincides with its norm as a subspace of $\mathfrak{B}_*$. Therefore every Von Neumann algebra provides a possible model for a physical system in which the set of states is represented by $\mathfrak{B}_*^+$, the set $\mathcal{Q}$ of simple observables is represented by elements A of $\mathfrak{B}$ such that $0 \leqq A \leqq 1_X$ and the set $E(\mathcal{Q})$ of extreme simple observables is represented by the set of extreme points of $[0, 1_X]$ which is the set of projections in $\mathfrak{B}$. Therefore, in this case $E(\mathcal{Q})$ forms a lattice. There exists a bijection $H \mapsto E_H$ from the set

$\Pi'(\mathfrak{B}_*^+)$ of norm closed faces of $\mathfrak{B}_*^+$ onto $E(\mathcal{Q})$ defined by the property that E_H is the smallest element of $E(\mathcal{Q})$ such that $E_H f E_H = f, \forall f \in H$. The bijection sends the complete Boolean algebra $\Gamma(\mathfrak{B}_*^+)$ onto the complete Boolean algebra of central projections in $\mathfrak{B}$ and in this case $E_H = p_{H^*}(1_X)$ from which it follows that $CE(\mathcal{Q})$ is the set of central projections in $\mathfrak{B}$. Therefore, $C(\mathfrak{B}^h)$ is the self-adjoint part of the centre of $\mathfrak{B}$.

Proposition 7.1. *In the Von Neumann algebra model defined by* $\mathfrak{B}$, *the complete Boolean algebras* $\Gamma(\mathfrak{B}_*^+)$, $\Gamma(\mathfrak{B}^+)$, $\mathscr{P}_{SRW}$ *and* $\mathscr{P}_{SRW}^*$ *are all isomorphic to the complete Boolean algebra* $CE(\mathcal{Q})$ *of central projections in* $\mathfrak{B}$. $C(\mathfrak{B}^h)$ *is the self-adjoint part of the centre of* $\mathfrak{B}$ *and* $(\mathfrak{B}^h, \mathfrak{B}^+, 1_X)$ *is a factor if and only if* $\mathfrak{B}$ *is a factor.*

$P\Gamma(\mathfrak{B}_*^+)$ is sent into the family of minimal projections in the centre of $\mathfrak{B}$ under the isomorphism $H \mapsto E_H$ and it follows that $H \in P\Gamma(\mathfrak{B}_*^+)$ if and only if $E_H \mathfrak{B} E_H$ is a factor. Notice that for $f \in \mathfrak{B}_*^+$, E_{H_f} is the smallest projection in the centre of $\mathfrak{B}$ such that $E_{H_f} f E_{H_f} = f$ and hence E_{H_f} is the central support of f. It follows that for $f, g \in \mathfrak{B}_*^+$, f, g are quasi-equivalent if and only if they possess the same central support. Further f is primary if and only if $E_{H_f} \mathfrak{B} E_{H_f}$ is a factor. Let $H \in P_I\Gamma(\mathfrak{B}_*^+)$ and let $f \in H \cap E(\mathfrak{B}_*^+)$, $f \neq 0$. The smallest element of $\Pi(\mathfrak{B}_*^+)$ containing f is $L_f = \{\lambda f : \lambda \geqq 0\}$ and hence E_{L_f} is a minimal projection in the factor $E_H \mathfrak{B} E_H$ which is therefore of Type I. Conversely, if $H \in P\Gamma(\mathfrak{B}_*^+)$ is such that $E_H \mathfrak{B} E_H$ is of Type I, since $E_H \mathfrak{B} E_H$ is isomorphic to $\mathfrak{L}(Y)$ for some Hilbert space Y, it follows that $E(H) = E((E_H \mathfrak{B} E_H)_*^+) \neq \{0\}$. Therefore $H \in P_I\Gamma(\mathfrak{B}_*^+)$.

