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Reduction of Associate Classes for Block Designs and Related Combinatorial Arrangements

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Contents

0. Introduction and summary

The concept of the Kronecker product for matrices was first introduced to the experimental designs by Vartak [58]. He defined the Kronecker product of designs and the reduced designs, but did not discuss explicitly the association schemes (Bose and Shimamoto [10]) concerning those designs. When there exists an arrangement with the parameters of a partially balanced incomplete block (**PBIB**) design first introduced by Bose and Nair [9] and generalized by Nair and Rao [38], it is important to find the association scheme matching its design in relation to the problem of determining the uniqueness of the association scheme and also of characterizing association schemes.

An association scheme was originally studied in relation to the definition of a **PBIB** design which has been derived from describing relations among treatments in terms of the structure of treatment-block incidence of the design. Bose and Shimamoto [10] rephrased the definition of a **PBIB** design so as to stress that the relations among treatments are determined only by the parameters n_i and p^i_{jk} $(i, j, k=1, 2,\dots, m)$. Bose and Mesner [8] studied the algebraic structure concerning an association scheme of a **PBIB** design. An association scheme, however, has been defined and characterized independently of treatment-block incidence of the design.

When the parameters λ_i $(i=1, 2,\dots, m)$ of a **PBIB** design are not all different, the m associate classes of the **PBIB** design based on an association scheme may not be all distinct. Two approaches will be considered for reductions of the number of associate classes. One (cf. [28; 58]) consists in using the parameters λ_i and the second kind of parameters p^i_{jk} of the **PBIB** design N being a standard approach generalized by Kageyama [28]. Another (cf. [23]) consists in using λ_i and the latent roots of the matrix NN'. The former approach is much useful to the discussions, for reduction on associate classes of an association scheme, which will appear in this paper. To a group of **PBIB** designs of a certain Kronecker product type, we may be encouraged to apply the latter approach in preference to the former. The relationship between two approaches is studied in this paper. The theorems on the reductions give some criteria to determine whether **PBIB** designs with l associate classes of the various types are reducible to those with l_1 distinct associate classes $(l_1 < l)$ when their parameters $\lambda_1, \lambda_2,\dots, \lambda_l$ are not all different.

We, usually, deal with a **PBIB** design with equi-replications and equi-size blocks. From a practical point of view, however, it may not be possible to design the equi-size blocks accommodating the equi-replication of each treatment in all the blocks. For incomplete block designs in such a situation, a balanced block (**BB**) design was introduced by Rao [47]. A partially balanced block (**PBB**) design was essentially introduced by Kishen [32] and was explicitly defined by

Ishii and Ogawa [20]. From a combinatorial point of view of a **BB** design and a **PBB** design, the constructions and combinatorial properties are reported by several authors throughout the literatures [3; 4; 20; 21; 32; 33; 43; 45]. In addition to the discussion about reductions of the number of associate classes for a **PBB** design and a **PBIB** design, we shall deal with these combinatorial aspects of a **BB** design and a **PBB** design.

This paper consists of three parts. In Part I, the reductions of the number of associate classes for **PBIB** designs are treated. Section 1 gives definitions of an association scheme, a **PBIB** design and a **BIB** design, and describes some of their properties. Section 2 presents some necessary and sufficient conditions that a **PBIB** design with m associate classes is reducible to a **PBIB** design with m_1 associate classes ($m_1 < m$). Section 3 contains three **PBIB** designs giving the validity of the necessary and sufficient condition in Section 2. Section 4 deals with the relation between generalized Vartak's and Kageyama's condition. The relation for a general **PBIB** design is studied through the properties of the latent roots of the matrix $\mathfrak{P}_k = \| p^i_{jk} \|$ ($k=0, 1,\ldots, m$). Section 5 gives some necessary and sufficient conditions for **PBIB** designs of certain Kronecker product types to be reducible. By using generalized Vartak's and Kageyama's condition, Section 6 is devoted to clarify the algebraic structures of **PBIB** designs constructed by generalization of Sillitto's product, and contains Table useful to investigate the reducibility in an F_3 type association scheme of v ($=v_1v_2v_3$) treatments. An interesting association scheme will be presented.

In Part II, some series of association schemes reducible by combining some associate classes, and the reductions of two association schemes with four associate classes are discussed independently of treatment-block incidence of the design. Furthermore, we shall find a note concerning a series of not reducible association schemes. Sections 7 and 8 cover a series of reducible N_m type association schemes and a series of reducible orthogonal Latin square type association schemes, respectively. Section 9 treats a series of reducible F_p type association schemes and gives remarks on a series of C_p type association schemes being a special case of an F_p type association scheme. Section 10 deals with a series of reducible m-associate cyclical type of association schemes, and with a series of not reducible T_m type association schemes. We give properties of reduction for a generalized right angular association scheme and for a rectangular lattice type association scheme reducible to association schemes of two and three associate classes. These two association schemes may not correspond to any of the known association schemes. Section 11 gives remarks on the reductions of associate classes for association schemes and for **PBIB** designs based on certain association schemes.

In Part III, the constructions and combinatorial properties of **BB** designs and **PBB** designs are discussed. Section 12 presents another useful description for the definitions of a **BB** design and a **PBB** design. Section 13 characterizes a **BB** design and a **PBB** design. Sections 14 and 15 deal with the constructions and some examples of **BB** designs and **PBB** designs. It is noted that the complements of a **BB** design and a **PBB** design are generally not a **BB** design and a **PBB** design, respectively. Section 16 treats the constructions and some examples of $(\mu_1, \mu_2,\dots, \mu_t)$-resolvable **BB** designs and **PBB** designs. Section 17 deals with the reduction procedures of the number of associate classes for **PBB** designs based on the results in Parts I and II, and gives some examples. Section 18 contains inequalities to hold for a **BB** design and/or a **PBB** design. A general inequality for an equireplicate **PBB** design based on an association scheme with m associate classes is presented. The bound on the latent roots for the C-matrix of a **PBB** design is also given.

For convenience, the notations and symbols shown below are used throughout this paper. Unless stated otherwise, their meanings are as follows:

- I_s : The unit matrix of order s.
- $E_{s\times t}$: An $s\times t$ matrix whose elements are all unity. As a special case, $E_{s\times s}$ is denoted by G_s.
- $O_{i\times j}$: An $i\times j$ matrix whose all elements are zero.
- A' : Transpose of the matrix A.
- $A\otimes B$: Kronecker product of the matrices $A=\|a_{ij}\|$ and B, i.e., $A\otimes B=\|a_{ij}B\|$.
- $[A:B]$: The juxtaposition of the matrices A and B.
- $\operatorname{tr} A$: The trace of the matrix A.
- $A^{\#}$: The superscript$^{\#}$ indicates that the matrix $A^{\#}$ is an idempotent matrix.
- z_{ij} : Latent roots of the matrix $\mathfrak{P}_j=\|p^k_{ij}\|$ $(i, k=0, 1,\dots, m)$ consisting of parameters p^k_{ij} in an association scheme with m associate classes.
- $\varepsilon(x)$: A function of x which assumes either the value zero or one according as x is zero or not.
- $\binom{n}{m}$: The binomial coefficient.
- $\operatorname{diag}\{a_1, a_2,\dots, a_l\}$: An $l\times l$ diagonal matrix with the diagonal elements $a_1, a_2,\dots, a_l$.
- $\mathfrak{A}=[A^{\#}_i; i=0, 1,\dots, m]$: An algebra generated by the linear closure of those commutative matrices indicated in the bracket [ ].

Part I. Reductions for the number of associate classes for PBIB designs

The Kronecker product of designs and the reduced designs were first defined by Vartak [58], who gave a necessary and sufficient condition that a **PBIB** design with m associate classes is reducible to a **PBIB** design with $m-1$ associate classes, but his condition was incomplete. The association schemes concerning those designs were not considered explicitly. From this point of view, Kageyama [23; 28; 30] dealt with the reduction of associate classes for **PBIB** designs constructed by the combinations of Kronecker products of **BIB** designs and he also considered together the association schemes matching their designs and the necessary and sufficient conditions for reductions. His approach uses the numbers, λ_i, of blocks containing a pair of treatments and the latent roots ρ_i of the matrix NN' for an incidence matrix N of a **PBIB** design. The approach is different from generalized Vartak's approach using λ_i and p^i_{jk} of a **PBIB** design. The problem considered in Parts I and II will be of both theoretical and practical importance with regard to constructing and analyzing certain **PBIB** designs.

1. Association schemes, PBIB designs and BIB designs

Given v treatments 1, 2,..., v, a relation satisfying the following three conditions is said to be an association scheme with m associate classes [10]:

(a) Any two treatments are either 1st, 2nd,..., or mth associates, the relation of association being symmetric, i.e., if treatment α is ith associate of treatment β, then β is ith associate of treatment α.

(b) Each treatment has n_i ith associates, the number n_i being independent of the treatment taken.

(c) If any two treatments α and β are ith associates, then the number of treatments which are jth associates of α and kth associates of β is p^i_{jk} and is independent of the pair of ith associates α and β.

Remark. It is shown by Bose and Clatworthy [7] that for an association scheme with two associate classes ($m=2$), the condition (c) could be replaced by

(c)′ If any two treatments α and β are ith associates, then, the number, p^i_{11} for $i=1$, 2 of treatments which are the first associates of α and the first associates of β is independent of the pair of ith associates α and β.

The numbers

$$v,\ n_i,\ p^i_{jk}; \qquad i, j, k = 1, 2, \ldots, m$$

are called the parameters of the association scheme; all must be nonnegative integers.

It is useful to make the convention that each treatment is the 0th associate of itself and of no other treatments. Then we must have

$$n_0 = 1,$$

$$
\begin{aligned}
p^0_{ij} &= p^0_{ji} = 0, && \text{if } i \neq j, \\
&= n_j, && \text{if } i = j, \\
p^i_{k0} &= p^i_{0k} = 0, && \text{if } i \neq k, \\
&= 1, && \text{if } i = k.
\end{aligned}
$$

Hence the following relations among the parameters are easily shown:

$$v = \sum_{i=0}^{m} n_i\,, \quad p^i_{jk} = p^i_{kj}\,, \quad \sum_{k=0}^{m} p^i_{jk} = n_j\,,$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}\,.$$

Let the association matrices $A_0, A_1, \ldots, A_m$ as a matrix representation of the association scheme be

$$A_i = \|a^{\beta}_{\alpha i}\|, \qquad \alpha, \beta = 1, 2, \ldots, v; \qquad i = 0, 1, \ldots, m,$$

where

$a^{\beta}_{\alpha i} = 1$, if αth and βth treatments are ith associates,

$= 0$, otherwise.

It will be clear that A_0 is nothing but the unit matrix of order v. Then from the very definition of the association matrices, it follows that they are all symmetric, linearly independent,

$$\text{(1.1)} \qquad \sum_{i=0}^{m} A_i = G_v, \quad A_i E_{v\times 1} = n_i E_{v\times 1} \quad \text{and} \quad A_i A_j = A_j A_i = \sum_{k=0}^{m} p^k_{ij} A_k\,.$$

Following Ogawa [40], if the association algebra generated by the matrices $A_0, A_1, \ldots, A_m$ may be denoted by $\mathfrak{A}$, which may be also expressed by indicating its ideal basis as $\mathfrak{A} = [A^{\#}_0, A^{\#}_1, \ldots, A^{\#}_m]$ provided the mutually orthogonal idempotents of $\mathfrak{A}$ are given by $A^{\#}_0, A^{\#}_1, \ldots, A^{\#}_m$, then (1.1) defines the regular representation of the association algebra:

$$(\mathfrak{A}):\ A_j \longrightarrow \mathfrak{P}_j\,,$$

where $\mathfrak{P}_j = \|p^k_{ij}\|$, $j = 0, 1, \ldots, m$.

We can choose a nonsingular real matrix

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_{10} & c_{11} & \cdots & c_{1m} \\ \vdots & \vdots & & \vdots \\ c_{m0} & c_{m1} & \cdots & c_{mm} \end{pmatrix}$$

which makes all $\mathfrak{P}_j$ diagonal simultaneously, in such a way that

$$C\mathfrak{P}_jC^{-1} = \begin{pmatrix} z_{0j} & & & \\ & z_{1j} & & 0 \\ & & \ddots & \\ 0 & & & z_{mj} \end{pmatrix}, \qquad j=0, 1,\dots, m, \tag{1.2}$$

where

$$z_{0j} = n_j, \qquad z_{00} = z_{10} = \cdots = z_{m0} = 1. \tag{1.3}$$

Furthermore, it is known (cf. [8; 40]) that

$$\sum_{j=0}^{m} z_{0j} = v; \quad \sum_{j=0}^{m} z_{ij} = 0, \qquad i = 1, 2,\dots, m, \tag{1.4}$$

$$z_{ui}z_{uj} = \sum_{k=0}^{m} p_{ij}^k z_{uk}, \qquad u = 0, 1,\dots, m, \tag{1.5}$$

and let the matrix Z, whose $(j+1)$st row is the diagonal elements of (1.2), be defined by

$$Z = \begin{pmatrix} z_{00} & z_{10} & \cdots & z_{m0} \\ z_{01} & z_{11} & \cdots & z_{m1} \\ \vdots & \vdots & & \vdots \\ z_{0m} & z_{1m} & \cdots & z_{mm} \end{pmatrix} \tag{1.6}$$

with (1.3), (1.4) and (1.5), then the matrix Z of order $m+1$ is nonsingular. Furthermore, a relation between A_i and $A_i^\#$ $(i=0, 1,\dots, m)$ is given by

$$\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_m \end{pmatrix} = [Z \otimes I_v] \begin{pmatrix} A_0^\# \\ A_1^\# \\ \vdots \\ A_m^\# \end{pmatrix}. \tag{1.7}$$

If we have an association scheme with m associate classes, then we get a partially balanced incomplete block (**PBIB**) design [9; 38] with b blocks, r replications, and block size k based on the association scheme, provided we can arrange the v treatments into b blocks such that

(i) each treatment occurs at most once in a block;

(ii) each block contains k distinct treatments;

(iii) each treatment occurs in exactly r blocks;

(iv) if two treatments α and β are ith associates, then they occur together in λ_i blocks (not all λ_i's equal), the number λ_i being independent of the particular pair of ith associates α and β $(1 \leqq i \leqq m)$.

The numbers

$$v,\ b,\ r,\ k,\ \lambda_i \qquad (i = 1, 2, \ldots, m)$$

are called the parameters of the design. The λ_i and p^i_{jk} are called the coincidence numbers and the second kind of parameters of the design, respectively.

A balanced incomplete block (**BIB**) design [62] with parameters v, b, r, k and λ is an arrangement of v treatments into b blocks such that

(i) each treatment occurs at most once in a block;
(ii) each block contains k distinct treatments;
(iii) each treatment occurs in exactly r blocks;
(iv) every pair of treatments occur in exactly λ blocks.

Note that, though (i), (ii) and (iv) lead to (iii), we follow the traditional definition of a **BIB** design. Among parameters v, b, r, k and λ, the following relations hold:

$$vr = bk, \quad \lambda(v-1) = r(k-1) \quad \text{and} \quad b \geqq v.$$

The last inequality is due to Fisher [15]. Incidentally, note that $vr=bk$ and $\lambda(v-1)=r(k-1)$ lead to $b-2r+\lambda=r(v-k)(v-k-1)/k(v-1)\geqq 0$, and that the equality $b-2r+\lambda=0$ holds when and only when the parameters of the original **BIB** design satisfy $v=k+1$. The number, $b-2r+\lambda$, is the coincidence number of the complement of a **BIB** design with parameters v, b, r, k and λ. On the other hand, from the known relations, i.e., $vr=bk$, $\sum_{i=1}^{m} n_i=v-1$, $\sum_{i=1}^{m} n_i\lambda_i=r(k-1)$, among the parameters of a **PBIB** design with m associate classes, we cannot derive the relation, $b-2r+\lambda_i\geqq 0$ for all i. Hence, this inequality is a necessary condition for the existence of a **PBIB** design [35]. The numbers, $b-2r+\lambda_i$, are the coincidence numbers of the complement of a **PBIB** design with parameters v, b, r, k and λ_i $(i=1, 2,\ldots, m)$.

After numbering v treatments and b blocks in some way, we can define the incidence matrix of a **PBIB** design or a **BIB** design to be the matrix:

$$N = \|n_{ij}\|; \qquad i = 1, 2,\ldots, v \quad \text{and} \quad j = 1, 2,\ldots, b,$$

where $n_{ij}=1$ or 0 according as the ith treatment occurs in the jth block or not. Then for the incidence matrix N of a **PBIB** design the following lemma is obtained (cf. [59]):

LEMMA A. *NN' belongs to the association algebra $\mathfrak{A}$ and can be expressed* as

$$NN' = \sum_{j=0}^{m} \lambda_j A_j = \sum_{i=0}^{m} \rho_i A_i^{\#},$$

where the last member of the expression is the spectral expansion of NN' in $\mathfrak{A}$. *The densities*

$$\rho_i = \sum_{j=0}^{m} \lambda_j z_{ij}, \qquad i = 0, 1,\ldots, m, \tag{1.8}$$

are the latent roots of NN'. In particular,

$$\rho_0 = rk = \sum_{i=0}^{m} n_i\lambda_i.$$

The ρ_i satisfy the inequalities

$$0 \leqq \rho_i \leqq rk, \qquad i = 0, 1,\ldots, m. \tag{1.9}$$

The multiplicity of ρ_i is the $\operatorname{tr} A_i^{\#}$.

Finally, since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix may be denoted by the same symbol throughout this paper.

2. Necessary and sufficient conditions for reductions

When the coincidence numbers $\lambda_1, \lambda_2,\ldots, \lambda_m$ of a **PBIB** design with m associate classes are not all different, the m associate classes of the **PBIB** design based on a certain association scheme may not be all distinct. Then when $\lambda_1 = \lambda_2$ in a **PBIB** design with m associate classes, Vartak [58] gave a necessary and sufficient condition for the **PBIB** design to be reducible to a **PBIB** design with $m-1$ associate classes. Moreover, he stated that repeated applications of the result to any **PBIB** design will ultimately give a **PBIB** design whose associate classes are all distinct. As indicated in the next section, however, we have reducible **PBIB** designs to which Vartak's iterative procedure does not apply. Then we need to generalize Vartak's condition. The following lemmas useful later give criteria to determine whether a **PBIB** design with m associate classes is reducible to a **PBIB** design with fewer distinct associate classes, when $\lambda_1, \lambda_2,\ldots, \lambda_m$ are not all different.

LEMMA 2.1 (Kageyama [28]). *Let a* **PBIB** *design N with m associate classes and with parameters*

$$v, b, r, k, \lambda_i, n_i, p^i_{jk}, \qquad i, j, k = 1, 2,\ldots, m,$$

be such that $\lambda_1, \lambda_2,\ldots, \lambda_m$ are not all different so that at least l of them are equal. Without loss of generality we can assume that $\lambda_1 = \lambda_2 = \cdots = \lambda_l$. In this case, the number of associate classes of the design N can be reduced from m to $m-l+1$ by combining its first l associate classes if and only if

$$
\begin{pmatrix}
\sum_{i,j=1}^{l} p_{ij}^{1}, & \sum_{i=1}^{l} p_{i,l+1}^{1},\ldots, & \sum_{i=1}^{l} p_{im}^{1} \\
\sum_{j=1}^{l} p_{l+1,j}^{1}, & p_{l+1,l+1}^{1},\ldots, & p_{l+1,m}^{1} \\
\vdots & \vdots & \vdots \\
\sum_{j=1}^{l} p_{mj}^{1}, & p_{m,l+1}^{1},\ldots, & p_{mm}^{1}
\end{pmatrix}
$$

$$
(2.1) \qquad = \begin{pmatrix}
\sum_{i,j=1}^{l} p_{ij}^{2}, & \sum_{i=1}^{l} p_{i,l+1}^{2},\ldots, & \sum_{i=1}^{l} p_{im}^{2} \\
\sum_{j=1}^{l} p_{l+1,j}^{2}, & p_{l+1,l+1}^{2},\ldots, & p_{l+1,m}^{2} \\
\vdots & \vdots & \vdots \\
\sum_{j=1}^{l} p_{mj}^{2}, & p_{m,l+1}^{2},\ldots, & p_{mm}^{2}
\end{pmatrix} = \cdots
$$

$$
\cdots = \begin{pmatrix}
\sum_{i,j=1}^{l} p_{ij}^{l}, & \sum_{i=1}^{l} p_{i,l+1}^{l},\ldots, & \sum_{i=1}^{l} p_{im}^{l} \\
\sum_{j=1}^{l} p_{l+1,j}^{l}, & p_{l+1,l+1}^{l},\ldots, & p_{l+1,m}^{l} \\
\vdots & \vdots & \vdots \\
\sum_{j=1}^{l} p_{mj}^{l}, & p_{m,l+1}^{l},\ldots, & p_{mm}^{l}
\end{pmatrix}.
$$

Furthermore, if (2.1) *holds, then the parameters of the reduced* **PBIB** *design with $m-l+1$ associate classes are as follows*:

$$
v' = v, \quad b' = b, \quad r' = r, \quad k' = k,
$$

$$
\lambda_1' = \lambda_1 = \lambda_2 = \cdots = \lambda_l, \qquad \lambda_2' = \lambda_{l+1},\ldots, \lambda_{m-l+1}' = \lambda_m,
$$

$$
n_1' = n_1 + n_2 + \cdots + n_l, \qquad n_2' = n_{l+1},\ldots, n_{m-l+1}' = n_m,
$$

$$
\|p_{uv}'^{1}\| = \begin{pmatrix}
\sum_{i,j=1}^{l} p_{ij}^{t}, & \sum_{i=1}^{l} p_{i,l+1}^{t},\ldots, & \sum_{i=1}^{l} p_{im}^{t} \\
\sum_{j=1}^{l} p_{l+1,j}^{t}, & p_{l+1,l+1}^{t},\ldots, & p_{l+1,m}^{t} \\
\vdots & \vdots & \vdots \\
\sum_{j=1}^{l} p_{mj}^{t}, & p_{m,l+1}^{t},\ldots, & p_{mm}^{t}
\end{pmatrix},
$$

$$
\|p_{uv}'^{w}\| = \begin{pmatrix}
\sum_{i,j=1}^{l} p_{ij}^{w+l-1}, & \sum_{i=1}^{l} p_{i,l+1}^{w+l-1},\ldots, & \sum_{i=1}^{l} p_{im}^{w+l-1} \\
\sum_{j=1}^{l} p_{l+1,j}^{w+l-1}, & p_{l+1,l+1}^{w+l-1},\ldots, & p_{l+1,m}^{w+l-1}, \\
\vdots & \vdots & \vdots \\
\sum_{j=1}^{l} p_{mj}^{w+l-1}, & p_{m,l+1}^{w+l-1},\ldots, & p_{mm}^{w+l-1}
\end{pmatrix},
$$

where $t=1, 2,\dots,$ *or* l; $w=2, 3,\dots, m-l+1$; $u, v=1, 2,\dots, m-l+1$.

