A resolvent estimate and a smoothing property of inhomogeneous Schrödinger equations

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1. Results. Throughout this paper, we always assume $n \geq 2$. Let $p(\xi) > 0$ be of the class $C^\infty(\boldsymbol{R}^n \backslash 0)$ and positively homogeneous of degree 1, and $P = p(D_x) = \mathscr{F}_\xi^{-1} p(\xi) \mathscr{F}_x$ the corresponding Fourier multiplier. Suppose that $\Sigma = \{\xi ; p(\xi) = 1\}$ has non-vanishing Gaussian curvature. The objective of this brief article is to show the following smoothing effect of inhomogeneous generalized Schrödinger equations:

Theorem 1.1. *Suppose* $1 - n/2 < s < 1/2$, $1 - n/2 < \alpha < 1/2$ *and let* $|x|^{1-s} f(t, x) \in L^2(\boldsymbol{R}_t \times \boldsymbol{R}_x^n)$. *Then there exists a unique solution* $u(t, x)$ *to*

$$(1.1) \qquad \begin{cases} (\partial_t + iP^2)\, u = f \\ u|_{t=0} = 0 \end{cases}$$

which satisfies $|x|^{\alpha-1} |D_x|^{s+\alpha} u(t, x) \in L^2(\boldsymbol{R}_t \times \boldsymbol{R}_x^n)$.

Theorem 1.1 says that the solution gains the regularity of order "s" in connection with the decay order of the inhomogeneous term f, plus an extra gain of order "$\alpha < 1/2$", in the sense of space-time norm. This is an improvement of the result in Hoshiro [3] which showed Theorem 1.1 with $P = |D_x|$ and $0 < \alpha = s < 1/2$.

Since Hoshiro's method deeply depends on the properties of special functions, it is not suitable for handling the general operator P. To remove this obstacle is also in our focus. The most essential part of the proof is the following resolvent estimate:

Theorem 1.2. *Suppose* $1 - n/2 < a < 1/2$ *and* $1 - n/2 < b < 1/2$. *Then we have*

$$(1.2) \qquad \sup_{\mathrm{Im}\lambda > 0} \left\| |x|^{a-1} |D|^{a+b} (P^2 - \lambda^2)^{-1} v(x) \right\|_{L^2(\boldsymbol{R}^n)} \leq C \left\| |x|^{1-b} v(x) \right\|_{L^2(\boldsymbol{R}^n)}.$$

Theorem 1.2 is partly proved in the master's thesis of the second author [7]. The main tools for the proof of it are the weighted L^2-boundedness of Fourier multipliers, the limiting absorption principle, and an estimate for the kernel of the resolvent, which enable us to treat general operators P. We shall explain the details in Section 2.

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2. *Proof.* To begin with, we shall prove Theorem 1.2. The argument here is based on [7]. Hereafter, we denote the norm $\|\cdot\|_{L^2(\boldsymbol{R}^n)}$ by $\|\cdot\|$. We remark

$$1/2 < 1 - a < n/2, \quad 1/2 < 1 - b < n/2, \quad 0 < a + b - 2 + n < n,$$

which will be used later frequently without any notice. Furthermore, we may assume

$$(3 - n)/2 \leq a + b.$$

The general case can be reduced to this special one because of the following:

Proposition 2.1 ([5, Theorem B*]). *Suppose* $k < n/2$, $l < n/2$, $0 < m < n$, *and* $k + l + m = n$. *Then we have*

$$\left\| |x|^{-l} |D|^{m-n} v \right\| = \left\| |x|^{-l} \int \frac{v(y)}{|x-y|^m} dy \right\| \leq C \left\| |x|^k v \right\|.$$

In fact, if $a + b < (3 - n)/2$, we have $(3 - n)/2 \leq (a + \delta) + b$ and $1 - n/2 < (a + \delta) < 1/2$, where $\delta = (3 - n)/2 - (a + b)$. We remark $0 < \delta < (n - 1)/2$. Then, by Proposition 2.1 and the estimate (1.2) with a replaced by $a + \delta$, we have

$$\begin{aligned} &\sup_{\mathrm{Im}\lambda > 0} \left\| |x|^{a-1} |D|^{a+b} (P^2 - \lambda^2)^{-1} v \right\| \\ &\leq C \sup_{\mathrm{Im}\lambda > 0} \left\| |x|^{(a+\delta)-1} |D|^{(a+\delta)+b} (P^2 - \lambda^2)^{-1} v \right\| \\ &\leq C \left\| |x|^{1-b} v \right\|, \end{aligned}$$

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which is the estimate (1.2).

Now, all we have to show is, by the scaling argument, the following two estimates:

$$(2.1)\quad \sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| |x|^{a-1}|D|^{a+b}(P^2-\lambda^2)^{-1} (1-\varphi\circ p)(D)v \right\| \le C \left\| |x|^{1-b}v \right\|,$$

$$(2.2)\quad \sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| |x|^{a-1}|D|^{a+b}(P^2-\lambda^2)^{-1} (\varphi\circ p)(D)v \right\| \le C \left\| |x|^{1-b}v \right\|,$$

where $\varphi(\rho) \in \mathscr{C}_0^\infty(\boldsymbol{R}_+)$ is a function which is equal to 1 near $\rho = 1$.