$E_{K_I} = \bigvee \{E_H : H \in P_I\Gamma(\mathfrak{B}_*^+)\} = \sum_{H \in P_I\Gamma(\mathfrak{B}_*^+)} E_H$ since $\{E_H : H \in P_I\Gamma(\mathfrak{B}_*^+)\}$ is a family of mutually orthogonal projections. It follows that $E_{K_I} \mathfrak{B} E_{K_I} = \prod_{H \in P_I\Gamma(\mathfrak{B}_*^+)} E_H \mathfrak{B} E_H$.

A Von Neumann algebra $\mathfrak{B}$ is said to be atomic if and only if the lattice of projections in $\mathfrak{B}$ is atomic. It follows that $\Gamma(\mathfrak{B}_*^+)$ is atomic if and only if the centre of $\mathfrak{B}$ is atomic. Suppose this is the case and in addition $\mathfrak{B}_*^+ = K_I$. Then, $P\Gamma(\mathfrak{B}_*^+) = P_I\Gamma(\mathfrak{B}_*^+)$ and for each $H \in P\Gamma(\mathfrak{B}_*^+)$, $E_H \mathfrak{B} E_H$ is a Type I factor. It follows from above that $\mathfrak{B}$ being the product of Type I factors is a Von Neumann algebra of Type I. Conversely, if $\mathfrak{B}$ is a Von Neumann algebra of Type I with atomic centre it follows that for each $H \in P\Gamma(\mathfrak{B}_*^+)$, $E_H \mathfrak{B} E_H$ is of Type I and hence $H \in P_I\Gamma(\mathfrak{B}_*^+)$, $P_I\Gamma(\mathfrak{B}_*^+) = P\Gamma(\mathfrak{B}_*^+)$, $\mathfrak{B}_*^+ = K_I$.

Proposition 7.2. *In the Von Neumann algebra model defined by* $\mathfrak{B}$,

(i) $H \in P\Gamma(\mathfrak{B}_*^+)$ *if and only if* $E_H \mathfrak{B} E_H$ *is a factor.*

(ii) $H \in P_I\Gamma(\mathfrak{B}_*^+)$ *if and only if* $E_H \mathfrak{B} E_H$ *is a Type I factor.*

(iii) $E_{K_I} = \sum_{H \in P_I\Gamma(\mathfrak{B}_*^+)} E_H$.

(iv) $E_{K_I}\mathfrak{B}E_{K_I}=\prod_{H\in P_I\Gamma(\mathfrak{B}_*^+)} E_H\mathfrak{B}E_H$.

(v) *If the centre of* $\mathfrak{B}$ *is atomic,* $K_I=\mathfrak{B}_*^+$ *if and only if* $\mathfrak{B}$ *is of Type I.*

For $A\in\mathcal{Q}$, $K_A=\{A\}_0\cap\mathfrak{B}_*^+$ and using Proposition 4.2 and Proposition 4.2 of [13], $K^H_{pH*(A)}=E_HE_{K_A}\mathfrak{B}_*^+E_{K_A}E_H$, or using the notation of [13], $P_{pH*(A)}=E_HP_A$, $Q_{pH*(A)}=E_HQ_A$.

For $H\in\Gamma(\mathfrak{B}_*^+)$ and for $j\in\mathscr{P}$, let E^H_j be the smallest element of $E(\mathcal{Q})$ such that $E^H_j f E^H_j=f$, $\forall f\in j(H)$. It is clear that $j\in \boldsymbol{P}^H$ if and only if $E^H_j\leqq E_H$, $E^{H'}_j\leqq E_{H'}$.

The remarks above and Propositions 4.3, 4.4 of [13] lead immediately to the following result.

Proposition 7.3. (i) *For* $j\in\mathscr{P}$, $j\in\boldsymbol{P}^H$ *if and only if* $E^H_j\leqq E_H$, $E^{H'}_j\leqq E_{H'}$.