LEMMA 2.2 (Kageyama [28]). *Let a* **PBIB** *design* N *with* m *associate classes and with parameters*

$$v,\ b,\ r,\ k,\ \lambda_i,\ n_i,\ p^i_{jk}, \qquad i, j, k = 1, 2,\dots, m,$$

be such that $\lambda_1, \lambda_2,\dots, \lambda_m$ *are not all different so that each of* l_j λ*'s* $(j=1,2,\dots,t)$ *is equal. Without loss of generality we can assume that for* $1\leqq\theta_1<\theta_1+l_1\leqq\theta_2<\cdots<\theta_{t-1}+l_{t-1}\leqq\theta_t\leqq m$,

$$\begin{aligned}
&\lambda_{\theta_1} = \lambda_{\theta_1+1} = \cdots = \lambda_{\theta_1+l_1-1},\\
&\lambda_{\theta_2} = \lambda_{\theta_2+1} = \cdots = \lambda_{\theta_2+l_2-1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\lambda_{\theta_t} = \lambda_{\theta_t+1} = \cdots = \lambda_{\theta_t+l_t-1}.
\end{aligned} \tag{2.2}$$

In this case, the number of associate classes of the design N *can be reduced from* m *to* $m-\sum_{f=1}^{t} l_f+t$ *by combining its* l_j *associate classes for each* l_j, *if and only if*

$$\text{(i)}\quad \sum_{i=\theta_p}^{\theta_p+l_p-1}\ \sum_{j=\theta_q}^{\theta_q+l_q-1} p^{\theta_u}_{ij} = \sum_{i=\theta_p}^{\theta_p+l_p-1}\ \sum_{j=\theta_q}^{\theta_q+l_q-1} p^{\theta_u+1}_{ij} = \cdots$$

$$\cdots = \sum_{i=\theta_p}^{\theta_p+l_p-1}\ \sum_{j=\theta_q}^{\theta_q+l_q-1} p^{\theta_u+l_u-1}_{ij}$$

$$\textit{for}\ \ u, p, q = 1, 2,\dots, t;$$

$$\text{(ii)}\quad \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u}_{i,q+a_{j-1}-j+1} = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+1}_{i,q+a_{j-1}-j+1} = \cdots$$

$$\cdots = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+l_u-1}_{i,q+a_{j-1}-j+1}$$

$$\textit{for}\ \ j, u = 1, 2,\dots, t;\ \theta_{j-1}-a_{j-2}+j-1 \leqq q \leqq \theta_j-a_{j-1}+j-2; \tag{2.3}$$

$$\text{(iii)}\quad \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u}_{i,q+a_t-t} = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+1}_{i,q+a_t-t} = \cdots$$

$$\cdots = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+l_u-1}_{i,q+a_t-t}$$

$$\textit{for}\ \ j, u = 1, 2,\dots, t;\ \theta_t-a_{t-1}+t \leqq q \leqq m-a_t+t;$$

$$\text{(iv)}\quad p^{\theta_u}_{\alpha\beta} = p^{\theta_u+1}_{\alpha\beta} = \cdots = p^{\theta_u+l_u-1}_{\alpha\beta} \qquad \textit{for}\ \ u=1, 2,\dots, t,$$

$$1 \leqq \alpha,\ \beta \notin \bigcup_{i=1}^{t} \{\theta_i, \theta_i+1, \ldots, \theta_i+l_i-1\} \leqq m,$$

where the notation $\alpha \notin \bigcup_{i=1}^{t} \{\theta_i, \theta_i+1\}$ *means that an integer* α *does not belong to the set* $\{\theta_1, \theta_1+1, \theta_2, \theta_2+1, \ldots, \theta_t, \theta_t+1\}$;

(v) *the above conditions* (i), (ii), (iii) *and* (iv) *remain true under any permutation of two subscripts of* p^i_{jk}, *where* $a_s = \sum_{f=1}^{s} l_f$, $s = j-2, j-1, t-1, t$.

Furthermore, if (2.3) *holds, then the parameters of the reduced* **PBIB** *design with* $m - \sum_{f=1}^{t} l_f + t$ *associate classes are as follows:*

$$v' = v, \quad b' = b, \quad r' = r, \quad k' = k,$$

$$\lambda'_{\theta_j - a_{j-1}+j-1} = \lambda_{\theta_j} = \lambda_{\theta_j+1} = \cdots = \lambda_{\theta_j+l_j-1},$$

$$\lambda'_i = \begin{cases} \lambda_{i+a_{j-1}-j+1} & \text{if } \ \theta_{j-1} - a_{j-2}+j-1 \leqq i \leqq \theta_j - a_{j-1}+j-2, \\ \lambda_{i+a_t-t} & \text{if } \ \theta_t - a_{t-1}+t \leqq i \leqq m - a_t + t, \end{cases}$$

$$n'_{\theta_j - a_{j-1}+j-1} = n_{\theta_j} + n_{\theta_j+1} + \cdots + n_{\theta_j+l_j-1},$$

$$n'_i = \begin{cases} n_{i+a_{j-1}-j+1} & \text{if } \ \theta_{j-1} - a_{j-2}+j-1 \leqq i \leqq \theta_j - a_{j-1}+j-2, \\ n_{i+a_t-t} & \text{if } \ \theta_t - a_{t-1}+t \leqq i \leqq m - a_t + t, \end{cases}$$

for $j = 1, 2, \ldots, t$. $\|p'^x_{yz}\|$ *can be written in a form similar to that of Lemma* 2.1 *and hence omitted here.*

Though Lemma 2.1 is a special case of Lemma 2.2 when $\theta_1 = 1$, $\theta_2 = l+1$, $\theta_3 = l+2, \ldots, \theta_t = l+t-1$; $l_1 = l$, $l_2 = 1, \ldots, l_t = 1$, it has been written especially because of its frequent use in the subsequent sections. We shall refer to the conditions in Lemmas 2.1 and 2.2 as generalized Vartak's condition.

3. PBIB designs validating necessary and sufficient conditions for reductions

DESIGN (I). When N_i are **BIB** designs with parameters v_i, b_i, r_i, k_i and λ_i, $i = 1, 2, 3$, we consider the 7-associate **PBIB** design $N = N_1 \otimes N_2 \otimes N_3$ based on an F_3 type association scheme given in Kageyama [23] with the following parameters:

$$v' = v_1 v_2 v_3, \quad b' = b_1 b_2 b_3, \quad r' = r_1 r_2 r_3, \quad k' = k_1 k_2 k_3,$$

$$\lambda'_1 = r_1 \lambda_2 r_3, \quad \lambda'_2 = \lambda_1 r_2 r_3, \quad \lambda'_3 = \lambda_1 \lambda_2 r_3, \quad \lambda'_4 = r_1 r_2 \lambda_3,$$

$$\lambda'_5 = r_1 \lambda_2 \lambda_3, \quad \lambda'_6 = \lambda_1 r_2 \lambda_3, \quad \lambda'_7 = \lambda_1 \lambda_2 \lambda_3, \quad n_1 = v_2 - 1,$$

$$n_2 = v_1 - 1, \quad n_3 = (v_1 - 1)(v_2 - 1), \quad n_4 = v_3 - 1, \quad n_5 = (v_2 - 1)(v_3 - 1),$$

$$n_6 = (v_3 - 1)(v_1 - 1), \quad n_7 = (v_1 - 1)(v_2 - 1)(v_3 - 1),$$

$$\|p_{ij}^1\| = \begin{pmatrix} v_2 - 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & v_1 - 1 & 0 & 0 & 0 & 0 \\ & & (v_1 - 1)(v_2 - 2) & 0 & 0 & 0 & 0 \\ & & & 0 & v_3 - 1 & 0 & 0 \\ & \text{Sym.} & & & (v_3 - 1)(v_2 - 2) & 0 & 0 \\ & & & & & 0 & (v_1 - 1)(v_3 - 1) \\ & & & & & & (v_1 - 1)(v_2 - 2)(v_3 - 1) \end{pmatrix},$$

$$\|p_{ij}^2\| = \begin{pmatrix} 0 & 0 & v_2 - 1 & 0 & 0 & 0 & 0 \\ & v_1 - 2 & 0 & 0 & 0 & 0 & 0 \\ & & (v_1 - 2)(v_2 - 1) & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & v_3 - 1 & 0 \\ & \text{Sym.} & & & 0 & 0 & (v_2 - 1)(v_3 - 1) \\ & & & & & (v_1 - 2)(v_3 - 1) & 0 \\ & & & & & & (v_1 - 2)(v_2 - 1)(v_3 - 1) \end{pmatrix},$$

$$\|p_{ij}^3\| = \begin{pmatrix} 0 & 1 & v_2 - 2 & 0 & 0 & 0 & 0 \\ & 0 & v_1 - 2 & 0 & 0 & 0 & 0 \\ & & (v_1 - 2)(v_2 - 2) & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & v_3 - 1 \\ & \text{Sym.} & & & 0 & v_3 - 1 & (v_2 - 2)(v_3 - 1) \\ & & & & & 0 & (v_1 - 2)(v_3 - 1) \\ & & & & & & (v_1 - 2)(v_2 - 2)(v_3 - 1) \end{pmatrix},$$

$$\|p_{ij}^4\| = \left(\begin{matrix} 0 & 0 & 0 & 0 & v_2 - 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & v_1 - 1 & 0 \\ & & 0 & 0 & 0 & 0 & (v_1 - 1)(v_2 - 1) \end{matrix}\right.$$

$$
\left.\begin{array}{cccc}
v_3-2 & 0 & 0 & 0 \\
 & (v_2-1)(v_3-2) & 0 & 0 \\
\text{Sym.} & & (v_1-1)(v_3-2) & 0 \\
 & & & (v_1-1)(v_2-1)(v_3-2)
\end{array}\right),
$$

$$
\|p^5_{ij}\| = \begin{pmatrix}
0 & 0 & 0 & 1 & v_2-2 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & v_1-1 \\
 & & 0 & 0 & 0 & v_1-1 & (v_1-1)(v_2-2) \\
 & & & 0 & v_3-2 & 0 & 0 \\
 & \text{Sym.} & & & (v_2-2)(v_3-2) & 0 & 0 \\
 & & & & & 0 & (v_1-1)(v_3-2) \\
 & & & & & & (v_1-1)(v_2-2)(v_3-2)
\end{pmatrix},
$$

$$
\|p^6_{ij}\| = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & v_2-1 \\
 & 0 & 0 & 1 & 0 & v_1-2 & 0 \\
 & & 0 & 0 & v_2-1 & 0 & (v_1-2)(v_2-1) \\
 & & & 0 & 0 & v_3-2 & 0 \\
 & & & & 0 & 0 & (v_2-1)(v_3-2) \\
 & \text{Sym.} & & & & (v_1-2)(v_3-2) & 0 \\
 & & & & & & (v_1-2)(v_2-1)(v_3-2)
\end{pmatrix},
$$

$$
\|p^7_{ij}\| = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & v_2-2 \\
 & 0 & 0 & 0 & 1 & 0 & v_1-2 \\
 & & 0 & 1 & v_2-2 & v_1-2 & (v_1-2)(v_2-2) \\
 & & & 0 & 0 & 0 & v_3-2 \\
 & & & & 0 & v_3-2 & (v_2-2)(v_3-2) \\
 & & \text{Sym.} & & & 0 & (v_1-2)(v_3-2) \\
 & & & & & & (v_1-2)(v_2-2)(v_3-2)
\end{pmatrix},
$$

where $v_i \geqq 2$, $i=1, 2, 3$.

In this design, it follows from the form of λ'_i $(i=1,2,\ldots,7)$ and Lemma 2.1 that by assuming a relation $\lambda'_i=\lambda'_j$ $(i\neq j;\ =1,2,4$ or $3,5,6)$ only, the design N is not reducible to a 6-associate **PBIB** design. Thus, every combination of two associate classes in the sense of Vartak does not lead to a reduced **PBIB** design, so that Vartak's iterative procedure is impossible. Some other combinations of associate classes, however, lead to reduced **PBIB** designs under certain restrictions. It follows from Lemmas 2.1 and 2.2 that all the cases of reductions are as follows:

(i) When $\lambda'_1=\lambda'_2$ and $\lambda'_5=\lambda'_6$, if $v_1=v_2$, (ii) when $\lambda'_1=\lambda'_4$ and $\lambda'_3=\lambda'_6$, if $v_2=v_3$, or (iii) when $\lambda'_2=\lambda'_4$ and $\lambda'_3=\lambda'_5$, if $v_3=v_1$, then the design is reducible to 5-associate **PBIB** designs and vice versa. (iv) When $\lambda'_1=\lambda'_2=\lambda'_4$ and $\lambda'_3=\lambda'_5=\lambda'_6$, if $v_1=v_2=v_3$, then the design is reducible to a 3-associate **PBIB** design with the cubic association scheme [46] and vice versa.

DESIGN (II). Consider a 5-associate **PBIB** design based on the hypercubic association scheme [34] (or the C_5 type association scheme which will be described in Section 9) with the following parameters:

$$v = s^5,\quad b,\ r,\ k,\ \lambda_i\ (i=1,2,\ldots,5),\quad n_1 = 5(s-1),$$

$$n_2 = 10(s-1)^2,\quad n_3 = 10(s-1)^3,\quad n_4 = 5(s-1)^4,\quad n_5 = (s-1)^5,$$

$$\|p^1_{ij}\| = \begin{pmatrix} s-2 & 4(s-1) & 0 & 0 & 0 \\ & 4(s-1)(s-2) & 6(s-1)^2 & 0 & 0 \\ & & 6(s-1)^2(s-2) & 4(s-1)^3 & 0 \\ & \text{Sym.} & & 4(s-1)^3(s-2) & (s-1)^4 \\ & & & & (s-1)^4(s-2) \end{pmatrix},$$

$$\|p^2_{ij}\| = \begin{pmatrix} 2 & 2(s-2) & 3(s-1) & 0 & 0 \\ & (s-2)^2+6(s-1) & 6(s-1)(s-2) & 3(s-1)^2 & 0 \\ & & 3(s-1)(s-2)^2+6(s-1)^2 & 6(s-1)^2(s-2) & (s-1)^3 \\ & \text{Sym.} & & 3(s-1)^2(s-2)^2+2(s-1)^3 & 2(s-1)^3(s-2) \\ & & & & (s-1)^3(s-2)^2 \end{pmatrix},$$

$$\|p^3_{ij}\| = \begin{pmatrix} 0 & 3 & 3(s-2) & 2(s-1) & 0 \\ & 6(s-2) & 3(s-2)^2+6(s-1) & 6(s-1)(s-2) & (s-1)^2 \\ & & (s-2)^3+12(s-1)(s-2) & 6(s-1)(s-2)^2+3(s-1)^2 & 3(s-1)^2(s-2) \\ & \text{Sym.} & & 2(s-1)(s-2)^3+6(s-1)^2(s-2) & 3(s-1)^2(s-2)^2 \\ & & & & (s-1)^2(s-2)^3 \end{pmatrix},$$

$$\|p_{ij}^4\| = \begin{bmatrix} 0 & 0 & 4 & 4(s-2) & s-1 \\ & 6 & 12(s-2) & 6(s-2)^2+4(s-1) & 4(s-1)(s-2) \\ & & 12(s-2)^2+6(s-1) & 4(s-2)^3+12(s-1)(s-2) & 6(s-1)(s-2)^2 \\ & \text{Sym.} & & (s-2)^4+12(s-1)(s-2)^2 & 4(s-1)(s-2)^3 \\ & & & & (s-1)(s-2)^4 \end{bmatrix},$$

$$\|p_{ij}^5\| = \begin{bmatrix} 0 & 0 & 0 & 5 & 5(s-2) \\ & 0 & 10 & 20(s-2) & 10(s-2)^2 \\ & & 30(s-2) & 30(s-2)^2 & 10(s-2)^3 \\ & \text{Sym.} & & 20(s-2)^3 & 5(s-2)^4 \\ & & & & (s-2)^5 \end{bmatrix},$$

where $s \geqq 2$.

In this design, it follows from Lemmas 2.1 and 2.2 that this design is not reducible to a 4-associate **PBIB** design. Thus, every combination of two associate classes in the sense of Vartak does not lead to a reduced **PBIB** design, so that Vartak's iterative procedure is impossible. Some other combinations of associate classes, however, lead to reduced **PBIB** designs under certain restrictions. For example, it follows from Lemmas 2.1 and 2.2 that when (i) $\lambda_1=\lambda_3=\lambda_5$, (ii) $\lambda_1=\lambda_2$ and $\lambda_3=\lambda_4$, (iii) $\lambda_1=\lambda_3$ and $\lambda_2=\lambda_4$, (iv) $\lambda_1=\lambda_4$ and $\lambda_2=\lambda_3$, or (v) $\lambda_1=\lambda_5$ and $\lambda_2=\lambda_4$, if $s=2$, then the design is reducible to 3-associate **PBIB** designs and vice versa. (vi) When $\lambda_1=\lambda_4$ and $\lambda_2=\lambda_5$, if $s=3$, then the design is reducible to a 3-associate **PBIB** design and vice versa. (vii) When $\lambda_1=\lambda_2=\lambda_3=\lambda_4$, if $s=2$, or (viii) when $\lambda_2=\lambda_4$ and $\lambda_1=\lambda_3=\lambda_5$, if $s=4$, then the design is reducible to 2-associate **PBIB** designs and vice versa.

DESIGN (III). Consider a 4-associate **PBIB** design based on the association scheme given by Adhikary [1] with the following parameters:

$$v = m_1(m_2+1)(m_3+1), \qquad b,\ r,\ k,\ \lambda_i\ (i=1,2,3,4),$$

$$n_1 = m_2, \quad n_2 = m_3, \quad n_3 = m_2 m_3,$$

$$n_4 = (m_1-1)(m_2+1)(m_3+1),$$

$$\|p_{ij}^1\| = \begin{bmatrix} m_2-1 & 0 & 0 & 0 \\ & 0 & m_3 & 0 \\ & & m_3(m_2-1) & 0 \\ & \text{Sym.} & & (m_1-1)(m_2+1)(m_3+1) \end{bmatrix},$$

$$\|p_{ij}^2\| = \begin{pmatrix} 0 & 0 & m_2 & 0 \\ & m_3-1 & 0 & 0 \\ & & m_2(m_3-1) & 0 \\ & \text{Sym.} & & (m_1-1)(m_2+1)(m_3+1) \end{pmatrix},$$

$$\|p_{ij}^3\| = \begin{pmatrix} 0 & 1 & m_2-1 & 0 \\ & 0 & m_3-1 & 0 \\ & \text{Sym.} & (m_2-1)(m_3-1) & 0 \\ & & & (m_1-1)(m_2+1)(m_3+1) \end{pmatrix},$$

$$\|p_{ij}^4\| = \begin{pmatrix} 0 & 0 & 0 & m_2 \\ & 0 & 0 & m_3 \\ & \text{Sym.} & 0 & m_2 m_3 \\ & & & (m_1-2)(m_2+1)(m_3+1) \end{pmatrix},$$

where $m_1 \geqq 2$.

All the cases of reductions in this design are as follows:

(i) When $\lambda_1 = \lambda_2$, if $m_2 = m_3$, then the design is reducible to a 3-associate **PBIB** design and vice versa. (ii) When $\lambda_1 = \lambda_3$, or (iii) when $\lambda_2 = \lambda_3$, the design is reducible to 3-associate **PBIB** designs. When (iv) $\lambda_1 = \lambda_2 = \lambda_3$, (v) $\lambda_1 = \lambda_3 = \lambda_4$, or (vi) $\lambda_2 = \lambda_3 = \lambda_4$, the design is reducible to 2-associate **PBIB** designs.

This design has an interesting property, which is that Vartak's iterative procedure depends on the order of combining some associate classes. For example, in Case (v), though we cannot apply Vartak's procedure in combining the 4th associate class and another associate class, we can apply Vartak's iterative procedure in combining the 1st associate class and the 3rd associate class, and then combining the 4th associate class.

4. Relationship among coincidence numbers, latent roots and second kind parameters with respect to reductions

On the derivation of conditions for the reduction of associate classes for certain **PBIB** designs, Vartak's approach [58] (i.e., generalized Vartak's condition given in Section 2) uses the coincidence numbers and the second kind of parameters of the **PBIB** design N, while Kageyama's approach [23] uses the coincidence numbers and the latent roots of the matrix NN'. As a necessary and sufficient condition for reductions in certain cases, Kageyama's condition is more practically useful than generalized Vartak's one. If the two conditions are equivalent, then we may be encouraged to use Kageyama's condition in preference to Vartak's. For this reason, the relation between generalized Vartak's condition and Kageyama's one is generally studied through the properties of latent roots of the matrix $\mathfrak{P}_k = \|p^i_{jk}\|$ for $k=0, 1,\ldots, m$.

4.1. Reductions for PBIB designs of Kronecker product type

Let N_i be **BIB** designs with parameters v_i, b_i, r_i, k_i, λ_i and N_i^* be complementary **BIB** designs with parameters $v_i^* = v_i$, $b_i^* = b_i$, $r_i^* = b_i - r_i$, $k_i^* = v_i - k_i$ and $\lambda_i^* = b_i - 2r_i + \lambda_i$ of N_i $(i=1, 2,\ldots, m)$. Consider the Kronecker product of these designs in the forms $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ and $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$. Then the following theorems are obtained:

THEOREM A (Kageyama [23]). *Given the* **BIB** *designs N_i with parameters v_i, b_i, r_i, k_i and λ_i $(i=1,2,\ldots, m)$, a necessary and sufficient condition for the Kronecker product* **PBIB** *design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$, which has at most $2^m - 1$ associate classes having the F_m type association scheme, to be reducible to a* **PBIB** *design with only m distinct associate classes having the hypercubic association scheme is that*

$$v_1 = v_2 = \cdots = v_m, \qquad k_1 = k_2 = \cdots = k_m. \tag{4.1}$$

THEOREM B (Kageyama [23]). *Given the* **BIB** *designs N_i with parameters v_i, b_i, r_i, k_i and λ_i $(i=1, 2)$, a necessary and sufficient condition for a* **PBIB** *design which has at most three associate classes having the rectangular association scheme and which is constructed by the Kronecker product $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ to be reducible to a* **PBIB** *design with only two distinct associate classes having the L_2 association scheme is that*

$$v_1 = v_2, \quad b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1), \quad b_i \neq 4(r_i - \lambda_i), \quad i = 1, 2.$$

In the derivation of Theorem A, condition (4.1) was obtained by equalizing all those among the latent roots of NN' and among all the coincidence numbers which may be equal to each other. On the other hand, the matrices $P_i = \|p^i_{jk}\|$ of the second kind of parameters of Kronecker product **PBIB** design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ can be constructed by repeated applications of Theorem 4.2 of Vartak [58] (or by Theorem 2 of Surendran [54] or (9.2)).

In design N, when all those which may be equal to each other among all the coincidence numbers are set to be equal, a necessary and sufficient condition for the **PBIB** design N which has at most $2^m - 1$ associate classes to be reducible to a **PBIB** design with only m distinct associate classes is given by Lemma 2.2.

