RESEARCH PROBLEMS

1. Paul Slepian: *Problems on polynomials.*

(1) Let $0<A<1$. Let B be the set of all positive integers n such that there exist n positive numbers $a_1, a_2, \cdots, a_n$ such that the polynomial

$$(x^2 - 2Ax + 1) \prod_{i=1}^{n} (x + a_i)$$

has all non-negative coefficients. It is known that B is nonempty. (See P. M. Lewis, *The concept of the one in voltage transfer synthesis*, IRE Trans. Vol. CT-2, pp. 316–319, December, 1955.) Find the least element of B.

(2) Let $0<A<1$ and let N be the smallest integer in B, as described in (1) above. Does there exist $b>0$ such that

$$(x^2 - 2Ax + 1)(x + b)^N$$

has all non-negative coefficients?

(3) Generalize the questions raised in (1) and (2) above to the case where the factor $x^2-2Ax+1$ is replaced by an arbitrary real polynomial, say $\sum_{i=0}^{m} C_i x^i$, having no positive real roots. (Received September 13, 1957.)

2. Louis Weinberg: *Decomposition of Hurwitz polynomials.*

Let $q(s)=\sum_{k=0}^{n} a_k s^k$ represent a Hurwitz polynomial with real coefficients, i.e., all of its zeros have negative real parts. Can $q(s)$ be divided into the arithmetic sum of two polynomials,

$$q(s) = q_1(s) + q_2(s)$$

each of which has positive coefficients and only nonpositive real roots? This can easily be done in particular cases; for example, if $q(s)=(s^2+2s+5)(s+4)=s^3+6s^2+13s+20$, then $q_1(s)=s^3+6s^2+11s+6=(s+1)(s+2)(s+3)$ and $q_2(s)=2s+14$. If this can be shown to be impossible in the general case, can the decomposition always be carried out with three polynomials,

$$q(s) = q_1(s) + q_2(s) + q_3(s),$$

each of which again has positive coefficients and only nonpositive real roots? (Received September 19, 1957.)

3. R. E. Bellman: *Number theory.* I.

The problem of generating the integer solutions of the equation $x^2+y^2=1 \pmod p$ by means of the formula $x_n=\cos n\theta$, $y_n=\sin n\theta$, where (x_1, y_1) is a fundamental solution which we can write symbolically in the form $x_1=\cos\theta_1$, $y_1=\sin\theta_1$, has been extensively studied. What are the corresponding results for the equations $x_1^2+x_2^2+\cdots x_n^2=1 \bmod p$?

In particular, for the equation $x_1^2+x_2^2+x_3^2=1 \bmod p$, what subset of solutions do we obtain by means of the formulas

$$x_1 = \cos k\theta_1 \cos l\theta_2,$$
$$x_2 = \cos k\theta_1 \sin l\theta_2,$$
$$x_3 = \sin k\theta_1,$$

where $k, l=0, 1, \cdots$, and θ_1, θ_2 correspond to certain primitive solutions? (Received May 22, 1957.)

4. R. E. Bellman: *Number theory*. II.

Consider the same type of problem for the multiplicative form

$$x^3 + y^3 + z^3 - 3xyz$$

and for the circulant functions of higher order. (Received May 22, 1957.)

5. R. E. Bellman: *Number theory*. III.

Consider the equation $y^2 = 4x^3 - g_2x - g_3$ which may be uniformized by means of the Weierstrassian elliptic functions $x = p(u)$, $y = p'(u)$. What subset of solutions of the congruence $y^2 = 4x^3 - g_2x - g_3 \pmod p$ can be obtained by means of the formulas $x = p(mu + nv)$, $y = p'(mu + nv)$, $m, n = 0, 1, 2, \cdots$, (not both zero simultaneously), where u and v correspond to certain primitive solutions?

Consider the similar problem for $y^2 = (1 - x^2)(1 - k^2x^2)$ which can be uniformized by means of Jacobian elliptic functions. (Received May 22, 1957.)

6. R. E. Bellman: *Number theory*. 1.

Let x be an irrational number in $[0, 1]$ and let $g(y; a, b)$, $0 \leqq a < b \leqq 1$ be a periodic function of y with period 1 defined by the conditions $g(y; a, b) = 1$, $a \leqq y \leqq b$, $g(y; a, b) = 0$ elsewhere for y in $[0, 1]$. Define the function

$$f_N(z, x) = g(z; a, b) + g(z + x; a, b) + \cdots + g(z + Nx; a, b)$$

for $1 \geqq z \geqq 0$, equal to the number of elements of the finite sequence $\{nx + z\}$, $n = 0, 1, \cdots, N$, falling inside $[a, b]$, modulo one.

The Weyl equidistribution theorem asserts that $f_N(z, x)/(N+1) \sim b - a$ as $N \to \infty$. It is easy to show via Fourier series that

$$\int_0^1 \int_0^1 [f_N(z, x) - (N + 1)(b - a)]^2 dzdx \sim (N + 1)c_1(a, b)$$

as $N \to \infty$.

This suggests that the quantity

$$u_N(z, x) = \frac{f_N(z, x) - (N + 1)(b - a)}{(N + 1)^{1/2}}$$

possesses asymptotic moments of all orders.

Does

$$\lim_{N \to \infty} \int_0^1 \int_0^1 \left(\frac{f_N(z, x) - (N + 1)(b - a)}{(N + 1)^{1/2}}\right)^{2k} dzdx,$$

$k = 2, 3, \cdots$ exist, and if so what is the limiting distribution? (Received July 15, 1957.)

7. R. E. Bellman: *Number theory*. 2.

Consider the same problem for the function

$$f_N(z, y, x) = g(z; a, b) + g(z + 2y + x; a, b) + \cdots + g(z + 2Ny + N^2x; a, b)$$

with x irrational, y and z in $[0, 1]$. As above, it is easy to show via Fourier series that

$$\int_0^1 \int_0^1 \int_0^1 [f_N(z, y, x) - (N + 1)(b - a)]^2 dxdydz \sim (N + 1)c_1(a, b)$$

as $N \to \infty$.

Does

$$\lim_{N\to\infty} \int_0^1 \int_0^1 \int_0^1 \left(\frac{f_N(z, y, x) - (N+1)(b-a)}{(N+1)^{1/2}}\right)^{2k} dxdzdy$$

exist for $k=1, 2, \cdots$, and if so, what is its value?

There are corresponding versions of this problem for polynomials of all orders.
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8. R. E. Bellman: *Differential equations.*

Consider the second order linear differential equation $u''+(1+\lambda g(x))u=0$, where λ is a real constant and $\int_0^\infty |g(x)|\,dx<\infty$. Let $u_1(x)$ be the solution specified by $u_1(0)=0$, $u_1'(0)=1$. It is known that $u\sim r(\lambda)\sin(x+\theta(\lambda))$ as $x\to\infty$.

Taking λ to be complex variable, what are the analytic properties of the functions $r(\lambda)$ and $\theta(\lambda)$? In particular, where are the singularities nearest the origin?

If $g(x)>0$ for $x\geqq 0$, is the singularity nearest the origin on the negative axis?
(Received August 16, 1957.)