(ii) $j\in\boldsymbol{P}^H$ *is repeatable with complementary repeatable operation* $j'\in\boldsymbol{P}^H$ *if and only if*

$$E^H_j\leqq E_HP_{T(j)},\ E^H_{j'}\leqq E_HP_{T(j')},\ E^{H'}_j\leqq E_{H'}P_{T(j)},\ E^{H'}_{j'}\leqq E_{H'}P_{T(j')}\,.$$

(iii) *The following conditions on* $j\in\boldsymbol{P}^H_R$ *and its complementary operation* j' *are equivalent,*

(a) j *is* $(a)^H$*-repeatable,*

(b) $E^H_j+E^H_{j'}=E_H$, $E^{H'}_j+E^{H'}_{j'}=E_{H'}$,

(c) j *is* (c)*-repeatable,*

and if any one of these conditions holds, $E^H_j=E_HT(j)$, $E^H_{j'}=E_HT(j')$, $E^{H'}_j=E_{H'}T(j)$, $E^{H'}_{j'}=E_{H'}T(j')$.

For $j\in\mathscr{P}$, $H\in\Gamma(\mathfrak{B}_*^+)$, let F^H_j be the smallest element of $E(\mathcal{Q})$ such that $j(F^H_jfF^H_j)=F^H_jfF^H_j$, $\forall f\in H$. For $j\in\boldsymbol{P}^H$, $j\in\boldsymbol{P}^H_F$ if and only if $F^H_j=E_HP_{T(j)}$, $F^{H'}_j=E_{H'}P_{T(j')}$. The following result is a consequence of Proposition 4.5 of [13].

Proposition 7.4. *Let* $j\in\boldsymbol{P}^H_R$ *have complementary operation* $j'\in\boldsymbol{P}^H_R$. *Then* $j\in\boldsymbol{P}^H_{SRa}=\boldsymbol{P}^H_{SRc}$ *if and only if* $F^H_j+F^H_{j'}=E_H$, $F^{H'}_j+F^{H'}_{j'}=E_{H'}$.

For $A\in\mathcal{Q}$, let j be defined by $j(f)=A^{\frac12}fA^{\frac12}$, $\forall f\in\mathfrak{B}_*^+$. Then $T(j)=A$ and since $\Gamma(\mathfrak{B}_*^+)$ is precisely the set of norm closed invariant faces of $\mathfrak{B}_*^+$, it is clear that $j\in\boldsymbol{P}$. Therefore, the condition of Proposition 5.1 are satisfied in the Von Neumann algebra model. The following two results are consequences of § 5 and Propositions 7.3, 7.4.

Proposition 7.5. (i) *For* $j\in\mathscr{P}$, $j\in\boldsymbol{P}$ *if and only if* $E^H_j\leqq E_H$, $\forall H\in\Gamma(\mathfrak{B}_*^+)$.

(ii) $j\in\boldsymbol{P}$ *is repeatable with complementary repeatable operation* $j'\in\boldsymbol{P}$ *if and only if* $E^H_j\leqq E_HP_{T(j)}$, $E^H_{j'}\leqq E_HP_{T(j')}$, $\forall H\in\Gamma(\mathfrak{B}_*^+)$.

(iii) *The following conditions on* $j\in\boldsymbol{P}_R$ *with complementary operation* $j'\in\boldsymbol{P}_R$ *are equivalent,*

(a) j *is* (a)*-repeatable,*

(b) $E^H_j+E^H_{j'}=E_H$, $\forall H\in\Gamma(\mathfrak{B}_*^+)$,

(c) j *is* (c)*-repeatable,*

and if any one of these conditions holds, $E_j^H = E_H T(j)$, $E_{j'}^H = E_H T(j')$, $\forall H \in \Gamma(\mathfrak{B}_*^+)$.

Proposition 7.6. *Let* $j \in \boldsymbol{P}_R$ *have complementary operation* $j' \in \boldsymbol{P}_R$. *Then* $j \in \boldsymbol{P}_{SRa} = \boldsymbol{P}_{SRc}$ *if and only if* $F_j^H + F_{j'}^H = E_H, \forall H \in \Gamma(\mathfrak{B}_*^+)$.