Furthermore, from Lemma 2.2 and the method of constructing the matrices P_i, it follows that this necessary and sufficient condition is equivalent to $v_1 = v_2 = \cdots = v_m$. Then from $v_1 = v_2 = \cdots = v_m$ and the fact that all those which may be equal to each other among all the coincidence numbers are set to be equal, it is clear that all those which may be equal to each other among the latent roots of the matrix NN' become equal. Therefore from Theorem A we have

THEOREM 4.1. *Vartak's and Kageyama's condition are equivalent for Kronecker product* **PBIB** *design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ which is reducible to a* **PBIB** *design with only m distinct associate classes.*

As an illustration, leaving aside certain trivial or uninteresting cases, the case $m=3$ is considered. In design $N = N_1 \otimes N_2 \otimes N_3$ with parameters $v' = v_1 v_2 v_3$, $b' = b_1 b_2 b_3$, $r' = r_1 r_2 r_3$, $k' = k_1 k_2 k_3$, $\lambda_1' = r_1 \lambda_2 r_3$, $\lambda_2' = \lambda_1 r_2 r_3$, $\lambda_3' = \lambda_1 \lambda_2 r_3$, $\lambda_4' = r_1 r_2 \lambda_3$, $\lambda_5' = r_1 \lambda_2 \lambda_3$, $\lambda_6' = \lambda_1 r_2 \lambda_3$, $\lambda_7' = \lambda_1 \lambda_2 \lambda_3$, $n_1 = v_2 - 1$, $n_2 = v_1 - 1$, $n_3 = (v_1 - 1)(v_2 - 1)$, $n_4 = v_3 - 1$, $n_5 = (v_2 - 1)(v_3 - 1)$, $n_6 = (v_3 - 1)(v_1 - 1)$ and $n_7 = (v_1 - 1)(v_2 - 1)(v_3 - 1)$, the relations obtained by equalizing all those among the latent roots of NN' and among all the coincidence numbers which may be equal to each other are as follows (cf. [23]):

(among the coincidence numbers)

$$r_1 \lambda_2 = r_2 \lambda_1, \quad r_2 \lambda_3 = r_3 \lambda_2, \quad r_1 \lambda_3 = r_3 \lambda_1, \tag{4.2}$$

(among the latent roots)

$$r_1 k_1 (r_2 - \lambda_2) = r_2 k_2 (r_1 - \lambda_1), \quad r_2 k_2 (r_3 - \lambda_3) = r_3 k_3 (r_2 - \lambda_2), \tag{4.3}$$
$$r_1 k_1 (r_3 - \lambda_3) = r_3 k_3 (r_1 - \lambda_1).$$

The matrices $P_i = \| p_{jk}^i \|$ $(i, j, k = 1, 2, \ldots, 7)$ are shown in Design (I) of Section 3. As shortly mentioned there, since under (4.2) we have distinct coincidence numbers $\lambda_1' = \lambda_2' = \lambda_4'$, $\lambda_3' = \lambda_5' = \lambda_6'$ and λ_7', from Lemma 2.2 a necessary and sufficient condition that the **PBIB** design N with at most seven associate classes is reducible to a **PBIB** design with only three distinct associate classes is that

(a) $$\sum_{i,j=1,2,4} p_{ij}^1 = \sum_{i,j} p_{ij}^2 = \sum_{i,j} p_{ij}^4, \quad \sum_{i,j=1,2,4} p_{ij}^3 = \sum_{i,j} p_{ij}^5 = \sum_{i,j} p_{ij}^6,$$

(b) $$\sum_{\substack{i=1,2,4\\ j=3,5,6}} p_{ij}^1 = \sum_{i,j} p_{ij}^2 = \sum_{i,j} p_{ij}^4, \quad \sum_{\substack{i=1,2,4\\ j=3,5,6}} p_{ij}^3 = \sum_{i,j} p_{ij}^5 = \sum_{i,j} p_{ij}^6,$$

(c) $$\sum_{i=1,2,4} p_{i7}^1 = \sum_{i} p_{i7}^2 = \sum_{i} p_{i7}^4, \quad \sum_{i=1,2,4} p_{i7}^3 = \sum_{i} p_{i7}^5 = \sum_{i} p_{i7}^6,$$

(d) $$\sum_{i,j=3,5,6} p_{ij}^1 = \sum_{i,j} p_{ij}^2 = \sum_{i,j} p_{ij}^4, \quad \sum_{i,j=3,5,6} p_{ij}^3 = \sum_{i,j} p_{ij}^5 = \sum_{i,j} p_{ij}^6,$$

(e) $\sum_{i=3,5,6} p_{i7}^{1} = \sum_{i} p_{i7}^{2} = \sum_{i} p_{i7}^{4}, \quad \sum_{i=3,5,6} p_{i7}^{3} = \sum_{i} p_{i7}^{5} = \sum_{i} p_{i7}^{6},$

(f) $p_{77}^{1} = p_{77}^{2} = p_{77}^{4}, \quad p_{77}^{3} = p_{77}^{5} = p_{77}^{6}.$

Substituting the elements of P_i ($i=1, 2,\dots, 7$) into the above conditions (a) to (f), it is clear that conditions (a) to (f) are equivalent to a condition

(4.4) $$v_1 = v_2 = v_3.$$

Now if (4.2) and (4.3) are satisfied, then from the relations among the parameters of the **BIB** designs, i.e., $\lambda_i(v_i-1)=r_i(k_i-1)$, $i=1, 2, 3$, we can obtain $v_1=v_2=v_3$ and $k_1=k_2=k_3$, and hence (4.4) holds. On the other hand, it follows from $\lambda_i(v_i-1)=r_i(k_i-1)$, $i=1, 2, 3$ that if (4.2) and (4.4) are satisfied, then $r_i\lambda_j=r_j\lambda_i$ and $v_i=v_j$ lead to $r_ik_i(r_j-\lambda_j)=r_jk_j(r_i-\lambda_i)$ for i, j $(i\neq j)=1, 2, 3$ and hence (4.3) holds. Therefore generalized Vartak's condition obtained by using the coincidence numbers and the second kind of parameters of **PBIB** design N are equivalent to Kageyama's one obtained by using the coincidence numbers and the latent roots of the matrix NN' for **PBIB** design $N=N_1\otimes N_2\otimes N_3$.

In a similar way, from the derivation of Theorem B we have

THEOREM 4.2. *Vartak's and Kageyama's condition are equivalent for* **PBIB** *design* $N=N_1\otimes N_2+N_1^*\otimes N_2^*$ *which is reducible to a* **PBIB** *design with only two distinct associate classes.*

4.2. Reductions for general PBIB designs

When there exists equality relation (2.2) among coincidence numbers λ_i of a **PBIB** design N with m associate classes, a necessary and sufficient condition for the **PBIB** design N to be reducible to a **PBIB** design with $m-\sum_{f=1}^{t} l_f+t$ associate classes is given by (2.3) in Lemma 2.2. Then it is clear that there are the equality relations among latent roots ρ_i of the matrix NN' corresponding to its coincidence numbers λ_i. Thus generalized Vartak's condition always leads to Kageyama's one. Therefore in the rest of this section, the converse of this fact will be discussed. That is, do the relations among coincidence numbers λ_i and among latent roots ρ_i (i.e., Kageyama's condition) lead to the relations among coincidence numbers λ_i and among the second kind of parameters like (2.3) (i.e., generalized Vartak's condition)?

Since it is sufficient to consider the form (2.2) as equality relations among coincidence numbers λ_i, the equality relations among latent roots ρ_i of the matrix NN' $(=\sum_{j=0}^{m}\lambda_jA_j=\sum_{i=0}^{m}\rho_iA_i^\#$ in Lemma A) corresponding to its coincidence numbers λ_i are also considered as the assumption. We now begin by separating this into two cases (i.e., $t=1$ and $t\geqq 2$ in (2.2)).

Case I (type $t=1$ in (2.2)), i.e., the case of Lemma 2.1. The conditions to be assumed are $\lambda_1=\lambda_2=\cdots=\lambda_l$ and $\rho_1=\rho_2=\cdots=\rho_l$.

(i) Case $l=m$: In this case it suffices to assume $\lambda_1=\lambda_2=\cdots=\lambda_m$ only. For, this condition leads to $\rho_1=\rho_2=\cdots=\rho_m$ by (1.3), (1.4) and (1.8). From (1.3) and (1.4) we have

$$\sum_{j=1}^{m} z_{1j}=\sum_{j=1}^{m} z_{2j}=\cdots=\sum_{j=1}^{m} z_{mj}. \tag{4.5}$$

Furthermore, from (1.3) and (1.5) we have

$$(\sum_{j=1}^{m} z_{1j})(\sum_{j=1}^{m} z_{1j})=\sum_{i=1}^{m} n_i+(\sum_{i,j=1}^{m} p_{ij}^1)z_{11}+(\sum_{i,j} p_{ij}^2)z_{12}+\cdots+(\sum_{i,j} p_{ij}^m)z_{1m},$$

$$(\sum_{j=1}^{m} z_{2j})(\sum_{j=1}^{m} z_{2j})=\sum_{i=1}^{m} n_i+(\sum_{i,j=1}^{m} p_{ij}^1)z_{21}+(\sum_{i,j} p_{ij}^2)z_{22}+\cdots+(\sum_{i,j} p_{ij}^m)z_{2m},$$

....

$$(\sum_{j=1}^{m} z_{mj})(\sum_{j=1}^{m} z_{mj})=\sum_{i=1}^{m} n_i+(\sum_{i,j=1}^{m} p_{ij}^1)z_{m1}+(\sum_{i,j} p_{ij}^2)z_{m2}+\cdots+(\sum_{i,j} p_{ij}^m)z_{mm},$$

which from (4.5) lead to

$$(z_{12}-z_{22})(\sum_{i,j=1}^{m} p_{ij}^2-\sum_{i,j=1}^{m} p_{ij}^1)+(z_{13}-z_{23})(\sum_{i,j} p_{ij}^3-\sum_{i,j} p_{ij}^1)+\cdots$$
$$\cdots+(z_{1m}-z_{2m})(\sum_{i,j} p_{ij}^m-\sum_{i,j} p_{ij}^1)=0,$$

$$(z_{12}-z_{32})(\sum_{i,j=1}^{m} p_{ij}^2-\sum_{i,j=1}^{m} p_{ij}^1)+(z_{13}-z_{33})(\sum_{i,j} p_{ij}^3-\sum_{i,j} p_{ij}^1)+\cdots$$
$$\cdots+(z_{1m}-z_{3m})(\sum_{i,j} p_{ij}^m-\sum_{i,j} p_{ij}^1)=0,$$

....

$$(z_{12}-z_{m2})(\sum_{i,j=1}^{m} p_{ij}^2-\sum_{i,j=1}^{m} p_{ij}^1)+(z_{13}-z_{m3})(\sum_{i,j} p_{ij}^3-\sum_{i,j} p_{ij}^1)+\cdots$$
$$\cdots+(z_{1m}-z_{mm})(\sum_{i,j} p_{ij}^m-\sum_{i,j} p_{ij}^1)=0.$$

Since the matrix Z is nonsingular as described in Section 1, it follows that the determinant

$$\begin{vmatrix} z_{12}-z_{22}, & z_{13}-z_{23},\dots, z_{1m}-z_{2m} \\ z_{12}-z_{32}, & z_{13}-z_{33},\dots, z_{1m}-z_{3m} \\ \dots\dots & \dots\dots \\ z_{12}-z_{m2}, & z_{13}-z_{m3},\dots, z_{1m}-z_{mm} \end{vmatrix} \neq 0.$$

Therefore we obtain $\sum_{i,j=1}^{m} p_{ij}^1 = \sum_{i,j=1}^{m} p_{ij}^2 = \cdots = \sum_{i,j=1}^{m} p_{ij}^m$ which coincides with (2.1) in this case. That is, Kageyama's condition leads to generalized Vartak's one in this type. Note that if $\lambda_1=\lambda_2=\cdots=\lambda_m$ holds in a **PBIB** design with m associate classes, then the **PBIB** design reduces to a **BIB** design. This is well known.

(ii) Case $l=m-1$: $\lambda_1=\lambda_2=\cdots=\lambda_{m-1}$ ($=\lambda$, say); $\rho_1=\rho_2=\cdots=\rho_{m-1}$. From (1.3), (1.4) and (1.8) we have

$$\rho_1 = \lambda_0-\lambda-z_{1m}(\lambda-\lambda_m),$$

$$\rho_2 = \lambda_0-\lambda-z_{2m}(\lambda-\lambda_m),$$

$$\dots\dots\dots\dots$$

$$\rho_{m-1} = \lambda_0-\lambda-z_{m-1,m}(\lambda-\lambda_m).$$

Hence from $\rho_1=\rho_2=\cdots=\rho_{m-1}$ and $\lambda-\lambda_m\neq 0$ we have

$$z_{1m} = z_{2m} = \cdots = z_{m-1,m}, \tag{4.6}$$

$$\sum_{j=1}^{m-1} z_{1j} = \sum_{j=1}^{m-1} z_{2j} = \cdots = \sum_{j=1}^{m-1} z_{m-1,j}. \tag{4.7}$$

Furthermore, from (1.3) and (1.5) we have

$$(\sum_{j=1}^{m-1} z_{1j})(\sum_{j=1}^{m-1} z_{1j}) = \sum_{i=1}^{m-1} n_i + (\sum_{i,j=1}^{m-1} p_{ij}^1)z_{11} + (\sum_{i,j} p_{ij}^2)z_{12} + \cdots + (\sum_{i,j} p_{ij}^m)z_{1m},$$

$$(\sum_{j=1}^{m-1} z_{2j})(\sum_{j=1}^{m-1} z_{2j}) = \sum_{i=1}^{m-1} n_i + (\sum_{i,j=1}^{m-1} p_{ij}^1)z_{21} + (\sum_{i,j} p_{ij}^2)z_{22} + \cdots + (\sum_{i,j} p_{ij}^m)z_{2m},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(\sum_{j=1}^{m-1} z_{m-1,j})(\sum_{j=1}^{m-1} z_{m-1,j}) = \sum_{i=1}^{m-1} n_i + (\sum_{i,j=1}^{m-1} p_{ij}^1)z_{m-1,1} + (\sum_{i,j} p_{ij}^2)z_{m-1,2} + \cdots + (\sum_{i,j} p_{ij}^m)z_{m-1,m},$$

which from (4.6) and (4.7) lead to

$$(z_{12}-z_{22})(\sum_{i,j=1}^{m-1} p_{ij}^2-\sum_{i,j=1}^{m-1} p_{ij}^1)+(z_{13}-z_{23})(\sum_{i,j} p_{ij}^3-\sum_{i,j} p_{ij}^1)+$$
$$\cdots+(z_{1,m-1}-z_{2,m-1})(\sum_{i,j} p_{ij}^{m-1}-\sum_{i,j} p_{ij}^1)=0,$$

$$(z_{12}-z_{32})(\sum_{i,j=1}^{m-1} p_{ij}^2-\sum_{i,j=1}^{m-1} p_{ij}^1)+(z_{13}-z_{33})(\sum_{i,j} p_{ij}^3-\sum_{i,j} p_{ij}^1)+$$
$$\cdots+(z_{1,m-1}-z_{3,m-1})(\sum_{i,j} p_{ij}^{m-1}-\sum_{i,j} p_{ij}^1)=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(z_{12}-z_{m-1,2})(\sum_{i,j=1}^{m-1} p_{ij}^2-\sum_{i,j=1}^{m-1} p_{ij}^1)+(z_{13}-z_{m-1,3})(\sum_{i,j} p_{ij}^3-\sum_{i,j} p_{ij}^1)+$$
$$\cdots+(z_{1,m-1}-z_{m-1,m-1})(\sum_{i,j} p_{ij}^{m-1}-\sum_{i,j} p_{ij}^1)=0\,.$$

Since it follows from the property of the matrix Z that

$$\text{(4.8)}\qquad \begin{vmatrix} z_{12}-z_{22}, & z_{13}-z_{23},\ldots, & z_{1,m-1}-z_{2,m-1} \\ z_{12}-z_{32}, & z_{13}-z_{33},\ldots, & z_{1,m-1}-z_{3,m-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ z_{12}-z_{m-1,2}, & z_{13}-z_{m-1,3},\ldots, & z_{1,m-1}-z_{m-1,m-1} \end{vmatrix} \neq 0\,,$$

we obtain

$$\text{(4.9)}\qquad \sum_{i,j=1}^{m-1} p_{ij}^1=\sum_{i,j=1}^{m-1} p_{ij}^2=\cdots=\sum_{i,j=1}^{m-1} p_{ij}^{m-1}\,.$$

From (1.5) we have

$$(\sum_{j=1}^{m-1} z_{1j})z_{1m}=(\sum_{i=1}^{m-1} p_{im}^1)z_{11}+(\sum_i p_{im}^2)z_{12}+\cdots+(\sum_i p_{im}^m)z_{1m}\,,$$

$$(\sum_{j=1}^{m-1} z_{2j})z_{2m}=(\sum_{i=1}^{m-1} p_{im}^1)z_{21}+(\sum_i p_{im}^2)z_{22}+\cdots+(\sum_i p_{im}^m)z_{2m},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\sum_{j=1}^{m-1} z_{m-1,j})z_{m-1,m}=(\sum_{i=1}^{m-1} p_{im}^1)z_{m-1,1}+(\sum_i p_{im}^2)z_{m-1,2}+\cdots+(\sum_i p_{im}^m)z_{m-1,m}.$$

Hence from (4.6), (4.7) and (4.8) we similarly obtain

$$\text{(4.10)}\qquad \sum_{i=1}^{m-1} p_{im}^1=\sum_{i=1}^{m-1} p_{im}^2=\cdots=\sum_{i=1}^{m-1} p_{im}^{m-1}\,.$$

From (1.3) and (1.5) we have

$$z_{1m}z_{1m}-z_{2m}z_{2m}=(z_{11}-z_{21})p_{mm}^{1}+(z_{12}-z_{22})p_{mm}^{2}+\cdots+(z_{1,m-1}-z_{2,m-1})p_{mm}^{m-1},$$

$$z_{1m}z_{1m}-z_{3m}z_{3m}=(z_{11}-z_{31})p_{mm}^{1}+(z_{12}-z_{32})p_{mm}^{2}+\cdots+(z_{1,m-1}-z_{3,m-1})p_{mm}^{m-1},$$

....

$$z_{1m}z_{1m}-z_{m-1,m}z_{m-1,m}=(z_{11}-z_{m-1,m})p_{mm}^{1}+(z_{12}-z_{m-1,2})p_{mm}^{2}+\cdots+(z_{1,m-1}-z_{m-1,m-1})p_{mm}^{m-1}.$$

Hence from (4.6), (4.7) and (4.8) we similarly obtain

$$p_{mm}^{1}=p_{mm}^{2}=\cdots=p_{mm}^{m-1}. \tag{4.11}$$

Conditions (4.9), (4.10) and (4.11) coincide with (2.1) in this case. Therefore Kageyama's condition leads to generalized Vartak's one in this type.

(iii) Case $l=m-2$: $\lambda_1=\lambda_2=\cdots=\lambda_{m-2}$ $(=\lambda$, say); $\rho_1=\rho_2=\cdots=\rho_{m-2}$. From (1.3), (1.4) and (1.8) we have

$$\rho_1=\lambda_0-\lambda-z_{1,m-1}(\lambda-\lambda_{m-1})-z_{1m}(\lambda-\lambda_m),$$

$$\rho_2=\lambda_0-\lambda-z_{2,m-1}(\lambda-\lambda_{m-1})-z_{2m}(\lambda-\lambda_m),$$

....

$$\rho_{m-2}=\lambda_0-\lambda-z_{m-2,m-1}(\lambda-\lambda_{m-1})-z_{m-2,m}(\lambda-\lambda_m).$$

If we suppose a condition

$$z_{1,m-1}=z_{2,m-1}=\cdots=z_{m-2,m-1}, \tag{4.12}$$

then from $\rho_1=\rho_2=\cdots=\rho_{m-2}$ and $\lambda-\lambda_i\neq 0$ $(i=m-1, m)$ we have

$$z_{1m}=z_{2m}=\cdots=z_{m-2,m}, \tag{4.13}$$

$$\sum_{j=1}^{m-2}z_{1j}=\sum_{j=1}^{m-2}z_{2j}=\cdots=\sum_{j=1}^{m-2}z_{m-2,j}. \tag{4.14}$$

From (1.3) and (1.5) we have

$$\Big(\sum_{j=1}^{m-2}z_{1j}\Big)\Big(\sum_{j=1}^{m-2}z_{1j}\Big)=\sum_{i=1}^{m-2}n_i+\Big(\sum_{i,j=1}^{m-2}p_{ij}^{1}\Big)z_{11}+\cdots+\Big(\sum_{i,j=1}^{m-2}p_{ij}^{m}\Big)z_{1m},$$

$$(\sum_{j=1}^{m-2} z_{2j})(\sum_{j=1}^{m-2} z_{2j}) = \sum_{i=1}^{m-2} n_i + (\sum_{i,j=1}^{m-2} p_{ij}^1) z_{21} + \cdots + (\sum_{i,j=1}^{m-2} p_{ij}^m) z_{2m},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\sum_{j=1}^{m-2} z_{m-2,j})(\sum_{j=1}^{m-2} z_{m-2,j}) = \sum_{i=1}^{m-2} n_i + (\sum_{i,j=1}^{m-2} p_{ij}^1) z_{m-2,1} + \cdots + (\sum_{i,j=1}^{m-2} p_{ij}^m) z_{m-2,m},$$

which from (4.12), (4.13) and (4.14) lead to

$$(z_{12} - z_{22})(\sum_{i,j=1}^{m-2} p_{ij}^2 - \sum_{i,j=1}^{m-2} p_{ij}^1) + (z_{13} - z_{23})(\sum_{i,j} p_{ij}^3 - \sum_{i,j} p_{ij}^1) +$$
$$\cdots + (z_{1,m-2} - z_{2,m-2})(\sum_{i,j} p_{ij}^{m-2} - \sum_{i,j} p_{ij}^1) = 0,$$

$$(z_{12} - z_{32})(\sum_{i,j=1}^{m-2} p_{ij}^2 - \sum_{i,j=1}^{m-2} p_{ij}^1) + (z_{13} - z_{33})(\sum_{i,j} p_{ij}^3 - \sum_{i,j} p_{ij}^1) +$$
$$\cdots + (z_{1,m-2} - z_{3,m-2})(\sum_{i,j} p_{ij}^{m-2} - \sum_{i,j} p_{ij}^1) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(z_{12} - z_{m-2,2})(\sum_{i,j=1}^{m-2} p_{ij}^2 - \sum_{i,j=1}^{m-2} p_{ij}^1) + (z_{13} - z_{m-2,3})(\sum_{i,j} p_{ij}^3 - \sum_{i,j} p_{ij}^1) +$$
$$\cdots + (z_{1,m-2} - z_{m-2,m-2})(\sum_{i,j} p_{ij}^{m-2} - \sum_{i,j} p_{ij}^1) = 0.$$

Since it follows from the property of the matrix Z that under (4.12)

$$(4.15) \qquad \begin{vmatrix} z_{12} - z_{22}, & z_{13} - z_{23}, \ldots, & z_{1,m-2} - z_{2,m-2} \\ z_{12} - z_{32}, & z_{13} - z_{33}, \ldots, & z_{1,m-2} - z_{3,m-2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ z_{12} - z_{m-2,2}, & z_{13} - z_{m-2,3}, \ldots, & z_{1,m-2} - z_{m-2,m-2} \end{vmatrix} \neq 0,$$

we obtain

$$(4.16) \qquad \sum_{i,j=1}^{m-2} p_{ij}^1 = \sum_{i,j=1}^{m-2} p_{ij}^2 = \cdots = \sum_{i,j=1}^{m-2} p_{ij}^{m-2}.$$

From (1.5) we have

$$(\sum_{j=1}^{m-2} z_{1j}) z_{1,m-1} = (\sum_{i=1}^{m-2} p_{i,m-1}^1) z_{11} + \cdots + (\sum_{i=1}^{m-2} p_{i,m-1}^m) z_{1m},$$

$$(\sum_{j=1}^{m-2} z_{2j}) z_{2,m-1} = (\sum_{i=1}^{m-2} p_{i,m-1}^1) z_{21} + \cdots + (\sum_{i=1}^{m-2} p_{i,m-1}^m) z_{2m},$$

....

$$(\sum_{j=1}^{m-2} z_{m-2,j})z_{m-2,m-1}=(\sum_{i=1}^{m-2} p_{i,m-1}^{1})z_{m-2,1}+\cdots+(\sum_{i=1}^{m-2} p_{i,m-1}^{m})z_{m-2,m}\,.$$

Hence from (4.12), (4.13), (4.14) and (4.15) we similarly obtain

$$(4.17)\qquad \sum_{i=1}^{m-2} p_{i,m-1}^{1}=\sum_{i=1}^{m-2} p_{i,m-1}^{2}=\cdots=\sum_{i=1}^{m-2} p_{i,m-1}^{m-2}\,.$$

Similarly, from $(\sum_{j=1}^{m-2} z_{1j})z_{1m}=(\sum_{j=1}^{m-2} z_{2j})z_{2m}=\cdots=(\sum_{j=1}^{m-2} z_{m-2,j})z_{m-2,m}$ we obtain

$$(4.18)\qquad \sum_{i=1}^{m-2} p_{im}^{1}=\sum_{i=1}^{m-2} p_{im}^{2}=\cdots=\sum_{i=1}^{m-2} p_{im}^{m-2}\,.$$

Further similarly, from $z_{1,m-1}z_{1,m-1}-z_{2,m-1}z_{2,m-1}=z_{1,m-1}z_{1,m-1}-z_{3,m-1}z_{3,m-1}$ $=\cdots=z_{1,m-1}z_{1,m-1}-z_{m-2,m-1}z_{m-2,m-1}=0,\quad z_{1,m-1}z_{1m}-z_{2,m-1}z_{2m}=z_{1,m-1}z_{1m}$ $-z_{3,m-1}z_{3m}=\cdots=z_{1,m-1}z_{1m}-z_{m-2,m-1}z_{m-2,m}=0$ and $z_{1m}z_{1m}-z_{2m}z_{2m}=z_{1m}z_{1m}$ $-z_{3m}z_{3m}=\cdots=z_{1m}z_{1m}-z_{m-2,m}z_{m-2,m}=0$, we obtain respectively

$$(4.19)\qquad \begin{aligned} &p_{m-1,m-1}^{1}=p_{m-1,m-1}^{2}=\cdots=p_{m-1,m-1}^{m-2}\,,\\ &p_{m-1,m}^{1}=p_{m-1,m}^{2}=\cdots=p_{m-1,m}^{m-2}\,,\\ &p_{mm}^{1}=p_{mm}^{2}=\cdots=p_{mm}^{m-2}\,. \end{aligned}$$

Conditions (4.16), (4.17), (4.18) and (4.19) coincide with (2.1) in this case. Therefore Kageyama's condition leads to generalized Vartak's one in this type provided that (4.12) holds. Note that, though (4.12) is a general assumption, there may be an association scheme satisfying (4.12).

