(4.3)
$$\begin{array}{ccc} \widetilde{Tr}_H(k) & \xrightarrow{\tilde{\tau}} & C_+(k) \\ & \searrow \quad \nearrow & \\ & \tilde{\mathscr{E}}_H(k) & \end{array}$$

where $\tau : Tr_H(k) \to C_+(k)$ is given by

(4.4) $\tau(t) = (\tan(A/2), \tan(B/2), \tan(C/2))$.

All other notation in (4.3) should be self-explanatory and the proof goes similarly as before.

Examples and comments. When $k = \boldsymbol{Q}$, elements of $Tr_H(\boldsymbol{Q})$ are called "rational triangles" or "Heron triangles" ([1] Chap. V). Heron of Alexandria noted that $t = (13, 14, 15)$ belongs to $Tr_H(\boldsymbol{Q})$ with $\Delta = 84$. By our map (4.4) it corresponds to the point (1/2, 4/7, 2/3) of the quadric $C_+(\boldsymbol{Q})$. On the other hand, by our map (1.6) it corresponds to the point (1/4, 16/49, 4/9) of the quartic $S_+(\boldsymbol{Q})$.

Obviously, every right triangle $t = (a, b, c) \in Tr(k)$ belongs to $Tr_H(k)$. Assume that $C = \pi/2$; hence $a^2 + b^2 = c^2$. Then $\tau(t) = (a/(b+c), b/(a+c), 1)$ and $\theta(t) = (a^2/(b+c)^2, b^2/(a+c)^2, 1)$. In both cases the image of right triangles with $C = \pi/2$ is the intersection of the surface in k_+^3 and the plane $z = 1$ (or $w = 1$).

Needless to say, all equilateral triangles $t = (a, a, a)$, $a \in k_+$, are similar and so they correspond to a single point in the quartic surface. If k does not contain $3^{\frac{1}{2}}$, then $t \notin Tr_H(k)$ because $\Delta_t = (3^{\frac{1}{2}}/4)a$.

§5. An involution. For $t = (a, b, c) \in Tr(k)$, put

(5.1) $t' = (a', b', c')$ with $a' = a(s-a)$,
$b' = b(s-b)$, $c' = c(s-c)$, $s = \dfrac{1}{2}(a+b+c)$.

Then one finds

(5.2) $s' - a' = (s-b)(s-c)$, $s' - b' = (s-c)(s-a)$, $s' - c' = (s-a)(s-b)$,

with $s' = \dfrac{1}{2}(a' + b' + c')$. By (5.1), (5.2), we obtain a map: $Tr(k) \to Tr(k)$. Furthermore, for the image $t'' = (a'', b'', c'')$ of $t' = (a', b', c')$, we get

(5.3) $a'' = a'(s' - a') = ad$, $b'' = bd$, $c'' = cd$,
with $d = (s-a)(s-b)(s-c)$.

In other words, we have $t'' \sim t$ and so the map $t \mapsto t'$ induces an involution * of $\widetilde{Tr}(k)$. The only fixed point of * is the class of equilateral triangle. By the diagram (2.5), we can transplant * on $S_+(k)$ and $\tilde{\mathscr{E}}(k)$. On the surface $S_+(k)$, the involution $P = (x, y, z) \mapsto P = (x^*, y^*, z^*)$ is determined by the relation:

(5.4) $xx^* = yy^* = zz^* = (xyz)/(x(yz)^{\frac{1}{2}} + y(zx)^{\frac{1}{2}} + z(xy)^{\frac{1}{2}})$.

Example (Heron). Let $k = \boldsymbol{Q}$ and $t = (a, b, c) = (13, 14, 15) \in Tr(\boldsymbol{Q})$. We have $s = 21$, $s - a = 8$, $s - b = 7$, $s - c = 6$, $\Delta = (s(s-a)(s-b)(s-c))^{\frac{1}{2}} = 84$, hence $t \in Tr_H(\boldsymbol{Q})$. Next, by (5.1), we have $t' = (a', b', c') = (104, 98, 90)$, $s' = 146$ and $(\Delta')^2 = 16482816 = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$, which means that $t' \notin Tr_H(\boldsymbol{Q})$; in other words, the involution * of $\widetilde{Tr}(\boldsymbol{Q})$ does not respect the subset $\widetilde{Tr}_H(\boldsymbol{Q})$. Passing to the surface $S_+(\boldsymbol{Q})$, we have

$\theta(t) = (1/2^2, 2^4/7^2, 2^2/3^2)$
$\theta(t)^* = (2^5/73, 7^3/(2\cdot 73), (2\cdot 3^2)/73)$.

As for triples of elliptic curves, denoting by $[P, Q]$ for the curve of type (3.1), we have

$\boldsymbol{E}_t = (E_a, E_b, E_c) = ([126, -84^2], [99, 〃], [70, 〃])$,
$\boldsymbol{E}_t^* = (E_{a'}, E_{b'}, E_{c'}) = ([3444, -2^9\cdot 3^2\cdot 7^2\cdot 73], [4656, 〃], [6160, 〃])$.

References

[1] Dickson, L. E.: History of the Theory of Numbers. vol. 2, Chelsea, New York (1971).
[2] Ono, T.: Triangles and elliptic curves. I ~ VI. Proc. Japan Acad., **70A**, 106–108(1994): **70A**, 223–225 (1994); **70A**, 311–314 (1994); **71A**, 104–106 (1995); **71A**, 137–139 (1995); **71A**, 184–186 (1995).

I take this opportunity to make a correction to my paper (VI). On p. 186, in (4.6), $x^3 + 4x^2 - 3x$ should read $x^3 + 2x^2 - 3x$.