The results of § 6 show that in order to study pure operations it suffices to examine the case in which $\mathfrak{B}$ is a Type I factor and therefore isomorphic to $\mathfrak{L}(Y)$ for some Hilbert space Y.

Lemma 7.7. *Let* $\phi : \mathfrak{B}_1 \to \mathfrak{B}_2$ *be an isomorphism of the Von Neumann algebras* $\mathfrak{B}_1, \mathfrak{B}_2$ *and let* ϕ_* *denote the isometry from* $\mathfrak{B}_{2*}$ *onto* $\mathfrak{B}_{1*}$ *whose adjoint is* ϕ. *If* $\boldsymbol{P}_P^1$, $\boldsymbol{P}_P^2$ *are the sets of pure operations in the models defined by* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *respectively, and for* $j \in \boldsymbol{P}_P^1$, $\phi(j) = \phi_*^{-1} j \phi_*$, *then* $j \mapsto \phi(j)$ *is a bijection from* $\boldsymbol{P}_P^1$ *onto* $\boldsymbol{P}_P^2$.

Proof. Notice that every isomorphism of Von Neumann algebras is normal and hence ϕ_* is well-defined. Further, both ϕ_* and ϕ_*^{-1} are positive and isometric from which it follows that for $j \in \boldsymbol{P}_P^1$, $\phi(j)$ is an element of $\mathscr{P}^2$, the set of operations in the model corresponding to $\mathfrak{B}_2$. For $p \in \mathscr{P}_{SRW}^1$, $\phi(p)^2 = \phi(p)$ and for $f \in \mathfrak{B}_{2*}^+$, $0 \leqq \phi(p)(f) = \phi_*^{-1} p \phi_*(f) \leqq \phi_*^{-1} \phi_*(f) = f$ from which it follows that $\phi(p) \in \mathscr{P}_{SRW}^2$. Similarly, for $p \in \mathscr{P}_{SRW}^2$, $\phi^{-1}(p) \in \mathscr{P}_{SRW}^1$ and hence $p \mapsto \phi(p)$ is a bijection from $\mathscr{P}_{SRW}^1$ onto $\mathscr{P}_{SRW}^2$. $j \in \boldsymbol{P}^1$ if and only if $jp = pj$, $\forall p \in \mathscr{P}_{SRW}^1$ and therefore, if and only if $\phi(j) \in \boldsymbol{P}^2$. Therefore, for $j \in \boldsymbol{P}_P^1$, $\phi(j) \in \boldsymbol{P}^2$. Suppose $f \in E(\mathfrak{B}_{2*}^+)$ and then since ϕ_*, ϕ_*^{-1} are isometric affine mappings from $\mathfrak{B}_{2*}^+$ onto $\mathfrak{B}_{1*}^+$ and from $\mathfrak{B}_{1*}^+$ onto $\mathfrak{B}_{2*}^+$ respectively, $\phi(j)(f) = \phi_*^{-1} j \phi_*(f)$ which lies in $E(\mathfrak{B}_{2*}^+)$. It follows that $\phi(j) \in \boldsymbol{P}_P^2$. A similar argument shows that for $j \in \boldsymbol{P}_P^2$, $\phi^{-1}(j) \in \boldsymbol{P}_P^1$ and completes the proof of the Lemma.

For a Hilbert space Y, let Y^c denote the conjugate space and let $c : Y \to Y^c$ denote the conjugate mapping (see [10], p. 9). The following lemma follows from Lemma 5.4 of [34] and Theorem 3.1 of [6].

Lemma 7.8. *Let* $\mathfrak{B} = \mathfrak{L}(Y)$ *for some Hilbert space* Y *and let* $j \in \boldsymbol{P}_P$. *Then* j *has one of the following three forms.*

(i) $j(f) = U^* f U, \forall f \in \mathfrak{B}_*^+, U \in \mathfrak{B}, U^* U = j^*(1_Y) = T(j)$,

(ii) $j(f) = c^* U^* f U c, \forall f \in \mathfrak{B}_*^+, U \in \mathfrak{B}, U^* U = j^*(1_Y) = T(j)$,

(iii) $j(f) = f(j^*(1_Y))\, \omega_y$ *where* ω_y *is the pure vector state of* $\mathfrak{B}$ *defined by* $y \in Y$, $\|y\| = 1$.