(iv) Case $l=m-q$ $(q\geqq 3)$: $\lambda_1=\lambda_2=\cdots=\lambda_{m-q}$; $\rho_1=\rho_2=\cdots=\rho_{m-q}$. When a positive integer q is equal to 3, if two conditions like (4.12) are assumed, then in a similar way as in Case (iii), we can get conditions like (2.1) corresponding to this case. In general, if $q-1$ conditions like (4.12) are assumed, then the required conditions like (2.1) can be similarly obtained.

Therefore, for the case in which $l=m-q$ $(q\geqq 2)$, Kageyama's condition leads to generalized Vartak's one with some additional assumptions.

Case II (type $t\geqq 2$ in Lemma 2.2). The conditions to be assumed are

$$(4.20)\qquad \begin{aligned} &\lambda_{\theta_1}=\lambda_{\theta_1+1}=\cdots=\lambda_{\theta_1+l_1-1},\quad \rho_{\theta_1}=\rho_{\theta_1+1}=\cdots=\rho_{\theta_1+l_1-1}\,,\\ &\lambda_{\theta_2}=\lambda_{\theta_2+1}=\cdots=\lambda_{\theta_2+l_2-1},\quad \rho_{\theta_2}=\rho_{\theta_2+1}=\cdots=\rho_{\theta_2+l_2-1}\,,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\ &\lambda_{\theta_t}=\lambda_{\theta_t+1}=\cdots=\lambda_{\theta_t+l_t-1},\quad \rho_{\theta_t}=\rho_{\theta_t+1}=\cdots=\rho_{\theta_t+l_t-1}\,. \end{aligned}$$

Since the conditions of each row to hold in (4.20) are of the conditions of types in Case I, from the discussion of (iii) and (iv) in Case I, we can see that in type (4.20) if some conditions like (4.12) are further assumed, then a necessary and sufficient condition for the reduction of associate classes like (2.3) is obtained. Therefore in Case II Kageyama's condition leads to generalized Vartak's one with some additional assumptions like (4.12).

We shall conclude this section by giving an effective example. Consider a **PBIB** design with five associate classes satisfying the conditions such that $\lambda_1 = \lambda_2$ ($=s_1$, say), $\lambda_3 = \lambda_4$ ($=s_2$, say); $\rho_1 = \rho_2$, $\rho_3 = \rho_4$. From (1.3) and (1.8) we have

$$\rho_1 = \lambda_0 + s_1(z_{11}+z_{12}) + s_2(z_{13}+z_{14}) + \lambda_5 z_{15},$$

$$\rho_2 = \lambda_0 + s_1(z_{21}+z_{22}) + s_2(z_{23}+z_{24}) + \lambda_5 z_{25},$$

$$\rho_3 = \lambda_0 + s_1(z_{31}+z_{32}) + s_2(z_{33}+z_{34}) + \lambda_5 z_{35},$$

$$\rho_4 = \lambda_0 + s_1(z_{41}+z_{42}) + s_2(z_{43}+z_{44}) + \lambda_5 z_{45}.$$

If we impose a condition

$$z_{15} = z_{25}, \qquad z_{35} = z_{45}, \tag{4.21}$$

then from $\rho_1 = \rho_2$, $\rho_3 = \rho_4$, $s_1 \neq s_2$ and (1.4) we have

$$\begin{aligned} z_{11}+z_{12} &= z_{21}+z_{22}, \quad z_{31}+z_{32} = z_{41}+z_{42}, \\ z_{13}+z_{14} &= z_{23}+z_{24}, \quad z_{33}+z_{34} = z_{43}+z_{44}. \end{aligned} \tag{4.22}$$

From (1.3) and (1.5) we have

$$(z_{11}+z_{12})(z_{11}+z_{12}) = \sum_{i=1}^{2} n_i + \Big(\sum_{i,j=1}^{2} p_{ij}^1\Big) z_{11} + \cdots + \Big(\sum_{i,j=1}^{2} p_{ij}^5\Big) z_{15},$$

$$(z_{21}+z_{22})(z_{21}+z_{22}) = \sum_{i=1}^{2} n_i + \Big(\sum_{i,j=1}^{2} p_{ij}^1\Big) z_{21} + \cdots + \Big(\sum_{i,j=1}^{2} p_{ij}^5\Big) z_{25},$$

$$(z_{31}+z_{32})(z_{31}+z_{32}) = \sum_{i=1}^{2} n_i + \Big(\sum_{i,j=1}^{2} p_{ij}^1\Big) z_{31} + \cdots + \Big(\sum_{i,j=1}^{2} p_{ij}^5\Big) z_{35},$$

$$(z_{41}+z_{42})(z_{41}+z_{42}) = \sum_{i=1}^{2} n_i + \Big(\sum_{i,j=1}^{2} p_{ij}^1\Big) z_{41} + \cdots + \Big(\sum_{i,j=1}^{2} p_{ij}^5\Big) z_{45},$$

which from (4.21) and (4.22) lead to

$$(z_{12}-z_{22})\Big(\sum_{i,j=1}^{2} p_{ij}^2 - \sum_{i,j=1}^{2} p_{ij}^1\Big) + (z_{14}-z_{24})\Big(\sum_{i,j=1}^{2} p_{ij}^4 - \sum_{i,j=1}^{2} p_{ij}^3\Big) = 0,$$

$$(z_{32}-z_{42})(\sum_{i,j=1}^{2} p_{ij}^{2}-\sum_{i,j=1}^{2} p_{ij}^{1})+(z_{34}-z_{44})(\sum_{i,j=1}^{2} p_{ij}^{4}-\sum_{i,j=1}^{2} p_{ij}^{3}) = 0.$$

If

$$\begin{vmatrix} z_{12}-z_{22}, & z_{14}-z_{24} \\ z_{32}-z_{42}, & z_{34}-z_{44} \end{vmatrix} \neq 0 \tag{4.23}$$

can be further assumed, then we obtain

$$\sum_{i,j=1}^{2} p_{ij}^{1} = \sum_{i,j=1}^{2} p_{ij}^{2}, \qquad \sum_{i,j=1}^{2} p_{ij}^{3} = \sum_{i,j=1}^{2} p_{ij}^{4}. \tag{4.24}$$

Similarly, from $(z_{13}+z_{14})(z_{13}+z_{14})=(z_{23}+z_{24})(z_{23}+z_{24})$, $(z_{33}+z_{34})(z_{33}+z_{34})=(z_{43}+z_{44})(z_{43}+z_{44})$ and (4.23) we obtain

$$\sum_{i,j=3}^{4} p_{ij}^{1} = \sum_{i,j=3}^{4} p_{ij}^{2}, \qquad \sum_{i,j=3}^{4} p_{ij}^{3} = \sum_{i,j=3}^{4} p_{ij}^{4}. \tag{4.25}$$

From $(z_{11}+z_{12})(z_{13}+z_{14})=(z_{21}+z_{22})(z_{23}+z_{24})$, $(z_{31}+z_{32})(z_{33}+z_{34})=(z_{41}+z_{42})(z_{43}+z_{44})$ and (4.23) we similarly obtain

$$\sum_{\substack{i=1,2\\ j=3,4}} p_{ij}^{1} = \sum_{\substack{i=1,2\\ j=3,4}} p_{ij}^{2}, \qquad \sum_{\substack{i=1,2\\ j=3,4}} p_{ij}^{3} = \sum_{\substack{i=1,2\\ j=3,4}} p_{ij}^{4}. \tag{4.26}$$

Furthermore, from $(z_{11}+z_{12})z_{15}=(z_{21}+z_{22})z_{25}$, $(z_{31}+z_{32})z_{35}=(z_{41}+z_{42})z_{45}$; $(z_{13}+z_{14})z_{15}=(z_{23}+z_{24})z_{25}$, $(z_{33}+z_{34})z_{35}=(z_{43}+z_{44})z_{45}$ and $z_{15}z_{15}=z_{25}z_{25}$, $z_{35}z_{35}=z_{45}z_{45}$, under (4.23) we obtain respectively

$$\begin{gathered} \sum_{i=1}^{2} p_{i5}^{1} = \sum_{i=1}^{2} p_{i5}^{2}, \qquad \sum_{i=1}^{2} p_{i5}^{3} = \sum_{i=1}^{2} p_{i5}^{4}, \\ \sum_{i=3}^{4} p_{i5}^{1} = \sum_{i=3}^{4} p_{i5}^{2}, \qquad \sum_{i=3}^{4} p_{i5}^{3} = \sum_{i=3}^{4} p_{i5}^{4}, \\ p_{55}^{1} = p_{55}^{2}, \qquad p_{55}^{3} = p_{55}^{4}. \end{gathered} \tag{4.27}$$

Conditions (4.24), (4.25), (4.26) and (4.27) coincide with (2.3) in this case. Therefore Kageyama's condition leads to generalized Vartak's one in this type provided that (4.21) and (4.23) hold. Note that these additional conditions (4.21) and (4.23) can be replaced by $z_{15}=z_{25}$, $z_{35}=z_{45}$, $z_{13}=z_{23}$, $z_{33}=z_{43}$ or $z_{15}=z_{25}=z_{35}=z_{45}=z_{55}$.

At the conclusion of Section 4, it might be said that generalized Vartak's condition is easier to use than Kageyama's one, since checking the conditions on the z_{ij}'s requires some calculations.

5. Reductions for a certain PBIB design

Let N_i be **BIB** designs with parameters v_i, b_i, r_i, k_i, λ_i and N_i^* be complementary **BIB** designs with parameters $v_i^*=v_i$, $b_i^*=b_i$, $r_i^*=b_i-r_i$, $k_i^*=v_i-k_i$, $\lambda_i^*=b_i-2r_i+\lambda_i$ of $N_i\,(i=1,2,\dots,m)$. Then Kageyama [23] gave a necessary and sufficient condition that a **PBIB** design $N_\gamma=N_1\otimes N_2+N_1^*\otimes N_2^*$ with at most three associate classes having an F_2 type association scheme is reducible to a **PBIB** design with only two distinct associate classes having an L_2 association scheme. Furthermore, Kageyama [30] showed necessary and sufficient conditions for **PBIB** designs $N_\alpha=N_1\otimes N_2\otimes N_3+N_1^*\otimes N_2^*\otimes N_3^*$ and $N_\beta=N_1\otimes N_2\otimes\cdots\otimes N_m+N_1^*\otimes N_2^*\otimes\cdots\otimes N_m^*$ to be reducible. N_α is different from $N_1\otimes N_2\otimes N_3+N_1\otimes N_2^*\otimes N_3^*+N_1^*\otimes N_2\otimes N_3^*+N_1^*\otimes N_2^*\otimes N_3$ constructed by Sillitto's product of N_γ and N_3, where N_γ is a **BIB** design provided $b_i=4(r_i-\lambda_i)$, $i=1, 2$ [50; 52]. A generalization of the Sillitto type of product will be treated in a subsequent section.

By use of Kageyama's condition, we have as a generalization of **PBIB** design N_γ the following

THEOREM 5.1 (Kageyama [30]). *Given the* **BIB** *designs N_i with parameters v, b_i, r_i, k and λ_i $(i=1,2,\dots,m)$, a necessary and sufficient condition for a* **PBIB** *design $N=N_1\otimes N_2\otimes\cdots\otimes N_m+N_1^*\otimes N_2^*\otimes\cdots\otimes N_m^*$ with at most 2^m-1 associate classes having the F_m type association scheme to be reducible to a* **PBIB** *design with the hypercubic association scheme of m associate classes is that*

$$b_i(r_j-\lambda_j) = b_j(r_i-\lambda_i) \tag{5.1}$$

hold simultaneously for every i, j $(i\neq j)=1, 2,\dots, m$.

For Kageyama [30], we remark that necessary and sufficient conditions for two distinct **PBIB** design based on the same association scheme to be reducible are generally different.

Further, note that (5.1) can be replaced by $r_i\lambda_j=r_j\lambda_i$, because $b_i(r_j-\lambda_j)=b_j(r_i-\lambda_i)$ is equivalent to $r_i\lambda_j=r_j\lambda_i$ under conditions $v_i=v_j$ and $k_i=k_j$. Since we can also see that $b_i(r_j-\lambda_j)=b_j(r_i-\lambda_i)$ is equivalent to $v_i=v_j$ under $r_i\lambda_j=r_j\lambda_i$ and $k_i=k_j$, as compared with Theorem 5.1 from a combinatorial point of view of the design we have

COROLLARY 5.2. *Given the* **BIB** *designs N_i with parameters v_i, b_i, r_i, k and λ_i $(i=1,2,3)$ satisfying $r_1\lambda_2=r_2\lambda_1$, $r_2\lambda_3=r_3\lambda_2$ and $r_1\lambda_3=r_3\lambda_1$, a necessary and sufficient condition for a* **PBIB** *design $N=N_1\otimes N_2\otimes N_3+N_1^*\otimes N_2^*\otimes N_3^*$ with at most seven associate classes having the F_3 type association scheme to be reducible to a* **PBIB** *design with only three distinct associate classes having the cubic association scheme is that*

$$v_1 = v_2 = v_3.$$

We will deal with the problem in Section 7 of Kageyama [23], i.e., the derivation of a necessary and sufficient condition for a **PBIB** design with at most 2^m-1 associate classes having the F_m type association scheme to be reducible to a **PBIB** design with m_1 associate classes for a positive integer m_1 such that $m< m_1<2^m-1$. Here we shall consider the case $m=3$ concerning Corollary 5.2.

As shown in Kageyama [30], when N_i are **BIB** designs with parameters v_i, b_i, r_i, k_i and λ_i, $i=1, 2, 3$, the parameters of **PBIB** design $N=N_1\otimes N_2\otimes N_3+N_1^*\otimes N_2^*\otimes N_3^*$ based on the F_3 type association scheme are given by

$$
\begin{aligned}
v' &= v_1v_2v_3, \quad b' = b_1b_2b_3,\\
r' &= r_1r_2r_3+(b_1-r_1)(b_2-r_2)(b_3-r_3),\\
k' &= k_1k_2k_3+(v_1-k_1)(v_2-k_2)(v_3-k_3),\\
\lambda_1' &= r_1\lambda_2r_3+(b_1-r_1)(b_2-2r_2+\lambda_2)(b_3-r_3),\\
\lambda_2' &= \lambda_1r_2r_3+(b_1-2r_1+\lambda_1)(b_2-r_2)(b_3-r_3),\\
\lambda_3' &= \lambda_1\lambda_2r_3+(b_1-2r_1+\lambda_1)(b_2-2r_2+\lambda_2)(b_3-r_3),\\
\lambda_4' &= r_1r_2\lambda_3+(b_1-r_1)(b_2-r_2)(b_3-2r_3+\lambda_3),\\
\lambda_5' &= r_1\lambda_2\lambda_3+(b_1-r_1)(b_2-2r_2+\lambda_2)(b_3-2r_3+\lambda_3),\\
\lambda_6' &= \lambda_1r_2\lambda_3+(b_1-2r_1+\lambda_1)(b_2-r_2)(b_3-2r_3+\lambda_3),\\
\lambda_7' &= \lambda_1\lambda_2\lambda_3+(b_1-2r_1+\lambda_1)(b_2-2r_2+\lambda_2)(b_3-2r_3+\lambda_3)\\
&\qquad +2(r_1-\lambda_1)(r_2-\lambda_2)(r_3-\lambda_3).
\end{aligned}
\tag{5.2}
$$

It follows from (5.2) and some calculations that

$$\lambda_1' = \lambda_4' \quad \text{and} \quad \lambda_3' = \lambda_6' \tag{5.3}$$

are equivalent to

$$b_2(r_3-\lambda_3) = b_3(r_2-\lambda_2), \tag{5.4}$$

under

$$v_2 = v_3 \quad \text{and} \quad k_2 = k_3. \tag{5.5}$$

In a similar way, if (5.5) is replaced by $v_1=v_2$ and $k_1=k_2$, then (5.3) is replaced by $\lambda_1'=\lambda_2'$ and $\lambda_5'=\lambda_6'$, while (5.4) by $b_1(r_2-\lambda_2)=b_2(r_1-\lambda_1)$. Furthermore, if

(5.5) is replaced by $v_1=v_3$ and $k_1=k_3$, then (5.3) is replaced by $\lambda'_2=\lambda'_4$ and $\lambda'_3=\lambda'_5$, while (5.4) by $b_1(r_3-\lambda_3)=b_3(r_1-\lambda_1)$. Therefore, since the properties for reductions of the F_3 type association scheme are given in Design (I) of Section 3, we can establish

THEOREM 5.3. *Given the* **BIB** *designs N_i with parameters v_i, b_i, r_i, k_i and λ_i $(i=1, 2, 3)$ satisfying*

$$v_2 = v_3 \quad \text{and} \quad k_2 = k_3, \tag{5.6}$$

then a necessary and sufficient condition for a **PBIB** *design $N=N_1\otimes N_2\otimes N_3+N_1^*\otimes N_2^*\otimes N_3^*$ with at most seven associate classes having the F_3 type association scheme to be reducible to a* **PBIB** *design with five associate classes having an association called a singular reduced F_3 type association scheme* [28] *is that*

$$b_2(r_3-\lambda_3) = b_3(r_2-\lambda_2), \tag{5.7}$$

which is equivalent to

$$r_2\lambda_3 = r_3\lambda_2. \tag{5.8}$$

REMARK. If (5.6) is replaced by $v_1=v_2$ and $k_1=k_2$, then (5.7) is replaced by $b_1(r_2-\lambda_2)=b_2(r_1-\lambda_1)$, while (5.8) by $r_1\lambda_2=r_2\lambda_1$. Furthermore, if (5.6) is replaced by $v_1=v_3$ and $k_1=k_3$, then (5.7) is replaced by $b_1(r_3-\lambda_3)=b_3(r_1-\lambda_1)$, while (5.8) by $r_1\lambda_3=r_3\lambda_1$.

Generalizations of Corollary 5.2 and Theorem 5.3 are easily given and hence they are omitted here.

6. Algebraic structures of PBIB designs obtained by generalization of Sillitto's product

In the previous section, we dealt with necessary and sufficient conditions that the **PBIB** design given by the Kronecker product of **BIB** designs in the form $N=N_1\otimes N_2\otimes\cdots\otimes N_n+N_1^*\otimes N_2^*\otimes\cdots\otimes N_n^*$ is reducible to a **PBIB** design with fewer associate classes. In this section we shall deal with the generalization of the Sillitto type of product concerning the product type stated above. That is, for usual Sillitto's product $N^{(1)}=N_1\otimes N_2+N_1^*\otimes N_2^*$, we study the algebraic structures of $N^{(2)}=N^{(1)}\otimes N_3+N^{(1)*}\otimes N_3^*$ and in general $N^{(n)}=N^{(n-1)}\otimes N_{n+1}+N^{(n-1)*}\otimes N_{n+1}^*$, where N_i's are **BIB** designs. The approach used here is standard, being the use of generalized Vartak's condition.

Let N_i be **BIB** designs with parameters v_i, b_i, r_i, k_i, λ_i $(i=1, 2,\ldots, n+1)$ and let the parameters of **PBIB** design $N^{(\alpha)}$ be denoted by $v^{(\alpha)}$, $b^{(\alpha)}$, $r^{(\alpha)}$, $k^{(\alpha)}$, $\lambda_i^{(\alpha)}$ and $n_i^{(\alpha)}$. Then it is known (cf. [23]) that the parameters of the **PBIB** design

$N^{(1)}=N_1\otimes N_2+N_1^*\otimes N_2^*$ based on a rectangular association scheme are as follows:

$$
\begin{aligned}
&v^{(1)} = v_1v_2, \quad b^{(1)} = b_1b_2,\\
&r^{(1)} = r_1r_2+(b_1-r_1)(b_2-r_2),\\
&k^{(1)} = k_1k_2+(v_1-k_1)(v_2-k_2),\\
&\lambda_1^{(1)} = r_1\lambda_2+(b_1-r_1)(b_2-2r_2+\lambda_2), \quad n_1^{(1)} = v_2-1,\\
&\lambda_2^{(1)} = r_2\lambda_1+(b_2-r_2)(b_1-2r_1+\lambda_1), \quad n_2^{(1)} = v_1-1,\\
&\lambda_3^{(1)} = \lambda_1\lambda_2+(b_1-2r_1+\lambda_1)(b_2-2r_2+\lambda_2)+2(r_1-\lambda_1)(r_2-\lambda_2),\\
&\qquad\qquad n_3^{(1)} = (v_1-1)(v_2-1).
\end{aligned}
\tag{6.1}
$$

Furthermore, as indicated in Section 5, $N^{(1)}$ is a **BIB** design when the parameters of the original **BIB** designs N_i satisfy $b_i=4(r_i-\lambda_i)$, $i=1, 2$. Since the algebraic structures of the **PBIB** design $N^{(1)}$ with at most three associate classes are discussed in Kageyama [23], we begin by considering the design $N^{(2)}=N^{(1)}\otimes N_3+N^{(1)*}\otimes N_3^*$ as the Sillitto type of product of $N^{(1)}$ and N_3.