The estimate (2.1) is a consequence of Proposition 2.1 and the following:

Proposition 2.2 ([6, Chapter 11, Theorem 5]). *Suppose $-n/2 < k < n/2$. Then we have*

$$\left\| |x|^k m(D)v \right\| \le C \sum_{|\gamma|\le n} \sup_{\xi\in R_n} \left| |\xi|^{|\gamma|} D^\gamma m(\xi) \right| \left\| |x|^k v \right\|.$$

In fact, setting $m_\lambda(\xi) = |\xi|^2(p(\xi)^2-\lambda^2)^{-1}(1-\varphi\circ p)(\xi)$, we have

$$\sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| |x|^{a-1}|D|^{(a+b-2+n)-n} m_\lambda(D)v \right\| \le C \sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| |x|^{1-b} m_\lambda(D)v \right\| \le C \left\| |x|^{1-b}v \right\|,$$

which is the estimate (2.1).

The estimate (2.2) is easily obtained from Proposition 2.2 and the estimate

$$(2.3)\quad \sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| |x|^{a-1}P^{a+b}(P^2-\lambda^2)^{-1} (\varphi\circ p)(D)v \right\| \le C \left\| |x|^{1-b}v \right\|,$$

which is a consequence of the following two propositions: (The curvature condition of Σ is necessary for Proposition 2.4 only).

Proposition 2.3 ([1, Theorem 14.2.2]). *Let $\Psi \in C_0^\infty(\boldsymbol{R}^n)$. Suppose $k > 1/2$ and $l > 1/2$. Then we have*

$$\sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| (1+|x|)^{-l}(P^2-\lambda^2)^{-1}\Psi(D)v \right\| \le C \left\| (1+|x|)^k v \right\|.$$

Proposition 2.4 ([4, Theorem 6.3]). *Let $\phi \in \mathscr{C}_0^\infty(\boldsymbol{R}_+)$. Then we have*

$$\sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left| \mathscr{F}^{-1}\left[(p(\xi)^2-\lambda^2)^{-1}(\phi\circ p)(\xi)\right](x) \right| \le C|x|^{-(n-1)/2}.$$

In fact, setting $\phi(\rho) = \rho^{a+b}\varphi(\rho)$ and $\Psi = \phi\circ p$, we have

$$\sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| (1-\chi)|x|^{a-1}(P^2-\lambda^2)^{-1}\Psi(D)(1-\chi)v \right\| \le C \left\| |x|^{1-b}v \right\|$$

by Proposition 2.3, where $\chi(x)$ is the characteristic function of the set $\{x\,;\,|x|\le 1\}$. On the other hand, since $(3-n)/2 \le a+b$ implies $1-a \le b+(n-1)/2 < n/2$, we have

$$\sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| \chi|x|^{a-1}(P^2-\lambda^2)^{-1}\Psi(D)v \right\| \le \sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| \chi|x|^{-b-(n-1)/2}(P^2-\lambda^2)^{-1}\Psi(D)v \right\| \le C \left\| |x|^{-b-(n-1)/2}\int \frac{|v(y)|}{|x-y|^{(n-1)/2}}dy \right\| \le C \left\| |x|^{1-b}v \right\|.$$

Here we have used Propositions 2.1 and 2.4. Similarly, since $(3-n)/2 \le a+b$ implies $1-b \le (n-1)/2+a < n/2$, we have

$$\sup_{\substack{\mathrm{Im}\lambda>0\\|\lambda|=1}} \left\| |x|^{a-1}(P^2-\lambda^2)^{-1}\Psi(D)\chi v \right\| \le C \left\| |x|^{(n-1)/2+a}\chi v \right\| \le C \left\| |x|^{1-b}v \right\|.$$

Thus we have obtained the estimate (2.3) and completed the proof of Theorem 1.2.

As is also explained in Hoshiro [2] and [3], we can construct the solution u to the inhomogeneous equation (1.1) by taking the weak limit of the functions

$$u_\varepsilon(t,x) = \frac{1}{i}\mathscr{F}_\tau^{-1}(P^2+(\tau-i\varepsilon))^{-1}\mathscr{F}_t f_+(t,x) + \frac{1}{i}\mathscr{F}_\tau^{-1}(P^2+(\tau+i\varepsilon))^{-1}\mathscr{F}_t f_-(t,x)$$

as $\varepsilon \searrow 0$ in an appropriate function spaces. Here $f_\pm$ denote the function f multiplied by the characteristic function of the set $\{t\,;\,\pm t \ge 0\}$. By Theorem 1.2 with $a=\alpha$, $b=s$, this argument can be justified, and we have Theorem 1.1.

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