Proposition 7.9. *Let* $\mathfrak{B}$ *be a Type I factor acting on the Hilbert space* X *and let* $j \in \boldsymbol{P}_P$. *Then* j *has one of the following three forms.*

(i) $j(f) = V^* f V, \forall f \in \mathfrak{B}_*^+, V \in \mathfrak{B}, V^* V = j^*(1_X) = T(j)$,

(ii) $j(f) = c^* V^* f V c, \forall f \in \mathfrak{B}_*^+, V \in \mathfrak{B}, V^* V = j^*(1_X) = T(j)$,

(iii) $j(f) = f(j^*(1_X))\, \omega_x, \forall f \in \mathfrak{B}_*^+, x \in X, \|x\| = 1$, *and* ω_x *is a pure state of* $\mathfrak{B}$.

Proof. This is merely a combination of Lemmas 7.7, 7.8.

The final result completely characterises all elements of $\boldsymbol{P}_P$ when $\mathfrak{B}$ is a Von Neumann algebra of Type I with atomic centre.

Proposition 7.10. *Let $\mathfrak{B}$ be a Type I Von Neumann algebra acting on the Hilbert space X and having atomic centre $\mathfrak{C}$. For $j \in \boldsymbol{P}_P$, there exist mutually disjoint subsets $\Lambda_1, \Lambda_2, \Lambda_3$ of $P\Gamma(\mathfrak{B}_*^+)$ with union $P\Gamma(\mathfrak{B}_*^+)$ and mutually orthogonal projections E_1, E_2, E_3 in $\mathfrak{C}$ such that $E_1 + E_2 + E_3 = 1_X$ defined by $E_k = \sum_{H \in \Lambda_k} E_H$, $k = 1, 2, 3$ and for $f \in \mathfrak{B}_*^+$,*

(i) *If $f = E_1 f E_1, j(f) = V^* f V, V \in E_1 \mathfrak{B} E_1, V^* V = j^*(E_1)$,*

(ii) *If $f = E_2 f E_2, j(f) = c^* V^* f V c, V \in E_2 \mathfrak{B} E_2, V^* V = j^*(E_2)$,*

(iii) *If $f = E_3 f E_3, j(f) = \sum_{H \in \Lambda_3} f(j^*(E_H))\, \omega_{x_H}, x_H \in E_H X, \|x_H\| = 1$ and ω_{x_H} is a pure state of $\mathfrak{B}$.*

For arbitrary $f \in \mathfrak{B}_*^+, j(f) = j(E_1 f E_1) + j(E_2 f E_2) + j(E_3 f E_3)$.

Proof. This follows immediately from Proposition 7.2 and Proposition 7.9.

§ 8. Concluding Remarks

At this stage it is convenient to discuss how far the initial programme has been successful. It was proposed to make some attempt at a reasonable definition of pure operation in the abstract situation in the hope that when applied to the usual models gives rise to recognisable results. In the event it was found to be necessary to study the structure of $\Gamma(K)$ in some detail. Whilst most of the results of § 3 are either well known or are easily derived from corresponding results in [5], as far as the author is aware some do not appear elsewhere in the literature. Propositions 3.6–3.8, 3.12, 3.14–3.19 fall into this category. Notice that one possible physical interpretation of $\Gamma(K)$ is that of forming the set of superselection rules of the system. In this case atomicity of $\Gamma(K)$ would correspond to the system possessing discrete minimal superselection rules. It became clear that a study of pure operations is only of any importance in the case $\Gamma(K)$ atomic and $P\Gamma(K) = P_I\Gamma(K)$. The details in this case were studied in § 5 culminating in the two results, Propositions 5.7, 6.1 most important from a physical point of view. These completely describe pure operations in the abstract situation up to a description of pure operations for systems without superselection rules and possessing pure states. Without placing more conditions on K it is unreasonable to expect to obtain much more information about pure operations. Notice that the results of § 4 and indeed some of the results of § 5 are incidental to the central theme but they do give some insight into the properties of strong operations whether pure or not.