Before a further consideration, we prepare the following lemma which plays an important role in this section.

LEMMA 6.1. *Let M_1 be a* **PBIB** *design with m associate classes and with parameters $v^{(1)}, b^{(1)}, r^{(1)}, k^{(1)}, \lambda_i^{(1)}, n_i^{(1)}, p_{jk}^{i(1)}$, $i, j, k=0, 1,\dots, m$, and let N_2 be a* **BIB** *design with parameters v_2, b_2, r_2, k_2 and λ_2. Then $N=M_1\otimes N_2+M_1^*\otimes N_2^*$ is a* **PBIB** *design with at most $2m+1$ associate classes and with parameters*

$$
\begin{aligned}
&v = v^{(1)}v_2, \quad b = b^{(1)}b_2,\\
&r = r^{(1)}r_2+(b^{(1)}-r^{(1)})(b_2-r_2),\\
&k = k^{(1)}k_2+(v^{(1)}-k^{(1)})(v_2-k_2),\\
&\lambda_1 = r^{(1)}\lambda_2+(b^{(1)}-r^{(1)})(b_2-2r_2+\lambda_2), \quad n_1=v_2-1,\\
&\lambda_{2i}=r_2\lambda_i^{(1)}+(b_2-r_2)(b^{(1)}-2r^{(1)}+\lambda_i^{(1)}), \quad n_{2i}=n_i^{(1)},\\
&\lambda_{2i+1}=\lambda_i^{(1)}\lambda_2+(b^{(1)}-2r^{(1)}+\lambda_i^{(1)})(b_2-2r_2+\lambda_2)\\
&\qquad +2(r^{(1)}-\lambda_i^{(1)})(r_2-\lambda_2), \quad n_{2i+1}=(v_2-1)n_i^{(1)},
\end{aligned}
$$

for $i=1, 2,\dots, m$. In addition, the following relations hold:

$\lambda_1 = \lambda_{2i}$ *if and only if* $b^{(1)}(r_2-\lambda_2) = b_2(r^{(1)}-\lambda_i^{(1)})$,

$\lambda_1 = \lambda_{2i+1}$ *if and only if* $(r^{(1)}-\lambda_i^{(1)})b_2 = 4(r^{(1)}-\lambda_i^{(1)})(r_2-\lambda_2)$,

$\lambda_{2i} = \lambda_{2j+1}$ *if and only if* $b_2(\lambda_j^{(1)}-\lambda_i^{(1)}) = (r_2-\lambda_2)[b^{(1)}-4(r^{(1)}-\lambda_j^{(1)})]$,

$\lambda_{2i} = \lambda_{2j}$ *if and only if* $\lambda_i^{(1)} = \lambda_j^{(1)}$,

$\lambda_{2i+1} = \lambda_{2j+1}$ *if and only if* $(\lambda_i^{(1)}-\lambda_j^{(1)})[b_2-4(r_2-\lambda_2)] = 0$

for all $i, j=1, 2,\ldots, m$.

Since it is clear that in general the complement of a **PBIB** design M with parameters v, b, r, k, λ_i, n_i and p_{jk}^i (if the design exists, then $b+\lambda_i \geqq 2r$ holds for all i) is also a **PBIB** design M^* with parameters $v^*=v$, $b^*=b$, $r^*=b-r$, $k^*=v-k$, $\lambda_i^*=b-2r+\lambda_i$, $n_i^*=n_i$ and $p_{jk}^{i*}=p_{jk}^i$ having the same association scheme as M, so that $N=M_1\otimes N_2+M_1^*\otimes N_2^*$ has the same association scheme as the design $M_1\otimes N_2$, the second kind of parameters p_{jk}^i of N coincide with those of $M_1\otimes N_2$. The latter can be found in Vartak [58] and hence we omit describing them here.

The proof of Lemma 6.1 is easily given by enumeration from the structure of $N=M_1\otimes N_2+M_1^*\otimes N_2^*$ and the combinatorial properties of M_1 and N_2, or with the help of association matrices for the purpose of the essential use of Lemma 5.1 due to Bose and Mesner [8]. The association matrices matching the design N can be also represented by the Kronecker products of those of designs M_1 and N_2. Note that in Lemma 6.1 if M_1 has an F_n type association scheme which will be given in Section 9, then N has an F_{n+1} type association scheme.

Since

$$(N_1 \otimes N_2+N_1^* \otimes N_2^*)^* = N_1 \otimes N_2^*+N_1^* \otimes N_2\,,$$

we have

$$N^{(2)} = N^{(1)} \otimes N_3+N^{(1)*} \otimes N_3^* \tag{6.2}$$

$$= N_1 \otimes N_2 \otimes N_3+N_1^* \otimes N_2^* \otimes N_3+N_1 \otimes N_2^* \otimes N_3^* + N_1^* \otimes N_2 \otimes N_3^*. \tag{6.3}$$

Remark. The complement of a design of the Sillitto type of product is easily made from a structural point of view, that is, it is essential to make the complement of the last **BIB** design only in each term consisting of Kronecker products of **BIB** designs. For example, the complement of $N^{(2)}$ is as follows:

$$N^{(2)*} = N_1 \otimes N_2 \otimes (N_3)^*+N_1^*\otimes N_2^* \otimes (N_3)^*$$

$$+N_1 \otimes N_2^* \otimes (N_3^*)^* + N_1^* \otimes N_2 \otimes (N_3^*)^*$$

$$= N_1 \otimes N_2 \otimes N_3^* + N_1^* \otimes N_2^* \otimes N_3^* + N_1 \otimes N_2^* \otimes N_3$$

$$+ N_1^* \otimes N_2 \otimes N_3 .$$

It is convenient to consider $N^{(2)}$ in the original form (6.2) rather than in an expansion form (6.3) for the sake of the easy use of Lemma 6.1. Thus, from (6.1) and Lemma 6.1 we have

PROPOSITION. *$N^{(2)} = N^{(1)} \otimes N_3 + N^{(1)*} \otimes N_3^* = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3 + N_1 \otimes N_2^* \otimes N_3^* + N_1^* \otimes N_2 \otimes N_3^*$ is a* **PBIB** *design with at most seven associate classes having an F_3 type association scheme and with parameters*

$$v^{(2)} = v_1 v_2 v_3, \quad b^{(2)} = b_1 b_2 b_3,$$

$$r^{(2)} = \{r_1 r_2 + (b_1 - r_1)(b_2 - r_2)\} r_3 + (b_1 r_2 + b_2 r_1 - 2r_1 r_2)(b_3 - r_3),$$

$$k^{(2)} = \{k_1 k_2 + (v_1 - k_1)(v_2 - k_2)\} k_3 + (v_1 k_2 + v_2 k_1 - 2k_1 k_2)(v_3 - k_3),$$

$$\lambda_1^{(2)} = b_1 b_2 \lambda_3 + (b_1 r_2 + b_2 r_1 - 2r_1 r_2)(b_3 - 2r_3), \quad n_1^{(2)} = v_3 - 1,$$

$$\lambda_{2i}^{(2)} = \lambda_i^{(1)} b_3 - (b_1 - 2r_1)(b_2 - 2r_2)(b_3 - r_3), \quad n_{2i}^{(2)} = n_i^{(1)},$$

$$\lambda_{2i+1}^{(2)} = b_1 b_2 (b_3 - 2r_3 + \lambda_3) + \lambda_i^{(1)} \{b_3 - 4(r_3 - \lambda_3)\}$$

$$-2\{r_1 r_2 + (b_1 - r_1)(b_2 - r_2)\}(b_3 - 3r_3 + 2\lambda_3),$$

$$n_{2i+1}^{(2)} = (v_3 - 1) n_i^{(1)},$$

for $i=1, 2, 3$, where $\lambda_i^{(1)}$ and $n_i^{(1)}$ are given in (6.1). *In addition, (i) when $b_i = 4(r_i - \lambda_i)$, $i=1, 2, 3$, $N^{(2)}$ is originally reducible to a* **BIB** *design. (ii) When $b_i = 4(r_i - \lambda_i)$, $i=1, 2$ and $b_3 \neq 4(r_3 - \lambda_3)$, $N^{(2)}$ is originally reducible to a* **PBIB** *design with at most three associate classes having a rectangular association scheme, and is further not reducible to a 2-associate* **PBIB** *design based on the L_2 association scheme.*

On the reduction of associate classes for $N^{(2)}$ as a **PBIB** design with seven associate classes having the F_3 type association scheme (see Section 3 for matrix expressions $P_i = \|p_{jk}^i\|$ of the second kind of parameters), the representations corresponding to the places of letters (A), α, (B) and (C) in the following proposition are given in the table:

PROPOSITION: *When there exists the relation (A) of equality among the coincidence numbers $\lambda_i^{(2)}$, a necessary and sufficient condition that a* **PBIB** *design $N^{(2)}$ with at most seven associate classes having the F_3 type association* *scheme is reducible to a* **PBIB** *design with* α *associate classes is condition* (*B*). *Furthermore, relation* (*A*) *holds when the parameters of the original* **BIB** *designs satisfy condition* (*C*).

Table

No.	relation (*A*) of $\lambda_i^{(2)}$	α	(*B*)	condition (*C*)
1	$4=5,\ 6=7$	5		$b_1=4(r_1-\lambda_1)$
2	$4=6,\ 5=7$	5		$b_1=4(r_1-\lambda_1)$
3	$2=3,\ 6=7$	5		$b_2=4(r_2-\lambda_2)$
4	$2=6,\ 3=7$	5		$b_2=4(r_2-\lambda_2)$
5	$1=3,\ 5=7$	5		$b_3=4(r_3-\lambda_3)$
6	$1=5,\ 3=7$	5		$b_3=4(r_3-\lambda_3)$
7	$1=2,\ 5=6$	5	$v_1=v_2$	$b_2(r_3-\lambda_3)=b_3(r_2-\lambda_2)$
8	$2=4,\ 3=5$	5	$v_1=v_3$	$b_1(r_2-\lambda_2)=b_2(r_1-\lambda_1)$
9	$1=4,\ 3=6$	5	$v_2=v_3$	$b_1(r_3-\lambda_3)=b_3(r_1-\lambda_1)$
10	$1=5,\ 2=6$	5	$v_1=v_2=2$	$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
11	$1=3,\ 4=6$	5	$v_2=v_3=2$	$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
12	$2=3,\ 4=5$	5	$v_1=v_3=2$	$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
13	$2=7,\ 3=6$	5	$v_2=v_3=2$	$b_2=4(r_2-\lambda_2)$
14	$4=7,\ 5=6$	5	$v_1=v_2=2$	$b_1=4(r_1-\lambda_1)$
15	$4=5=6=7$	4		$b_1=4(r_1-\lambda_1)$
16	$2=3=6=7$	4		$b_2=4(r_2-\lambda_2)$
17	$1=3=5=7$	4		$b_3=4(r_3-\lambda_3)$
18	$2=3,\ 4=5,\ 6=7$	4		$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
19	$2=3=4=5$	4	$v_1=v_3=2$	$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
20	$1=3=4=6$	4	$v_2=v_3=2$	$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
21	$1=2=5=6$	4	$v_1=v_2=2$	$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
22	$2=3,\ 4=5=6=7$	3		$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
23	$4=5,\ 2=3=6=7$	3		$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
24	$1=3,\ 4=5=6=7$	3		$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
25	$4=6,\ 1=3=5=7$	3		$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
26	$2=6,\ 1=3=5=7$	3		$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
27	$1=2=3,\ 5=6=7$	3		$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
28	$2=4=6,\ 3=5=7$	3		$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
29	$1=4=5,\ 3=6=7$	3		$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
30	$2=3=4=5,\ 6=7$	3	$v_1=v_3$	$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
31	$2=4,\ 1=3=5=7$	3	$v_1=v_3$	$b_i=4(r_i-\lambda_i),\ i=1,\ 2,\ 3$
32	$1=2,\ 4=5=6=7$	3	$v_1=v_2$	$b_1=4(r_1-\lambda_1),$ $b_2(r_3-\lambda_3)=b_3(r_2-\lambda_2)$
33	$3=7,\ 1=2=5=6$	3	$v_1=v_2$	$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
34	$5=7,\ 1=3=4=6$	3	$v_2=v_3$	$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
35	$1=2=4,\ 3=5=6$	3	$v_1=v_2=v_3$	$b_1(r_3-\lambda_3)=b_3(r_1-\lambda_1),$ $b_2(r_3-\lambda_3)=b_3(r_2-\lambda_2)$
36	$2=5=6,\ 3=4=7$	3	$v_1=v_2=2$	$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
37	$1=3=6,\ 2=5=7$	3	$v_2=v_3=2$	$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
38	$1=7,\ 2=3=4=5$	3	$v_1=v_3=2$	$b_i=4(r_i-\lambda_i),\ i=1,\ 2,\ 3$
39	$1=4=7,\ 3=5=6$	3	$v_1=2,\ v_2=v_3=4$	$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$
40	$1=2=7,\ 3=5=6$	3	$v_1=v_2=4,\ v_3=2$	$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
41	$2=3=4=5=6=7$	2		$b_1=4(r_1-\lambda_1),\ b_2=4(r_2-\lambda_2)$
42	$1=2=3=5=6=7$	2		$b_2=4(r_2-\lambda_2),\ b_3=4(r_3-\lambda_3)$
43	$1=3=4=5=6=7$	2		$b_1=4(r_1-\lambda_1),\ b_3=4(r_3-\lambda_3)$

(Continued)

No.	relation (A) of $\lambda_i^{(2)}$	α	(B)	condition (C)
44	1=2=3, 4=5=6=7	2		$b_i=4(r_i-\lambda_i)$, $i=1, 2, 3$
45	1=4=5, 2=3=6=7	2		$b_i=4(r_i-\lambda_i)$, $i=1, 2, 3$
46	1=2=3=4, 5=6=7	2	$v_1v_2=v_3$	$b_i=4(r_i-\lambda_i)$, $i=1, 2, 3$
47	1=2=4=5=6=7	2	$v_1=v_2=2$	$b_i=4(r_i-\lambda_i)$, $i=1, 2, 3$
48	1=4=7, 2=3=5=6	2	$v_1=4$, $v_2=v_3=2$	$b_i=4(r_i-\lambda_i)$, $i=1, 2, 3$

The numbers in the column of relation (A) denote suffices i of coincidence numbers $\lambda_i^{(2)}$ as follows; for example $4=5, 6=7$ means the relation $\lambda_4^{(2)}=\lambda_5^{(2)}$ and $\lambda_6^{(2)}=\lambda_7^{(2)}$, and $2=3=6=7$ means the relation $\lambda_2^{(2)}=\lambda_3^{(2)}=\lambda_6^{(2)}=\lambda_7^{(2)}$, and so on. The blanks in column (B) mean that condition (B) is automatically satisfied under relation (A). Condition (B) can be easily checked by generalized Vartak's condition. Though generally $v_i \geqq 2$ in **BIB** designs N_i, $i=1, 2, 3$, we omit the cases in which $v_1=v_2=v_3=2$ and which is reducible to a **BIB** design in Table, since they are not interesting for us.

Note that under the same condition (C), by combining associate classes in some ways, $N^{(2)}$ is reducible to **PBIB** designs with fewer associate classes. Of course, there are many cases other than those given in Table concerning the reduction of associate classes. In particular, there are many combinations of $\lambda_i^{(2)}$ for the cases reducible to **PBIB** designs with two or three associate classes. Okuno and Okuno [41] have also studied **PBIB** designs based on the F_3 type association scheme of $v=m_1m_2m_3$ treatments in some detail.

Further, note that the reduced design of No. 18 is a 4-associate **PBIB** design based on a generalized right angular association scheme which will be indicated in Section 10, and that the reduced designs of Nos. 27 and 35 are, respectively, 3-associate **PBIB** designs based on the F_2 type association scheme and the C_3 type association scheme which will be described in Section 9. There are some 2-associate **PBIB** designs based on the well known association schemes. For example, the reduced designs of Nos. 41, 42, 43, 44, 45 and 47 are 2-associate **PBIB** designs based on the N_2 type association schemes which will be described in Section 7. The reduced design of No. 46 is a 2-associate **PBIB** design based on the L_2 association scheme provided $v_1v_2=v_3\neq 4$ from the uniqueness of the L_2 association scheme [49]. The reduced design of No. 48 may be a 2-associate **PBIB** design based on the following association scheme with parameters

$$v = 16, \quad n_1 = 5, \quad n_2 = 10,$$

$$\|p_{ij}^1\| = \begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix}, \quad \|p_{ij}^2\| = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$$

for $i, j=1, 2$. Suppose that there are treatments represented by 5-tuples $(\alpha_1,$

$\alpha_2, \alpha_3, \alpha_4, \alpha_5)$ where $\alpha_i=0$ or 1 for $i=1, 2, 3, 4, 5$ and 5-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is identified with its complement $(1-\alpha_1, 1-\alpha_2, 1-\alpha_3, 1-\alpha_4, 1-\alpha_5)$. Among these 2^4 treatments, an association is defined as follows. Two treatments $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ are ith associates if $\sum_{k=1}^{5} \varepsilon(\alpha_k-\beta_k)=i$. Each treatment is the 0th associate of itself. In fact, we can see that this association satisfies three conditions (i.e., (a), (b) and (c)′ in Section 1) of the association scheme with two associate classes. This association scheme was suggested by Enomoto [14].

As a generalization of the Sillitto type of product, we have

$$\begin{aligned}
N^{(3)} &= N^{(2)} \otimes N_4 + N^{(2)*} \otimes N_4^* \\
&= N_1 \otimes N_2 \otimes N_3 \otimes N_4 + N_1^* \otimes N_2^* \otimes N_3 \otimes N_4 \\
&\quad + N_1 \otimes N_2^* \otimes N_3^* \otimes N_4 + N_1^* \otimes N_2 \otimes N_3^* \otimes N_4 \\
&\quad + N_1 \otimes N_2 \otimes N_3^* \otimes N_4^* + N_1^* \otimes N_2^* \otimes N_3^* \otimes N_4^* \\
&\quad + N_1 \otimes N_2^* \otimes N_3 \otimes N_4^* + N_1^* \otimes N_2 \otimes N_3 \otimes N_4^*
\end{aligned}$$

and for $n \geqq 1$

$$N^{(n)} = N^{(n-1)} \otimes N_{n+1} + N^{(n-1)*} \otimes N_{n+1}^* ,$$

where $N^{(n-1)*} = N^{(n-2)} \otimes N_n^* + N^{(n-2)*} \otimes N_n$, $N^{(0)} \equiv N_1$ (i.e., $v^{(0)}=v_1$, $b^{(0)}=b_1$, $r^{(0)}=r_1$, $k^{(0)}=k_1$, $\lambda_i^{(0)}=\lambda_1$) and then $N^{(n)}$ is developed into 2^n terms each consisting of Kronecker products of $n+1$ **BIB** designs.

From Lemma 6.1 and a **PBIB** design $N^{(n-1)}$ in which an F_n type association scheme can be introduced, it follows that $N^{(n)}$ is a **PBIB** design with at most $2^{n+1}-1$ associate classes having the F_{n+1} type association scheme and with parameters

$$\begin{aligned}
v^{(n)} &= v_1 v_2 \dots v_{n+1}, \\
b^{(n)} &= b_1 b_2 \dots b_{n+1}, \\
r^{(n)} &= r^{(n-1)} r_{n+1} + (b^{(n-1)} - r^{(n-1)})(b_{n+1} - r_{n+1}) , \\
k^{(n)} &= k^{(n-1)} k_{n+1} + (v^{(n-1)} - k^{(n-1)})(v_{n+1} - k_{n+1}) , \\
\lambda_1^{(n)} &= r^{(n-1)} \lambda_{n+1} + (b^{(n-1)} - r^{(n-1)})(b_{n+1} - 2r_{n+1} + \lambda_{n+1}) , \\
\lambda_{2i}^{(n)} &= \lambda_i^{(n-1)} r_{n+1} + (b^{(n-1)} - 2r^{(n-1)} + \lambda_i^{(n-1)})(b_{n+1} - r_{n+1}) , \\
\lambda_{2i+1}^{(n)} &= \lambda_i^{(n-1)} \lambda_{n+1} + (b^{(n-1)} - 2r^{(n-1)} + \lambda_i^{(n-1)})(b_{n+1} - 2r_{n+1} + \lambda_{n+1})
\end{aligned}$$

$$+2(r^{(n-1)}-\lambda_i^{(n-1)})(r_{n+1}-\lambda_{n+1}),$$

$$n_1^{(n)}=v_{n+1}-1,\ n_{2i}^{(n)}=n_i^{(n-1)},\quad n_{2i+1}^{(n)}=(v_{n+1}-1)n_i^{(n-1)},$$

for $i=1, 2,\ldots, 2^n-1$. If the parameters of $N^{(n)}$ are represented by the parameters of $n+1$ **BIB** designs N_1, $N_2,\ldots$, and N_{n+1}, then their expressions are very complicated. The **PBIB** design $N^{(n)}$, therefore, remains to be discussed with the parameters of $N^{(n-1)}$ and N_{n+1}. The following relations among coincidence numbers $\lambda_i^{(n)}$ can be obtained:

$$\lambda_1^{(n)}=\lambda_{2i}^{(n)}\quad\text{if and only if}\quad b_{n+1}(r^{(n-1)}-\lambda_i^{(n-1)})=b^{(n-1)}(r_{n+1}-\lambda_{n+1}),$$

$$\lambda_1^{(n)}=\lambda_{2i+1}^{(n)}\quad\text{if and only if}\quad (r^{(n-1)}-\lambda_i^{(n-1)})b_{n+1}$$
$$=4(r^{(n-1)}-\lambda_i^{(n-1)})(r_{n+1}-\lambda_{n+1}),$$

$$\lambda_{2i}^{(n)}=\lambda_{2j+1}^{(n)}\quad\text{if and only if}\quad (\lambda_j^{(n-1)}-\lambda_i^{(n-1)})b_{n+1}$$
$$=(r_{n+1}-\lambda_{n+1})[b^{(n-1)}-4(r^{(n-1)}-\lambda_j^{(n-1)})],$$

$$\lambda_{2i}^{(n)}=\lambda_{2j}^{(n)}\quad\text{if and only if}\quad \lambda_i^{(n-1)}=\lambda_j^{(n-1)},\quad\text{and}$$

$$\lambda_{2i+1}^{(n)}=\lambda_{2j+1}^{(n)}\ \text{if and only if}\quad (\lambda_i^{(n-1)}-\lambda_j^{(n-1)})[b_{n+1}-4(r_{n+1}-\lambda_{n+1})]=0$$

for all $i, j=1, 2,\ldots, 2^n-1$.

Using these relations and an F_{n+1} type association scheme, we may be able to make statements on the reduction of associate classes for a **PBIB** design $N^{(n)}$ such as Table concerning those of a **PBIB** design $N^{(2)}$. Furthermore, they also depend on the algebraic structures of a design $N^{(n-1)}$. For example, (i) when $b_i=4(r_i-\lambda_i)$, $i=1, 2,\ldots, n+1$, $N^{(n)}$ is originally reducible to a **BIB** design, (ii) when $b_i=4(r_i-\lambda_i)$, $i=1, 2,\ldots, n$ and $b_{n+1}\neq 4(r_{n+1}-\lambda_{n+1})$, $N^{(n)}$ is originally reducible to a **PBIB** design with at most three associate classes having a rectangular association scheme, and (iii) when $b_i=4(r_i-\lambda_i)$, $i=1, 2,\ldots, n-1$ and $b_j\neq 4(r_j-\lambda_j)$, $j=n, n+1$, $N^{(n)}$ is originally reducible to a **PBIB** design with at most seven associate classes having the F_3 type association scheme, because $N^{(n-1)}$ is a **PBIB** design with at most three associate classes having an F_2 type association scheme, and so on. Cases (ii) and (iii) are, respectively, included in the algebraic structures of **PBIB** designs $N^{(1)}=N_1\otimes N_2+N_1^*\otimes N_2^*$ and $N^{(2)}=N^{(1)}\otimes N_3+N^{(1)*}\otimes N_3^*$ as shown before.

The reducibility of associate classes for a **PBIB** design $N^{(n)}$ may be studied from exhaustive combinations of associate classes in the F_{n+1} type association scheme by Lemmas 2.1 and 2.2, similarly as for $N^{(2)}$, but they will not be carried out here.

COMPLEMENTARY REMARK. As a generalization of Lemma 6.1, when M_1 and N_2 are **PBIB** designs with s and t associate classes, respectively, we can give all the parameters of a **PBIB** design $N=M_1 \otimes N_2 + M_1^* \otimes N_2^*$ with at most $s+t+st$ associate classes. However, they are omitted here.

Part II. Some types of reducible association schemes

We shall discuss some series of association schemes, each series of which is reducible to an association scheme of the same type with fewer associate classes by combining some prescribed associate classes. The discussion is independent of treatment-block incidence of the design and is useful to the reductions of the number of associate classes for an incomplete block design based on a certain association scheme as seen in Part I and, also, useful to the characterization of the association scheme.

7. Nested type of association schemes

Following Yamamoto, Fujii and Hamada [61], suppose that there are $v_m = s_1 s_2 \ldots s_m$ treatments $\phi(\alpha_1, \alpha_2, \ldots, \alpha_m)$ indexed by m-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ where $\alpha_i = 1, 2, \ldots, s_i\ (\geqq 2)$ for $i=1, 2, \ldots, m$. Among these treatments, we can define the following association called here an N_m type association scheme (or an m-fold nested type association scheme) satisfying three conditions of the association scheme with m associate classes, which is also called a generalized group divisible association scheme with m associate classes by Raghavarao [42]:

DEFINITION: Two treatments $\phi(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\phi(\beta_1, \beta_2, \ldots, \beta_m)$ are ith associates if and only if $\alpha_j = \beta_j$ for all $j=1, 2, \ldots, m-i$ and $\alpha_{m-i+1} \neq \beta_{m-i+1}$. Each treatment is the 0th associate of itself.

After numbering v_m treatments lexicographical order, we can express ith association matrices as

$$(7.1) \qquad A_i = I_{v_{m-i}} \otimes (G_{s_{m-i+1}} - I_{s_{m-i+1}}) \otimes G_{s_{m-i+2}} \otimes \cdots \otimes G_{s_m}$$

for $i=0, 1, \ldots, m$, where $v_j = s_1 s_2 \ldots s_j$. Furthermore, the parameters of this association scheme are known to be

$$v_m = s_1 s_2 \ldots s_m,$$

$$n_i = s_m s_{m-1} \cdots s_{m-i+2}\, (s_{m-i+1} - 1),$$

$$
(7.2)\qquad P_i = \|p^i_{jk}\| = \left(\begin{array}{c:c:c} O_{(i-1)\times(i-1)} & \mathbf{x}_{(i-1)} & O_{(i-1)\times(m-i)} \\ \hdashline \mathbf{x}'_{(i-1)} & \multicolumn{2}{c}{} \\ \cdashline{1-1} O_{(m-i)\times(i-1)} & \multicolumn{2}{c}{D_{(m-i+1)\times(m-i+1)}} \end{array}\right)
$$

for $i, j, k=1, 2,\dots, m$, where $\mathbf{x}_{(i-1)}$ is the $(i-1)$st order column vector with elements $n_1, n_2,\dots, n_{i-1}$; $\mathbf{x}'_{(i-1)}$ is the transpose of $\mathbf{x}_{(i-1)}$; and $D_{(m-i+1)\times(m-i+1)}$ is a diagonal matrix with diagonal elements $s_m s_{m-1}\cdots s_{m-i+2}(s_{m-i+1}-2)$, n_{i+1}, $n_{i+2},\dots, n_m$ (for $i=0$ the first element is put equal to n_0), respectively.

We shall now show that an N_m type association scheme is reducible to an N_{m-1} type association scheme for a positive integer $m\geqq 3$ (for brevity, this fact is denoted hereafter by $N_m \supseteqq N_{m-1}$, $m\geqq 3$).

In the association matrices (7.1) of an N_m type association scheme of $v_m=s_1s_2\dots s_m$ treatments, for example, consider the following form:

$$
(7.3)\qquad A_{m-2}+A_{m-1} = I_{s_1}\otimes(G_{s_2s_3}-I_{s_2s_3})\otimes G_{s_4}\otimes\cdots\otimes G_{s_m}.
$$

Let $s_1=u_1$, $s_2s_3=u_2$, $s_4=u_3,\dots,$ and $s_m=u_{m-1}$. Then (7.1) and (7.3) imply that we can obtain the association matrices of an N_{m-1} type association scheme of $v_m=s_1s_2\dots s_m=u_1u_2\dots u_{m-1}$ treatments by combining $(m-2)$nd and $(m-1)$st associate classes in the N_m type association scheme. Furthermore, (7.2) yields

$$
P_{m-2} = \left(\begin{array}{cccc|ccc} \multicolumn{4}{c}{} & n_1 & 0 & 0 \\ \multicolumn{4}{c}{} & n_2 & 0 & 0 \\ \multicolumn{4}{c}{O_{(m-3)\times(m-3)}} & \vdots & \vdots & \vdots \\ \multicolumn{4}{c}{} & n_{m-3} & 0 & 0 \\ n_1 & n_2 & \dots & n_{m-3} & a & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & n_{m-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & n_m \end{array}\right),
$$

$$
P_{m-1} = \left(\begin{array}{cccccc} \multicolumn{4}{c}{} & n_1 & 0 \\ \multicolumn{4}{c}{} & n_2 & 0 \\ \multicolumn{4}{c}{O_{(m-2)\times(m-2)}} & \vdots & \vdots \\ \multicolumn{4}{c}{} & n_{m-2} & 0 \\ n_1 & n_2 & \dots & n_{m-2} & b & 0 \\ 0 & 0 & \dots & 0 & 0 & n_m \end{array}\right),
$$

where $a=s_ms_{m-1}\dots s_4(s_3-2)$ and $b=s_ms_{m-1}\dots s_3(s_2-2)$. In these matrices P_{m-2} and P_{m-1}, it is clear that the following relations hold:

$$\sum_{i,j=m-2}^{m-1} p_{ij}^{m-2} = \sum_{i,j=m-2}^{m-1} p_{ij}^{m-1},$$

$$\sum_{i=m-2}^{m-1} p_{il}^{m-2} = \sum_{i=m-2}^{m-1} p_{il}^{m-1} \qquad \text{for } l=1, 2,\dots, m-3, m,$$

$$p_{ij}^{m-2} = p_{ij}^{m-1} \qquad \text{for } i, j=1, 2,\dots, m-3, m,$$

which are a necessary and sufficient condition (i.e., generalized Vartak's condition) for an N_m type association scheme to be reducible to an association scheme with $m-1$ associate classes. Hence it follows that $N_m \supseteqq N_{m-1}$, $m \geqq 3$.

Incidentally, it is clear from (7.1) and (7.2) that this statement is also shown by combining other two consecutive associate classes. Moreover, it is useful to note that $N_m \supseteqq N_{m-l+1}$ holds, by combining l consecutive associate classes in the N_m type association scheme for $l=2, 3,\dots, m-1$. This note is described in Raghavarao [42].

REMARK. The N_m type association scheme was called a hierarchical group divisible association scheme with m associate classes (shortly, a HGD_m type association scheme) by Roy [48]. That is, the parameters of the HGD_m type association scheme expressed in a slightly different form can be identified with the parameters of the N_m type association scheme by renaming the different associates suitably.

8. Orthogonal Latin square type of association schemes

Following Yamamoto, Fujii and Hamada [61], suppose that there are $v=k^2$ treatments indexed by $1, 2,\dots, k^2$ and they are set forth in a square $\mathfrak{B}$ so that the $\{(i-1)k+j\}$th treatment lies in the jth column of the ith row. Suppose, further, there are $r-2$ mutually orthogonal Latin squares, $\mathfrak{B}_3, \mathfrak{B}_4,\dots, \mathfrak{B}_r$, of order k ($r \leqq k+1$). Among these treatments, an association called an OL_r^m type association scheme (or an orthogonal Latin square type association scheme with m associate classes), satisfying three conditions of the association scheme with m associate classes, is defined as follows:

DEFINITION: Two treatments α and β are 1st associates if they occur in the same row, 2nd associates if they occur in the same column, and ith associates if they correspond to the same letter of ith Latin square $\mathfrak{B}_i$ ($i=3, 4,\dots, r$). Otherwise they are $(r+1)$st associates. Each treatment is the 0th associate of itself. Note that if $r=k+1$, there is no pair of treatments which are neither 1st, 2nd,..., nor rth associates. The number of associate classes is therefore $m=\min(r+1, k+1)$.

Then we have the following association matrices:

$$(8.1)\qquad \begin{aligned} &A_0 = I_v, \quad A_i = F_iF_i' - I_v, \qquad i = 1, 2,\dots, r, \\ &A_{r+1} = G_v - \sum_{i=0}^{r} A_i, \end{aligned}$$

where A_0, A_i and A_{r+1} are 0th, ith and, if $r \leqq k$, $(r+1)$st association matrices, respectively. Furthermore, F_1 is a $v \times k$ incidence matrix for treatments vs. rows, F_2 is a $v \times k$ incidence matrix for treatments vs. columns and F_i are $v \times k$ incidence matrices for treatments vs. letters of the ith Latin squares ($i = 3, 4,\dots, r$) which satisfy the following relations:

$$F_1 = I_k \otimes E_{k\times 1}, \qquad F_2 = E_{k\times 1} \otimes I_k,$$

$$F_i'F_i = kI_k \qquad (i = 1, 2,\dots, r),$$

$$F_i'F_j = G_k \qquad (i \neq j;\ i, j = 1, 2,\dots, r).$$

The parameters of this association scheme are known to be

$$v = k^2, \quad n_0 = 1, \quad n_1 = n_2 = \cdots = n_r = k-1,$$

$$\text{if} \quad r \leqq k, \quad n_{r+1} = (k-1)(k-r+1),$$

$$(8.2)\qquad P_i = \| p^i_{j_1 j_2} \|$$

$$= \left(\begin{array}{c:c:c:c} G_{i-1} - I_{i-1} & 0_{(i-1)\times 1} & E_{(i-1)\times(r-i)} & (k-r+1)E_{(i-1)\times 1} \\ \hdashline 0_{1\times(i-1)} & k-2 & \multicolumn{2}{c}{0_{1\times(r-i+1)}} \\ \hdashline E_{(r-i)\times(i-1)} & 0_{(r-i)\times 1} & G_{r-i} - I_{r-i} & (k-r+1)E_{(r-i)\times 1} \\ \hdashline (k-r+1)E_{1\times(i-1)} & 0 & (k-r+1)E_{1\times(r-i)} & (k-r)(k-r+1) \end{array}\right)$$

of order $r+1$, for $i = 1, 2,\dots, r$;

$$P_{r+1} = \| p^{r+1}_{j_1 j_2} \| = \left(\begin{array}{c:c:c} G_{r-1} - I_{r-1} & E_{(r-1)\times 1} & \\ \hdashline E_{1\times(r-1)} & 0 & (k-r)E_{r\times 1} \\ \hdashline \multicolumn{2}{c:}{(k-r)E_{1\times r}} & (k-r)^2 + r - 2 \end{array}\right)$$

of order $r+1$, for $j_1, j_2 = 1, 2,\dots, r+1$; if $r = k+1$,

$$(8.3)\qquad P_i = \| p^i_{j_1 j_2} \|$$

$$= \left(\begin{array}{c:c:c:c} G_{i-1}-I_{i-1} & 0_{(i-1)\times 1} & E_{(i-1)\times(r-i-1)} & E_{(i-1)\times 1} \\ \hdashline 0_{1\times(i-1)} & k-2 & \multicolumn{2}{c}{0_{1\times(r-i)}} \\ \hdashline E_{(r-i-1)\times(i-1)} & & G_{r-i-1}-I_{r-i-1} & E_{(r-i-1)\times 1} \\ \cdashline{1-1}\cdashline{3-4} E_{1\times(i-1)} & 0_{(r-i)\times 1} & E_{1\times(r-i-1)} & 0 \end{array}\right)$$

of order r, for $i, j_1, j_2=1, 2,\ldots, r$.

We shall now show that $OL_r^m \supseteqq OL_r^{m-1}$ for a positive integer $m \geqq 4$ and, in particular, $m \geqq 3$ when $r=k+1$. This is shown by separating into two cases.

Case (I) when $r \leqq k$. Then $m=r+1$. It follows from (8.2) that

$$\sum_{i,j=m-1}^{m} p_{ij}^{m-1} = \sum_{i,j=m-1}^{m} p_{ij}^{m},$$

$$\sum_{i=m-1}^{m} p_{il}^{m-1} = \sum_{i=m-1}^{m} p_{il}^{m} \qquad \text{for } l=1, 2,\ldots, m-2,$$

$$p_{ij}^{m-1} = p_{ij}^{m} \qquad \text{for } i, j=1, 2,\ldots, m-2,$$

which are a necessary and sufficient condition (i.e., generalized Vartak's condition) for an OL_r^m type association scheme with $m=r+1$ associate classes to be reducible to an association scheme with $m-1$ associate classes. Furthermore, since we have from (8.1)

$$A_{m-1}+A_m = G_v - \sum_{i=0}^{m-2} A_i,$$

it is clear from the combination of $(m-1)$st and mth associate classes that $OL_r^m \supseteqq OL_r^{m-1}$ for $m \geqq 4$ when $r \leqq k$.

Note that when $m=3$, i.e., $r=2$, an OL_r^3 type association scheme is reducible to an L_r type association scheme [10] with two associate classes by combining 1st and 2nd associate classes.

Case (II) when $r=k+1$, i.e., the case in which there exists a complete set of mutually orthogonal Latin squares of order k. Then $m=r$. From (8.3) and the same argument as in Case (I), by combining $(m-1)$st and mth associate classes we can show that $OL_r^m \supseteqq OL_r^{m-1}$ when $r=k+1$. In this case, $r=k+1$ and $k \geqq 2$ lead to $m \geqq 3$.

Remark. It is known (cf. [61]) that for $r \leqq k$, $OL_r^m \supseteqq L_r$ holds by combining 1st, 2nd,..., and $m-1=r$th associate classes.

9. Factorial type of association schemes

Following Yamamoto, Fujii and Hamada [61], suppose that there are $v_p = s_1 s_2 \ldots s_p$ treatments $\phi(\alpha_1, \alpha_2, \ldots, \alpha_p)$ indexed by p-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_p)$ where $\alpha_i = 1, 2, \ldots, s_i\ (\geqq 2)$ for $i = 1, 2, \ldots, p$. Among these treatments, we can define the following association called an F_p type association scheme satisfying three conditions of an association scheme with $2^p - 1$ associate classes, which is also called an extended group divisible association by Hinkelmann [16] and Hinkelmann and Kempthorne [17]:

DEFINITION: Two treatments $\phi(\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\phi(\beta_1, \beta_2, \ldots, \beta_p)$ are $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$th associates if $[\varepsilon(\alpha_1 - \beta_1), \varepsilon(\alpha_2 - \beta_2), \ldots, \varepsilon(\alpha_p - \beta_p)] = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$. Each treatment is the $(0, 0, \ldots, 0)$th associate of itself.

After numbering v_p treatments lexicographical order, we can express $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$th association matrices as

$$A_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_p} = A_{\varepsilon_1} \otimes A_{\varepsilon_2} \otimes \cdots \otimes A_{\varepsilon_p}, \tag{9.1}$$

where

$$A_{\varepsilon_i} = (1 - \varepsilon_i) I_{s_i} + \varepsilon_i (G_{s_i} - I_{s_i}),$$

$$\varepsilon_i = 0 \quad \text{or} \quad 1, \qquad i = 1, 2, \ldots, p,$$

i.e., A_{ε_i} are the association matrices of an association scheme with one associate class (e.g., a **BIB** design). Similarly, we can give the matrix representation of parameters p^i_{jk} as follows:

$$P_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_p} = \| p^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_p}_{l_1 l_2} \| = P^{(1)}_{\varepsilon_1} \otimes P^{(2)}_{\varepsilon_2} \otimes \cdots \otimes P^{(p)}_{\varepsilon_p}, \tag{9.2}$$

for $l_1, l_2 = 0, 1, \ldots, 2^p - 1$, where

$$P^{(i)}_{\varepsilon_i} = (1 - \varepsilon_i) \begin{pmatrix} 1 & 0 \\ 0 & s_i - 1 \end{pmatrix} + \varepsilon_i \begin{pmatrix} 0 & 1 \\ 1 & s_i - 2 \end{pmatrix},$$

$$\varepsilon_i = 0 \quad \text{or} \quad 1, \qquad i = 1, 2, \ldots, p,$$

i.e., $P^{(i)}_0$ and $P^{(i)}_1$ are the matrix representations of parameters p^i_{jk} of an association scheme with one associate class (e.g., a **BIB** design).

Consider first an F_3 type association scheme of $v_3 = s_1 s_2 s_3$ treatments. Then, letting $s_1 = u_1$ and $s_2 s_3 = u_2$, we have from (9.1) and $v_3 = s_1 s_2 s_3 = u_1 u_2$

$$A_{000} = I_{s_1 s_2 s_3} = I_{u_1 u_2},$$

$$A_{001} + A_{010} + A_{011} = I_{s_1} \otimes (G_{s_2 s_3} - I_{s_2 s_3}) = I_{u_1} \otimes (G_{u_2} - I_{u_2}), \tag{9.3}$$

$$A_{100} = (G_{s_1} - I_{s_1}) \otimes I_{s_2 s_3} = (G_{u_1} - I_{u_1}) \otimes I_{u_2},$$

$$A_{101} + A_{110} + A_{111} = (G_{s_1} - I_{s_1}) \otimes (G_{s_2 s_3} - I_{s_2 s_3})$$
$$= (G_{u_1} - I_{u_1}) \otimes (G_{u_2} - I_{u_2}).$$

It is clear that the matrices in (9.3) are the association matrices of an F_2 type association scheme of $u_1 u_2$ treatments. Furthermore, we can show that by combining $(0, 0, 1)$th, $(0, 1, 0)$th and $(0, 1, 1)$th associate classes, and combining $(1, 0, 1)$th, $(1, 1, 0)$th and $(1, 1, 1)$th associate classes, a necessary and sufficient condition for an F_3 type association scheme with seven associate classes to be reducible to an F_2 type association scheme with three associate classes is satisfied. It holds that $F_3 \supseteq F_2$.

Moreover, it follows from the definition of the association and Lemma 2.1 that an F_2 type association scheme (or a rectangular association scheme) of $s_1 s_2$ treatments is (i) automatically reducible to an N_2 type association scheme (i.e., $F_2 \supseteq N_2$) and is (ii) also reducible to an L_2 association scheme with two associate classes provided $s_1 = s_2$ (i.e., $F_2 \supseteq L_2$, if $s_1 = s_2$). Since these reduced association schemes are different from a series of F_p type association schemes in the strict sense, we will consider an F_p type association scheme for $p \geqq 2$.

Next, we shall show that $F_p \supseteq F_{p-1}$ for a positive integer $p \geqq 3$.

In the association matrices (9.1) of an F_p type association scheme, consider the following form:

$$(9.4) \qquad A_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{p-2} 0 1} + A_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{p-2} 1 0} + A_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{p-2} 1 1}$$
$$= A_{\varepsilon_1} \otimes A_{\varepsilon_2} \otimes \cdots \otimes A_{\varepsilon_{p-2}} \otimes (G_{s_{p-1} s_p} - I_{s_{p-1} s_p})$$

for all 2^{p-2} possible combinations of $\varepsilon_1, \varepsilon_2, \ldots,$ and ε_{p-2}. Let $s_1 = u_1$, $s_2 = u_2, \ldots, s_{p-2} = u_{p-2}$ and $s_{p-1} s_p = u_{p-1}$. Then (9.1) and (9.4) imply that we can obtain the association matrices of an F_{p-1} type association scheme of $v_p = s_1 s_2 \ldots s_p = u_1 u_2 \ldots u_{p-1}$ treatments by combining three associate classes, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p-2}, 0, 1)$th, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p-2}, 1, 0)$th and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p-2}, 1, 1)$th, in the F_p type association scheme for all 2^{p-2} possible combinations of $\varepsilon_1, \varepsilon_2, \ldots,$ and ε_{p-2}. Furthermore, we have from (9.2)

$$P_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{p-2} 0 1} = P_{\varepsilon_1}^{(1)} \otimes P_{\varepsilon_2}^{(2)} \otimes \cdots \otimes P_{\varepsilon_{p-2}}^{(p-2)} \otimes P_0^{(p-1)} \otimes P_1^{(p)},$$

$$(9.5) \qquad P_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{p-2} 1 0} = P_{\varepsilon_1}^{(1)} \otimes P_{\varepsilon_2}^{(2)} \otimes \cdots \otimes P_{\varepsilon_{p-2}}^{(p-2)} \otimes P_1^{(p-1)} \otimes P_0^{(p)},$$

$$P_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{p-2} 1 1} = P_{\varepsilon_1}^{(1)} \otimes P_{\varepsilon_2}^{(2)} \otimes \cdots \otimes P_{\varepsilon_{p-2}}^{(p-2)} \otimes P_1^{(p-1)} \otimes P_1^{(p)}.$$

Since

$$P_0^{(p-1)} \otimes P_1^{(p)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & v_p-2 & 0 & 0 \\ 0 & 0 & 0 & v_{p-1}-1 \\ 0 & 0 & v_{p-1}-1 & (v_{p-1}-1)(v_p-2) \end{pmatrix},$$

$$P_1^{(p-1)} \otimes P_0^{(p)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & v_p-1 \\ 1 & 0 & v_{p-1}-2 & 0 \\ 0 & v_p-1 & 0 & (v_{p-1}-2)(v_p-1) \end{pmatrix},$$

$$P_1^{(p-1)} \otimes P_1^{(p)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & v_p-2 \\ 0 & 1 & 0 & v_{p-1}-2 \\ 1 & v_p-2 & v_{p-1}-2 & (v_{p-1}-2)(v_p-2) \end{pmatrix},$$

it is clear that the following relations hold:

$$\text{(9.6)} \qquad \begin{aligned} &\sum_{i,j=1}^{3} p_{ij}^{01} = \sum_{i,j=1}^{3} p_{ij}^{10} = \sum_{i,j=1}^{3} p_{ij}^{11}, \\ &\sum_{i=1}^{3} p_{i0}^{01} = \sum_{i=1}^{3} p_{i0}^{10} = \sum_{i=1}^{3} p_{i0}^{11}, \\ &p_{00}^{01} = p_{00}^{10} = p_{00}^{11}. \end{aligned}$$

Therefore, from (9.6) and a part of the same matrix representation $P_{\varepsilon_1}^{(1)} \otimes P_{\varepsilon_2}^{(2)} \otimes \cdots \otimes P_{\varepsilon_{p-2}}^{(p-2)}$ in (9.5), it follows that an F_p type association scheme is reducible to an F_{p-1} type association scheme by combining three associate classes, $(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_{p-2}, 0, 1)$th, $(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_{p-2}, 1, 0)$th and $(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_{p-2}, 1, 1)$th, for all 2^{p-2} possible combinations of ε_1, $\varepsilon_2\dots$, and ε_{p-2}, i.e., $F_p \supseteqq F_{p-1}$ for $p \geqq 3$.