The results of § 7 are self-explanatory from a physical point of view and in particular Proposition 7.10 describes pure operations in the Von Neumann algebra model provided that $\mathfrak{B}$ is of Type I and has an atomic centre. The form of pure operations in this case conforms with the usual convention. However, it is not clear that all physical situations are exhausted by consideration of such Von Neumann algebras. Indeed it is well known that the Type II and Type III cases occur. However, the general theory indicates that a discussion of pure operations in these cases is invalid and this is of course borne out by the fact that in these cases $E(\mathfrak{B}_*^+) = \{0\}$.

More specifically, in physical examples, $\mathfrak{B}$ is usually chosen to be some ultraweakly closed two-sided ideal in the Von Neumann envelope $\mathfrak{A}^{**}$ of a C^*-algebra $\mathfrak{A}$ with identity e. To be precise the elements $a \in \mathfrak{A}$ such that $0 \leqq a \leqq e$ represent the basic simple observables of the system whilst K is some norm closed invariant face in $\mathfrak{A}^{*+}$, the positive linear functionals on $\mathfrak{A}$, weak* dense in $\mathfrak{A}^{*+}$. Since $\mathfrak{A}^{*+}$ may be identified with the positive normal functionals on $\mathfrak{A}^{**}$ the general theory shows that K may be identified with the set of positive normal functionals on $E\mathfrak{A}^{**}E$ where E is a unique well-defined central projection in $\mathfrak{A}^{**}$. The Von Neumann algebra $\mathfrak{B}$ is identified with $E\mathfrak{A}^{**}E$. Since $\mathfrak{A}^{*+}$ is weak* closed the Krein-Milman theorem ensures that $E(\mathfrak{A}^{*+}) \neq \{0\}$ though it may still occur that $E(K) = \{0\}$. Therefore, even in this special case the choice of K is crucial. The point of course is that given the C^*-algebra $\mathfrak{A}$ a large number of different Von Neumann algebra models are then usually available. Only when there is a unique norm closed invariant face K of $\mathfrak{A}^{*+}$, weak* dense in $\mathfrak{A}^{*+}$ and hence equal to $\mathfrak{A}^{*+}$ is $\mathfrak{B}$ uniquely defined by $\mathfrak{A}$. This is in fact the case when $\mathfrak{A}$ is chosen to describe a quantum system with a finite number of degrees of freedom. It has often been conjectured that the possible Von Neumann algebra models which can be obtained from a given C^*-algebra in the way described above are all "physically" equivalent in some way. This assumption is equivalent to supposing that it is the weak* topology of $\mathfrak{A}^*$ not the norm topology which is important. This is certainly valid if only the basic simple observables are thought to be important. Under this assumption it is possible to make a choice of K which leads to $\mathfrak{B}$ being a Von Neumann algebra of Type I with atomic centre. Notice that $\mathfrak{A}^{*+}$ is a possible candidate for the set of states of the system and therefore it is possible to define $K = (\mathfrak{A}^{*+})_I$ the pure restriction of the system whose states are elements of $\mathfrak{A}^{*+}$. It is a consequence of the Krein-Milman theorem that K is weak* dense in $\mathfrak{A}^{*+}$. It then follows that $K_I = K$ and hence from the general theory that $\mathfrak{B}$ is a Von Neumann algebra of Type I with atomic centre. Therefore if one takes the C^*-algebra point of view to statistical physical situations Proposition 7.10 completely describes pure operations and these act on all states of the system. In general however pure operations only act on all states of the system in models in which $K_I = K$.

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