REMARK. As a special case of an F_p type association scheme, we have a hypercubic type of association schemes (cf. [34; 46; 61]). That is, suppose that there are $v_p = s^p$ treatments $\phi(\alpha_1, \alpha_2,\dots, \alpha_p)$ indexed by p-tuples $(\alpha_1, \alpha_2,\dots, \alpha_p)$, $(\alpha_i = 1, 2,\dots, s;\ i = 1, 2,\dots, p)$. Among these treatments, an association called a C_p type association scheme (or a p-dimensional hypercubic association scheme) is defined as follows:

DEFINITION: Two treatments $\phi(\alpha_1, \alpha_2,\dots, \alpha_p)$ and $\phi(\beta_1, \beta_2,\dots, \beta_p)$ are ith associates if and only if $\sum_{k=1}^{p} \varepsilon(\alpha_k - \beta_k) = i$. Each treatment is the 0th associate of itself.

Then it follows from the definition of the association that $F_p \supseteqq C_p$ provided $s_1 = s_2 = \cdots = s_p$ $(=s$, say). In this case, the association matrices of a C_p type association scheme of $v = s^p$ treatments can be expressed as follows:

$$\overset{(p)}{C_i} = \sum_{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p=i} A_{\varepsilon_1} \otimes A_{\varepsilon_2} \otimes \cdots \otimes A_{\varepsilon_p},$$

$$i = 0, 1, 2, \ldots, p,$$

where A_{ε_i} are given in (9.1).

If a C_p type association scheme of $v = s^p$ treatments is reducible to a C_l type association scheme of $v = t^l$ treatments for $l < p$, then we have first $s^p = t^l$ and then $s = n^{l_1}$ and $t = n^{p_1}$ for a positive integer n, where $(p, l) = \alpha$, $p = p_1\alpha$, $l = l_1\alpha$ and $(p_1, l_1) = 1$. Thus, the association matrices $\overset{(p)}{C_i}$ and $\overset{(l)}{C_i}$ are, respectively, Kronecker product representations of the matrices of order $s = n^{l_1}$ and $t = n^{p_1}$. These patterns of association matrices may imply that a C_p type association scheme is not reducible to a C_l type association scheme for $p > l \geqq 2$ (for brevity, $C_p \not\supseteqq C_l$). Practically, we can show that $C_3 \not\supseteqq C_2$; $C_4 \not\supseteqq C_3$, C_2; $C_5 \not\supseteqq C_4$, C_3, C_2 and $C_{p_1} \not\supseteqq C_{p_2}$ for many other prescribed integers p_1 and p_2 such that $p_1 > p_2 \geqq 2$. Incidentally, it is conjectured [28] that a necessary and sufficient condition for a C_p type association scheme of $v = s^p$ treatments to be reducible is that $s = 2$, 3 or 4. Indeed, this conjecture holds for $p = 3$, 4 and 5.

10. Other types of association schemes

We shall deal with the known types of two association schemes with four associate classes and of two instructive association schemes with m associate classes.

10.1. *An m-associate cyclical type of association schemes* defined by Nandi and Adhikary [39], which is a generalization of a cyclic type association scheme with two associate classes defined by Bose and Shimamoto [10]. For the definition of this association scheme refer to Nandi and Adhikary [39] for details. Since this association scheme includes an N_m type association scheme as a special case, it is clear from an argument in the N_m type association scheme of Section 7 that an m-associate cyclical type of association schemes is reducible to an $(m-1)$-associate cyclical type of association schemes, after renumbering the associates.

10.2. *A generalized right angular association scheme* with four associate classes of $v = pls$ treatments introduced by Tharthare [57], which leads to a right angular association scheme [56] provided $p=2$. That is, suppose that there are $v=pls$ treatments denoted by (α, β, γ) for $\alpha=1, 2,\dots, l$; $\beta=1, 2,\dots, p$; $\gamma=1, 2,\dots, s$. For the treatment (α, β, γ), 1st associates of it are those that differ in the third position; 2nd associates are those that differ in the second position while being the same or different in the third position; 3rd associates are those that have the same second position, a different first position, and the same or different third position; the others are 4th associates. Each treatment is the 0th associate of itself. The parameters of this association scheme are as follows:

$$v = pls, \quad n_1 = s-1, \quad n_2 = s(p-1),$$

$$n_3 = s(l-1), \quad n_4 = s(l-1)(p-1),$$

$$\|p_{ij}^1\| = \begin{pmatrix} s-2 & 0 & 0 & 0 \\ & s(p-1) & 0 & 0 \\ & \text{Sym.} & s(l-1) & 0 \\ & & & s(l-1)(p-1) \end{pmatrix},$$

$$\|p_{ij}^2\| = \begin{pmatrix} 0 & s-1 & 0 & 0 \\ & s(p-2) & 0 & 0 \\ & \text{Sym.} & 0 & s(l-1) \\ & & & s(l-1)(p-2) \end{pmatrix},$$

$$\|p_{ij}^3\| = \begin{pmatrix} 0 & 0 & s-1 & 0 \\ & 0 & 0 & s(p-1) \\ & \text{Sym.} & s(l-2) & 0 \\ & & & s(p-1)(l-2) \end{pmatrix},$$

$$\|p_{ij}^4\| = \begin{pmatrix} 0 & 0 & 0 & s-1 \\ & 0 & s & s(p-2) \\ & \text{Sym.} & 0 & s(l-2) \\ & & & s(l-2)(p-2) \end{pmatrix},$$

for $l, p, s \geqq 2$.

It is useful to note that as shown in Section 6, an F_3 type association scheme is reducible to the generalized right angular association scheme by combining three pairs of associate classes, the 2nd and 3rd, the 4th and 5th, and the 6th and 7th, referring to the matrix representations of the p^i_{jk}'s of Design (I) in Section 3. It follows from Lemmas 2.1 and 2.2 and the structure of an N_m type association scheme for $m=2$ and 3 that all the cases of reductions are as follows:

(1) By combining 2nd and 4th associate classes, or combining 3rd and 4th associate classes, it is reducible to an N_3 type association scheme.

(2) By combining 2nd and 3rd associate classes, it is reducible to an association scheme with three associate classes if and only if $l=p$.

(3) By combining 2nd, 3rd and 4th associate classes, or combining two pairs of associate classes, the 1st and 2nd, and the 3rd and 4th, or similarly combining the 1st and 3rd, and the 2nd and 4th, it is reducible to an N_2 type association scheme.

(4) By combining 1st and 4th associate classes, and combining 2nd and 3rd associate classes, it is reducible to an N_2 type association scheme if and only if $l=p=2$.

Note that the above cases (1) and (3) are stated in Tharthare [57] by the form of a generalized right angular design. The reduced association scheme in Case (2) may correspond to the association scheme matching a 3-associate **PBIB** design given by Nair [36] as follows:

Let $v=p^2s$. Assume them to be arranged as a three-dimensional lattice of points, p along x- and y-axes and s along z-axis. If the blocks are formed consisting of all treatments represented by points lying in planes parallel to the xz or yz coordinate planes, we get a **PBIB** design with three associate classes. Its parameters are given by

$$v = p^2s, \quad k = ps, \quad r = 2, \quad b = 2p,$$

$$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0,$$

$$n_1 = s-1, \quad n_2 = 2s(p-1), \quad n_3 = s(p-1)^2,$$

$$\|p^1_{ij}\| = \begin{pmatrix} s-2 & 0 & 0 \\ 0 & 2s(p-1) & 0 \\ 0 & 0 & s(p-1)^2 \end{pmatrix},$$

$$\|p^2_{ij}\| = \begin{pmatrix} 0 & s-1 & 0 \\ s-1 & s(p-2) & s(p-1) \\ 0 & s(p-1) & s(p-1)(p-2) \end{pmatrix},$$

$$\|p_{ij}^3\| = \begin{pmatrix} 0 & 0 & s-1 \\ 0 & 2s & 2s(p-2) \\ s-1 & 2s(p-2) & s(p-2)^2 \end{pmatrix}.$$

The parameters of this association scheme coincide with those of the reduced association schemes of Nos. 30, 33 and 34 in Table, after renaming the associates.

10.3. *A rectangular lattice type association scheme* with four associate classes of $v=s(s-1)$ treatments, which, though inherent in Nair's definition [37] as a simple rectangular lattice design, was explicitly introduced by Ishii and Ogawa [20] as an association scheme as follows:

Suppose that there are $v=s(s-1)$ treatments represented by the ordered pairs of two integers out of the set $(1, 2,\dots, s)$. That is, the $s(s-1)$ positions excluding the principal diagonal of an $s\times s$ square are filled by different treatments. Among these treatments, the association is defined as follows:

DEFINITION: For a treatment in (i, j) cell, 1st associates of it are the treatments in the ith row or in the jth column, 2nd associates are the treatments in the ith column or in the jth row (excluding the treatment in (j, i) cell). The 3rd associates are the treatments in the rows and columns excluding the i, jth rows and columns, 4th associate is the treatment in the (j, i) cell. Each treatment is the 0th associate of itself.

The parameters of this association scheme are given by

$$v = s(s-1), \quad n_1 = 2(s-2),$$

$$n_2 = 2(s-2), \quad n_3 = (s-2)(s-3), \quad n_4 = 1,$$

$$\|p_{ij}^1\| = \begin{pmatrix} s-3 & 1 & s-3 & 0 \\ & s-3 & s-3 & 1 \\ & \text{Sym.} & (s-3)(s-4) & 0 \\ & & & 0 \end{pmatrix},$$

$$\|p_{ij}^2\| = \begin{pmatrix} 1 & s-3 & s-3 & 1 \\ & 1 & s-3 & 0 \\ & \text{Sym.} & (s-3)(s-4) & 0 \\ & & & 0 \end{pmatrix},$$

$$\|p_{ij}^3\| = \begin{pmatrix} 2 & 2 & 2(s-4) & 0 \\ & 2 & 2(s-4) & 0 \\ & \text{Sym.} & (s-4)(s-5) & 1 \\ & & & 0 \end{pmatrix},$$

$$\|p_{ij}^4\| = \begin{pmatrix} 0 & 2(s-2) & 0 & 0 \\ & 0 & 0 & 0 \\ & \text{Sym.} & (s-2)(s-3) & 0 \\ & & & 0 \end{pmatrix},$$

for $s \geqq 4$.

It follows from Lemmas 2.1 and 2.2 that all the cases of reductions are as follows:

(1) By combining 1st and 2nd associate classes, it is reducible to an association scheme of three associate classes.

(2) By combining 1st, 2nd and 3rd associate classes, it is reducible to an association scheme of two associate classes.

(3) By combining 1st and 2nd associate classes, and combining 3rd and 4th associate classes, it is reducible to an N_2 type association scheme if and only if $s=4$.

REMARK. The reduced association schemes in the above cases (1) and (2) may not correspond to any of the known association schemes. Association schemes of Cases (2) and (3) can be also derived from further reductions of the reduced association scheme of Case (1). Incidentally, by renaming the associates, the parameters of the reduced association schemes in Cases (1) and (2) are respectively

$$v = s(s-1), \quad n_1 = 4(s-2), \quad n_2 = (s-2)(s-3), \quad n_3 = 1,$$

$$\|p_{ij}^1\| = \begin{pmatrix} 2(s-2) & 2(s-3) & 1 \\ 2(s-3) & (s-3)(s-4) & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

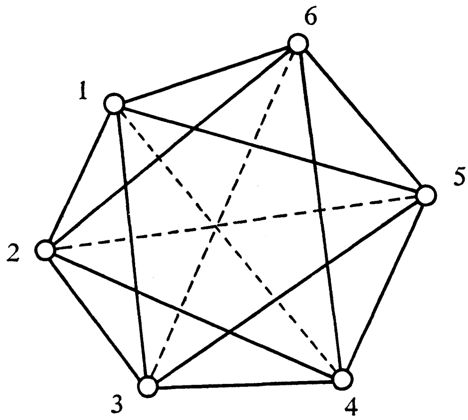
$$\|p_{ij}^2\| = \begin{pmatrix} 8 & 4(s-4) & 0 \\ 4(s-4) & (s-4)(s-5) & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\|p_{ij}^3\| = \begin{pmatrix} 4(s-2) & 0 & 0 \\ 0 & (s-2)(s-3) & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$v = s(s-1), \quad n_1 = (s-2)(s+1), \quad n_2 = 1,$$

$$\|p_{ij}^1\| = \begin{pmatrix} s^2-s-4 & 1 \\ 1 & 0 \end{pmatrix}, \quad \|p_{ij}^2\| = \begin{pmatrix} s^2-s-2 & 0 \\ 0 & 0 \end{pmatrix}.$$

From the octahedron in the following figure, we get a design (cf. [10]) by considering the faces as blocks and points as treatments, having the blocks, (1, 2, 3), (1, 2, 6), (1, 3, 5), (1, 5, 6), (2, 3, 4), (2, 4, 6), (3, 4, 5) and (4, 5, 6).



Through the treatment-block incidence of this design, we obtain a **PBIB** design with the following parameters:

$$v = 6, \quad b = 8, \quad r = 4, \quad k = 3,$$

$$\lambda_1 = 2, \quad \lambda_2 = 0,$$

$$n_1 = 4, \quad n_2 = 1,$$

$$\|p_{ij}^1\| = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \|p_{ij}^2\| = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

This corresponds to a special case of the above case (2) when $s=3$.

10.4. As another familiar association, there is *a T_m type association scheme* block (**BB**) design having respect to a **PBB** design in a combinatorial sense.

12. BB designs and PBB designs

Consider v treatments arranged in b blocks in a block design with incidence matrix $N=\|n_{ij}\|$, where n_{ij} denotes the number of experimental units in the jth block getting the ith treatment. If $n_{ij}=1$ or 0, the design is called a binary design; we deal with only binary designs in Part III. The ith treatment is replicated r_i times $(i=1,2,\dots,v)$ and the jth block is of size k_j $(j=1,2,\dots,b)$. Let T_i be the total yield for the ith treatment and B_j that for the jth block $(i=1,2,\dots,v;$ $j=1,2,\dots,b)$. On writing $T'=(T_1,\dots,T_v)$ and $B'=(B_1,\dots,B_b)$ in matrix notation, the normal equations (cf. [12; 31]) for estimating the vector of treatment effects t can be written under the usual assumptions as

$$Q = C\hat{t},$$

where $\hat{t}$ is the estimate of t,

$$Q = T-N\,\mathrm{diag}\{k_1^{-1},\dots,k_b^{-1}\}B$$

and

$$C = \mathrm{diag}\{r_1,\dots,r_v\}-N\,\mathrm{diag}\{k_1^{-1},\dots,k_b^{-1}\}N'. \tag{12.1}$$

The matrix C defined in (12.1) is well known as the C-matrix of the incomplete block design and is very useful in the theory of incomplete block designs.

Since each row (or column) of C adds up to zero, the rank of C is at most $v-1$, and $(v^{-\frac{1}{2}}, v^{-\frac{1}{2}},\dots, v^{-\frac{1}{2}})$ is the latent vector corresponding to the zero root. If the rank of C is $v-1$, the design is said to be connected (cf. Bose [6]). Throughout Part III we shall deal with only connected designs.

DEFINITION (cf. Rao [47]): A block design is said to be balanced if every elementary contrast is estimated with the same variance.

Rao [47] has shown that a necessary and sufficient condition for a block design to be balanced is that the C-matrix has $v-1$ equal latent roots other than zero. In this case, since

$$C = L'\begin{pmatrix} \rho & & & & 0 \\ & \rho & & & \\ & & \ddots & & \\ & & & \rho & \\ 0 & & & & 0 \end{pmatrix}L$$

for an orthogonal matrix

of $v_m=\binom{s}{m}$ treatments (or a triangular type association scheme with m associate classes). Suppose that there are $v_m=\binom{s}{m}$ treatments $\phi(\alpha_1, \alpha_2,\dots, \alpha_m)$ indexed by the subsets of m integers $(\alpha_1, \alpha_2,\dots, \alpha_m)$ out of the set of s integers $(1, 2,\dots, s)$. Among these treatments, the association is defined as follows:

DEFINITION: Two treatments $\phi(\alpha_1, \alpha_2,\dots, \alpha_m)$ and $\phi(\beta_1, \beta_2,\dots, \beta_m)$ are ith associates if their indices $(\alpha_1, \alpha_2,\dots, \alpha_m)$ and $(\beta_1, \beta_2,\dots, \beta_m)$ have $m-i$ integers in common. Each treatment is the 0th associate of itself.

Since this association scheme is well-defined for a positive integer m such that $2\leqq m\leqq s/2$, we can show that

$$\binom{s}{m}\neq\binom{s}{l} \quad \text{for} \quad l<m$$

under this restriction. It follows therefore that $T_m \not\supseteq T_l$ for positive integers m and l such that $m>l\geqq 2$.

11. Remarks

As mentioned in Sections 7, 8, 9 and 10, we can discuss without difficult the reductions of the number of associate classes by use of Lemmas 2.1 and 2.2 provided that the integral values of parameters p^i_{jk} in an association scheme are explicitly known or the patterns of p^i_{jk} are concretely known. Then, when there are association schemes described in this part, the reducibilities of the number of associate classes for **PBIB** designs based on the association schemes have only to check the coincidence numbers λ_i of the **PBIB** designs from Lemmas 2.1 and 2.2. For example, though we have $F_3\supseteq F_2$ as shown in Section 9, the **PBIB** design (I) with the F_3 type association scheme in Section 3 is not reducible to a **PBIB** design with the F_2 type association scheme, since relations $\lambda'_1=\lambda'_2=\lambda'_3$ and $\lambda'_5=\lambda'_6=\lambda'_7$ do not hold.

Part III. Combinatorial properties of a balanced or partially balanced block design

We dealt with incomplete block designs with the equi-replications and the equal block sizes in Part I. From a practical point of view, it may not be possible to design equi-size blocks accommodating the equi-replications of each treatment in all the blocks. We shall, here, deal with the block designs with unequal block sizes and/or different replications. Before considering a partially balanced block (**PBB**) design generalized a **PBIB** design in a sense, we shall discuss a balanced

$$L = \begin{bmatrix} L_{v-1} \\ v^{-\frac{1}{2}} E_{1\times v} \end{bmatrix},$$

we can write as

$$C = \rho\left(I_v - \frac{1}{v} G_v\right), \tag{12.2}$$

where $\rho = (n-b)/(v-1)$ and $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$.

Thus, a balanced block (**BB**) design is given by an incidence matrix N satisfying

$$\operatorname{diag}\{r_1, r_2, \ldots, r_v\} - N \operatorname{diag}\{k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1}\} N' = \rho\left(I_v - \frac{1}{v} G_v\right), \tag{12.3}$$

where $\rho = (\sum_{i=1}^{v} r_i - b)/(v-1)$.

Note that a **BIB** design is a special case of **BB** designs.

Following Ishii and Ogawa [20], suppose that the association matrices A_0, $A_1, \ldots, A_m$ are defined as mentioned in Section 1. Furthermore, $A_i^\#$, $i=0, 1, \ldots, m$, $\operatorname{rank} A_i^\# = \alpha_i$ are the mutually orthogonal idempotents of the association algebra $\mathfrak{A}$.

Definition: A block design is said to be partially balanced with m associate classes if the $C = D_r - ND_k^{-1}N'$ matrix of the design has the latent roots $0, \rho_1, \rho_2, \ldots, \rho_m$ with multiplicities $1, \alpha_1, \alpha_2, \ldots, \alpha_m$ and the linear space spanned by the latent vectors corresponding to a root ρ_i is equal to the linear space spanned by the column vectors of $A_i^\#$, $i=1, 2, \ldots, m$ (by a suitable change of order of ρ_i), where $D_r = \operatorname{diag}\{r_1, r_2, \ldots, r_v\}$ and $D_k^{-1} = \operatorname{diag}\{k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1}\}$ in (12.1).

In this case there exists an orthogonal matrix L such that

$$C = L' \begin{pmatrix} \left.\begin{matrix} \rho_1 & & \\ & \ddots & \\ & & \rho_1 \end{matrix}\right\} \alpha_1 & & & 0 \\ & \ddots & & \\ & & \left.\begin{matrix} \rho_m & & \\ & \ddots & \\ & & \rho_m \end{matrix}\right\} \alpha_m & \\ 0 & & & 0 \end{pmatrix} L,$$

$$L = [L_1 : L_2 : \cdots : L_m : v^{-\frac{1}{2}} E_{v\times 1}]',$$

where L_i are of order $v \times \alpha_i$ and each column of L_i is the independent latent vectors corresponding to ρ_i, $i=1, 2,\dots, m$,

$$L_i'L_i = I_{\alpha_i}$$

and L_iL_i' are the projection operators to the linear space spanned by L_i. Then

$$L_iL_i' = A_i^{\#}.$$

Hence

$$C = \rho_1 L_1 L_1' + \rho_2 L_2 L_2' + \cdots + \rho_m L_m L_m' \qquad (12.4)$$

$$= \rho_1 A_1^{\#} + \rho_2 A_2^{\#} + \cdots + \rho_m A_m^{\#}.$$

Thus, a partially balanced block (**PBB**) design with m associate classes is given by an incidence matrix N satisfying

$$C = \mathrm{diag}\{r_1, r_2,\dots, r_v\} - N\,\mathrm{diag}\{k_1^{-1}, k_2^{-1},\dots, k_b^{-1}\}N' \qquad (12.5)$$

$$= \rho_1 A_1^{\#} + \rho_2 A_2^{\#} + \cdots + \rho_m A_m^{\#}.$$

Furthermore, from relation (1.7), (12.4) can be written as

$$C = \mathrm{diag}\{r_1, r_2,\dots, r_v\} - N\,\mathrm{diag}\{k_1^{-1}, k_2^{-1},\dots, k_b^{-1}\}N' \qquad (12.6)$$

$$= a_0 A_0 + a_1 A_1 + \cdots + a_m A_m,$$

where

$$a_0 = (\sum_{i=1}^{v} r_i - b)/v \qquad (12.7)$$

and

$$a_i \leqq 0, \qquad i = 1, 2,\dots, m. \qquad (12.8)$$

For, from a comparison of diagonal elements in both sides of (12.6) we have (12.7). Furthermore, from a comparison of off-diagonal elements and $a_{\alpha i}^{\beta}=1$ or 0, we have (12.8).
Explicitly,

$$\rho_i = a_0 + a_1 z_{i1} + a_2 z_{i2} + \cdots + a_m z_{im}, \qquad i = 0, 1, 2,\dots, m$$

with $\rho_0=0$, or

$$a_i = \rho_1 z^{i1} + \rho_2 z^{i2} + \cdots + \rho_m z^{im}, \qquad i = 0, 1,\dots, m$$

where $\|z^{ij}\| = Z^{-1}$ for $Z=\|z_{ij}\|$ in (1.6).

For a **PBIB** design N with m associate classes we have from Lemma A

$$NN' = rkA_0^{\#} + \rho_1 A_1^{\#} + \cdots + \rho_m A_m^{\#}$$

and

$$I_v = A_0^{\#} + A_1^{\#} + \cdots + A_m^{\#} .$$

Then

$$\begin{aligned} C &= rI_v - \frac{1}{k} NN' \\ &= \left(r - \frac{\rho_1}{k}\right) A_1^{\#} + \cdots + \left(r - \frac{\rho_m}{k}\right) A_m^{\#} . \end{aligned}$$

Thus, a **PBIB** design is a special case of **PBB** designs.

Finally, though the incomplete block designs satisfying $1 \leqq r_i \leqq b$ and $1 \leqq k_j \leqq v$ are generally considered, we will not deal with the three cases in which $r_i = b$, $r_i = 1$ and $k_j = 1$ for all $i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, b$ throughout Part III.

13. Properties of BB designs and PBB designs

From a structural point of view in a **BB** design or a **PBB** design we have the followings:

THEOREM 13.1. *A* **BB** *design with a constant block size is a* **BIB** *design.*

PROOF. In this case, for a **BB** design $N = \|n_{ij}\|$ we have from (12.3)

$$\begin{aligned} &\operatorname{diag}\{r_1, r_2, \ldots, r_v\} - \frac{1}{k} NN' \\ &= \frac{bk - b}{v - 1}\left(I_v - \frac{1}{v} G_v\right). \end{aligned}$$

Hence from comparisons of diagonal elements or off-diagonal elements in the above both sides

$$r_i - \frac{r_i}{k} = \frac{bk - b}{v - 1}\left(1 - \frac{1}{v}\right), \qquad i = 1, 2, \ldots, v ,$$

and

$$-\frac{1}{k} \sum_{l=1}^{b} n_{il} n_{jl} = -\frac{bk - b}{v(v - 1)} ,$$

for all $i, j\ (i \neq j) = 1, 2, \ldots, v$, which lead respectively to

$$(13.1) \qquad r_i = \frac{b(v-1)k(k-1)}{v(v-1)}, \qquad i = 1, 2, \ldots, v,$$

and

$$(13.2) \qquad \sum_{l=1}^{b} n_{il} n_{jl} = \frac{bk(k-1)}{v(v-1)},$$

for all $i, j\ (i \neq j) = 1, 2, \ldots, v$. Therefore, from Section 1 (13.1) and (13.2) imply that a **BB** design N with a constant block size, k, is a **BIB** design. This result was essentially derived in Ishii [19] and Rao [47].

THEOREM 13.2. *A* **PBB** *design with a constant block size based on an association scheme of m associate classes is a* **PBIB** *design based on the same association scheme.*

PROOF. As shown in Section 12, for a **PBB** design $N = \|n_{ij}\|$ in this case, we have

$$(13.3) \qquad \mathrm{diag}\,\{r_1, r_2, \ldots, r_v\} - \frac{1}{k} NN'$$
$$= a_0 A_0 + a_1 A_1 + \cdots + a_m A_m .$$

Hence from a property of the association matrices described in Section 1 we obtain after a comparison of diagonal elements in (13.3)

$$r_i - \frac{r_i}{k} = a_0 \qquad \text{for all } i = 1, 2, \ldots, v,$$

which imply that the replication of each treatment is a constant. Furthermore, from a comparison of off-diagonal elements in (13.3) we obtain

$$\sum_{l=1}^{b} n_{il} n_{jl} = -ka_p, \qquad \text{for all} \quad i, j\ (i \neq j) = 1, 2, \ldots, v$$

provided the ith and jth treatments are pth associates $(p = 1, 2, \ldots, m)$. Setting $\lambda_p = -ka_p$ shows that a **PBB** design N with a constant block size, k, based on an association scheme of m associate classes is a **PBIB** design based on the same association scheme having the coincidence numbers λ_p $(p = 1, 2, \ldots, m)$.

As another property in a **BB** design, Bhaskararao [4] showed that an equireplicate **BB** design with $b = v$ is a symmetrical **BIB** design. The proposition to hold for a **PBB** design corresponding to Bhaskararao's result does not hold. In fact, there exists a **PBB** design with $v = b = 6$, $r = 3$, $k_j = 2$, 3 or 6 based on the F_2

type association scheme of $v=2\times 3$ treatments which will be seen in Example 18.6.

14. Construction of BB designs

We begin by describing trivial methods of constructing a **BB** design.

THEOREM 14.1.* *If N_i are* **BB** *designs with a common treatment number for $i=1, 2,\dots, l$, then juxtaposition of its designs*

$$N = [N_1 : N_2 : \cdots : N_l]$$

is a **BB** design.

Proof is given by the fact that the C-matrix of N is equal to the sum of C-matrices of N_i, $i=1, 2,\dots, l$. Simple examples are made when N_i are **BIB** designs with a common treatment number and different block sizes. Furthermore, in this case cyclic solutions of **BB** designs N are obtainable through more than one initial block (i.e., difference set). For example, juxtaposition $[N_1 : N_1^*]$ of a **BIB** design N_1, with parameters v, b, r, k, λ and $v \neq 2k$, generated by some initial blocks and its complement N_1^*. For difference sets generating **BIB** designs with the parameters of practically useful range, we refer to Takeuchi [55], Sillitto [53], Clatworthy and Lewyckyj [13] and Kageyama [25].

COROLLARY 14.2. *When N is a* **BB** *design,*

$$[N : I_v] \quad \text{and} \quad [N : E_{v\times 1}]$$

are **BB** *designs.*

COROLLARY 14.3. *When N and N_1 are* **BB** *designs in $N=[N_1 : N_2]$, N_2 is a* **BB** *design provided N_2 is connected.*

THEOREM 14.4 (cf. [20]). *Suppose that there are l* **PBB** *designs N_t ($t=1, 2,\dots, l$) based on the same association scheme of m associate classes, whose C-matrices are given as*

$$C_1 = \rho_1^{(1)}A_1^{\#} + \rho_2^{(1)}A_2^{\#} + \cdots + \rho_m^{(1)}A_m^{\#},$$

$$C_2 = \rho_1^{(2)}A_1^{\#} + \rho_2^{(2)}A_2^{\#} + \cdots + \rho_m^{(2)}A_m^{\#},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

*) After writing this paper we came to know that Theorem 14.1 was shown in a different expression by A. Hedayat and W. T. Federer (Pairwise and variance balanced incomplete block designs. To appear in Ann. Inst. Statist. Math.).

$$C_l = \rho_1^{(l)} A_1^{\#} + \rho_2^{(l)} A_2^{\#} + \cdots + \rho_m^{(l)} A_m^{\#} .$$

If

$$\rho = \rho_i^{(1)} + \rho_i^{(2)} + \cdots + \rho_i^{(l)} \qquad \text{for all } i = 1, 2, \ldots, m, \tag{14.1}$$

then juxtaposition of its **PBB** *designs* $N = [N_1 : N_2 : \cdots : N_l]$ *is a* **BB** *design.*

Proof is similar to that of Theorem 14.1 and hence omitted here. An example of this theorem is shown in Ishii and Ogawa [20] by use of **PBB** designs based on an N_2 type association scheme.

COROLLARY 14.5. *When N_t in Theorem* 14.4 *are* **PBIB** *designs with replications $r^{(t)}$, block sizes $k^{(t)}$ and coincidence numbers $\lambda_i^{(t)}$ for $t = 1, 2, \ldots, l$; $i = 1, 2, \ldots, m$, condition* (14.1) *can be replaced by a condition such that*

$$\frac{\lambda_i^{(1)}}{k^{(1)}} + \frac{\lambda_i^{(2)}}{k^{(2)}} + \cdots + \frac{\lambda_i^{(l)}}{k^{(l)}} = \text{constant} \tag{14.2}$$

for all $i = 1, 2, \ldots, m$.

PROOF. First, we shall show that condition (14.1) is equivalent to

$$\frac{\rho_i^{[1]}}{k^{(1)}} + \frac{\rho_i^{[2]}}{k^{(2)}} + \cdots + \frac{\rho_i^{[l]}}{k^{(l)}} = \text{constant } (= \alpha, \text{ say}) \tag{14.3}$$

for all $i = 1, 2, \ldots, m$, where $\rho_i^{[t]}$ are latent roots of $N_t N_t'$ for **PBIB** designs N_t for $t = 1, 2, \ldots, l$; $i = 1, 2, \ldots, m$.

From Lemma A in Section 1, we have

$$N_t N_t' = r^{(t)} k^{(t)} A_0^{\#} + \rho_1^{[t]} A_1^{\#} + \cdots + \rho_m^{[t]} A_m^{\#} .$$

Then the C-matrices of N_t are

$$\begin{aligned} C_t &= r^{(t)} I_v - \frac{1}{k^{(t)}} N_t N_t' \\ &= \sum_{i=1}^{m} \left(r^{(t)} - \frac{\rho_i^{[t]}}{k^{(t)}} \right) A_i^{\#} , \end{aligned}$$

for $t = 1, 2, \ldots, l$. Hence, the above implies that (14.1) is equivalent to (14.3).

Next, we shall show that condition (14.2) is equivalent to condition (14.3). From Lemma A we have

$$\rho_i^{[t]} = \sum_{j=0}^{m} \lambda_j^{(t)} z_{ij} , \qquad t = 1, 2, \ldots, l; i = 0, 1, \ldots, m .$$

As a matrix notation, we obtain

$$\left[\frac{\rho_0^{[t]}}{k^{(t)}}, \frac{\rho_1^{[t]}}{k^{(t)}}, \ldots, \frac{\rho_m^{[t]}}{k^{(t)}}\right] = \left[\frac{r^{(t)}}{k^{(t)}}, \frac{\lambda_1^{(t)}}{k^{(t)}}, \ldots, \frac{\lambda_m^{(t)}}{k^{(t)}}\right] Z,$$

where $Z = \|z_{ij}\|$ is given in (1.6). Then we can have

$$(14.4) \qquad \left[\sum_{t=1}^{l} \frac{\rho_0^{[t]}}{k^{(t)}}, \sum_{t=1}^{l} \frac{\rho_1^{[t]}}{k^{(t)}}, \ldots, \sum_{t=1}^{l} \frac{\rho_m^{[t]}}{k^{(t)}}\right]$$

$$= \left[\sum_{t=1}^{l} \frac{r^{(t)}}{k^{(t)}}, \sum_{t=1}^{l} \frac{\lambda_1^{(t)}}{k^{(t)}}, \ldots, \sum_{t=1}^{l} \frac{\lambda_m^{(t)}}{k^{(t)}}\right] Z.$$

Hence it clearly follows from (1.4) that condition (14.2) implies condition (14.3). Conversely, assume that condition (14.3) holds. Then, letting $\beta = \sum_{t=1}^{l} r^{(t)}/k^{(t)}$ and $x_i = \sum_{t=1}^{l} \lambda_i^{(t)}/k^{(t)}$, we have from (14.4)

$$\begin{aligned}
z_{11}x_1 + z_{12}x_2 + \cdots + z_{1m}x_m &= \alpha - \beta, \\
z_{21}x_1 + z_{22}x_2 + \cdots + z_{2m}x_m &= \alpha - \beta, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
z_{m1}x_1 + z_{m2}x_2 + \cdots + z_{mm}x_m &= \alpha - \beta.
\end{aligned}$$

In this case, it is sufficient to show that $x_1 = x_2 = \cdots = x_m$ and hence (14.2) holds. From (1.4) we now obtain

$$(14.5) \qquad \begin{aligned}
(z_{12} - z_{22})(x_2 - x_1) + (z_{13} - z_{23})(x_3 - x_1) + \cdots + (z_{1m} - z_{2m})(x_m - x_1) &= 0, \\
(z_{12} - z_{32})(x_2 - x_1) + (z_{13} - z_{33})(x_3 - x_1) + \cdots + (z_{1m} - z_{3m})(x_m - x_1) &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(z_{12} - z_{m2})(x_2 - x_1) + (z_{13} - z_{m3})(x_3 - x_1) + \cdots + (z_{1m} - z_{mm})(x_m - x_1) &= 0.
\end{aligned}$$

Since the matrix Z is nonsingular as seen in Section 1, it follows that the determinant

$$\begin{vmatrix}
z_{12} - z_{22}, & z_{13} - z_{23}, \ldots, & z_{1m} - z_{2m} \\
z_{12} - z_{32}, & z_{13} - z_{33}, \ldots, & z_{1m} - z_{3m} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
z_{12} - z_{m2}, & z_{13} - z_{m3}, \ldots, & z_{1m} - z_{mm}
\end{vmatrix} \neq 0.$$

Therefore from (14.5) we obtain

$$x_1 = x_2 = \cdots = x_m.$$

Thus, we have the required result.

Note that in Corollary 14.5 condition (14.2) can be replaced by condition (14.3).

COROLLARY 14.6. *Suppose that there exists an association scheme of* m *associate classes with parameters* v, n_i, p^i_{jk}, *whose association matrices are* $A_0, A_1,\dots,$ *and* A_m. *If*

$$\frac{p^l_{ii}}{n_i} + \frac{p^l_{jj}}{n_j} = \text{constant for all } l = 1, 2,\dots, m,$$

then $[A_i : A_j]$ *is a* **BB** *design.*

For, the association matrices are symmetrical **PBIB** designs with the same association scheme and hence from (1.1) we obtain the result.

REMARK. Considering a mixed type of a linear combination and juxtaposition of association matrices, Kageyama [29] has constructed a series of **BIB** designs under some restrictions. The method used in them may lead to the various **BB** designs under some restrictions, but they are omitted here.

THEOREM 14.7. *If* $b=(v+1)/2$, *then the following matrix is a* **BB** *design with parameters* $v'=v+2$, $b'=b+v+1$, $r'_i=v+1$ *or* $b+1$, $k'_j=2$ *or* $v+1$ $(i=1,2,\dots,v';\ j=1,2,\dots,b')$:

$$N = \left(\begin{array}{c:c:c} E_{1\times b} & 0_{1\times v} & 1 \\ \hdashline 0_{1\times b} & E_{1\times v} & 1 \\ \hdashline E_{v\times b} & I_v & 0_{v\times 1} \end{array}\right).$$

For, we have the C-matrix of N as

$$C = \begin{pmatrix} \frac{bv}{v+1}+\frac{1}{2}, & -\frac{1}{2}, & -\frac{b}{v+1},\dots, & -\frac{b}{v+1} \\ -\frac{1}{2}, & \frac{v}{2}+\frac{1}{2}, & -\frac{1}{2},\dots, & -\frac{1}{2} \\ -\frac{b}{v+1}, & -\frac{1}{2}, & & \\ \vdots & \vdots & \multicolumn{2}{c}{\left(b+\frac{1}{2}\right)I_v - \frac{b}{v+1}G_v} \\ -\frac{b}{v+1}, & -\frac{1}{2}, & & \end{pmatrix}.$$

THEOREM 14.8. *If there are a* **BIB** *design* N_1 *with parameters* v, b, r, k, λ *and a* **BB** *design* N_2 *with parameters* $v, b', r_i, k_j, n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b'} k_j$ *for positive integers* l *and* m *such that*

$$\frac{m}{l}=\frac{v(v-1)(r-\lambda)}{(n-b')(k+1)},$$

then there exists a **BB** *design* N *with parameters* $v^*=v+1$, $b^*=lb+mb'$, $r_{i'}^*=lb$ *or* $lr+mr_i$, $k_{j'}^*=k+1$ *or* k_j:

$$N=\left(\begin{array}{ccc:ccc} \underbrace{\begin{matrix} E_{1\times b} & \cdots & E_{1\times b} \\ N_1 & \cdots & N_1 \end{matrix}}_{l \text{ times}} & \underbrace{\begin{matrix} 0_{1\times b'} & \cdots & 0_{1\times b'} \\ N_2 & \cdots & N_2 \end{matrix}}_{m \text{ times}} \end{array}\right).$$

PROOF. The C-matrix of N is given by

$$C=\begin{pmatrix} lb-\dfrac{lb}{k+1}, & -\dfrac{lr}{k+1},\ldots, & -\dfrac{lr}{k+1} \\ -\dfrac{lr}{k+1}, & & \\ \vdots & lrI_v-\dfrac{l}{k+1}N_1N_1'+m(D_{r_2}-N_2D_{k_2}^{-1}N_2') & \\ -\dfrac{lr}{k+1}, & & \end{pmatrix},$$

where $D_{r_2}=\operatorname{diag}\{r_1, r_2,\ldots, r_v\}$ and $D_{k_2}^{-1}=\operatorname{diag}\{k_1^{-1}, k_2^{-1},\ldots, k_{b'}^{-1}\}$. Then N is a **BB** design, if and only if

$$-\frac{lr}{k+1}=-\frac{l\lambda}{k+1}-\frac{m\rho}{v} \tag{14.6}$$

and

$$lb-\frac{lb}{k+1}=lr-\frac{lr}{k+1}+m\rho\left(1-\frac{1}{v}\right), \tag{14.7}$$

where $\rho=(n-b')/(v-1)$.

Since l and m should be chosen so that both the above equations are satisfied simultaneously, we have

$$m/l=v(v-1)(r-\lambda)/(n-b')(k+1)$$

which is actually derived from (14.6), but it satisfies (14.7). Hence the theorem follows.

Note that when N_2 is a **BIB** design, Theorem 14.8 leads to the result of the binary type of Theorem 2.1 in Kulshreshtha, Dey and Saha [33]. When $l=m=1$, we have

COROLLARY 14.9. *If there are a* **BIB** *design* N_1 *with parameters* v, b, r, k, λ *and a* **BB** *design* N_2 *with parameters* v, b', r_i, k_j, $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b'} k_j$ *such that*

$$n-b' = v(v-1)(r-\lambda)/(k+1), \tag{14.8}$$

then the following matrix is a **BB** *design with parameters* $v^*=v+1$, $b^*=b+b'$, $r_{i'}^*=b$ *or* $r+r_i$, $k_{j'}^*=k_j$ *or* $k+1$:

$$\left(\begin{array}{c:c} E_{1\times b} & 0_{1\times b'} \\ \hdashline N_1 & N_2 \end{array}\right).$$

EXAMPLE 14.1. Consider a **BIB** design with parameters $v=5$, $b=10$, $r=4$, $k=2$, $\lambda=1$ of the unreduced type and the following **BB** design with $v=5$, $b'=12$ and $n=32$:

	1	2	3	4	5	6	7	8	9	10	11	12	r_i
1	1	1	1	1	0	0	1	1	0	0	0	1	7
2	1	0	0	0	1	1	0	0	1	1	1	1	7
3	0	1	0	0	1	1	1	1	1	0	0	0	6
4	0	0	1	0	1	1	1	1	0	1	0	0	6
5	0	0	0	1	1	1	1	1	0	0	1	0	6
k_j	2	2	2	2	4	4	4	4	2	2	2	2	32

, $C=5\left(I_5-\frac{1}{5}G_5\right)$.

Then these two designs satisfy condition (14.8) and hence Corollary 14.9 yields a **BB** design with parameters $v=6$, $b=22$, $r_i=10$ or 11, $k_j=2$, 3 or 4 ($i=1,2,\dots,6$; $j=1,2,\dots,22$).

COROLLARY 14.10. *If there exists a* **BB** *design* N *with parameters* v, b, r_i, k_j *and* $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$ *such that* $v(v-1)=2(n-b)$, *then the following matrix is a* **BB** *design with parameters* $v'=v+1$, $b'=v+b$, $r_0=v$, r_i+1, $k_0=2$, k_j *for* $i=1,2,\dots,v$; $j=1,2,\dots,b$:

$$\left(\begin{array}{c:c} E_{1\times v} & 0_{1\times b} \\ \hdashline I_v & N \end{array}\right).$$

Note that when N is a **BIB** design, Corollary 14.10 leads to the result of John [21].

EXAMPLE 14.2. Consider a **BB** design with parameters $v=5$, $b=6$ and $n=16$ satisfying $v(v-1)=2(n-b)$, which is constructed by the first six columns (blocks) in the **BB** design of Example 14.1, i.e.,

	1	2	3	4	5	6	r_i
1	1	1	1	1	0	0	4
2	1	0	0	0	1	1	3
3	0	1	0	0	1	1	3
4	0	0	1	0	1	1	3
5	0	0	0	1	1	1	3
k_j	2	2	2	2	4	4	16

$$C=\frac{5}{2}\left(I_5-\frac{1}{5}G_5\right).$$

Then Corollary 14.10 yields a **BB** design with parameters $v=6$, $b=11$, $r_i=4$ or 5, $k_j=2$ or 4 $(i=1,2,\ldots,6;\ j=1,2,\ldots,11)$.

Note that Theorem 14.8, Corollaries 14.9 and 14.10 imply the method of constructing a **BB** design with $v+1$ treatments from a **BB** design with v treatments. If we have I_v or $E_{v\times b}$ as a **BIB** design N_1 in Theorem 14.8, then it seems that a **BB** design with v treatments can be extended to a **BB** design with $v+1$ treatments when and only when Theorem 14.8, Corollaries 14.9 or 14.10 hold.

Pairwise balanced designs of index λ, introduced by Bose and Shrikhande [11] for constructing counterexamples to Euler's conjecture, may lead to **BB** designs, which are constructed by the juxtaposition of **BIB** designs with a common treatment number and different block sizes, and which are obtained by omitting some treatments in **BIB** designs. However, the **BB** designs so obtained may be included in Theorem 14.1.

Note that the complement of a **BB** design is generally not a **BB** design and that Corollary 14.10 also implies a method of constructing a **BIB** design with $v'=v+1$ and $k'=2$ of the unreduced type from a **BIB** design with $v^*=v$ and $k^*=2$ of the unreduced type. Incidentally, from Theorems 13.1 and 14.8 the following **BIB** design is constructed. That is, if there are two **BIB** designs, respectively, with parameters v, b, r, k, λ and with parameters v, b', r', $k'=k+1$, λ', then for positive integers l and m such that $(r-\lambda)/\lambda'=m/l$, there exists a **BIB** design with parameters $v^*=v+1$, $b^*=lb+mb'$, $r^*=lb$, $k^*=k+1$ and $\lambda^*=lr$.

Finally, we will review the construction of a **BB** design from a structural point of view. As stated in Section 12, a block design is balanced, if and only if C is of the form

$$C = (x-y)I_v + yG_v,$$

where x and y are two constants. Then from (12.3) a **BB** design $N=\|n_{ij}\|$ with parameters v, b, r_i and k_j is given if

$$r_i - \left(\frac{n_{i1}}{k_1} + \frac{n_{i2}}{k_2} + \cdots + \frac{n_{ib}}{k_b}\right) = \text{constant}, \tag{14.9}$$

for all $i=1, 2,\dots, v$ and

$$\frac{n_{i1}n_{j1}}{k_1} + \frac{n_{i2}n_{j2}}{k_2} + \cdots + \frac{n_{ib}n_{jb}}{k_b} = \text{constant}, \tag{14.10}$$

for all i, j $(i \neq j) = 1, 2,\dots, v$. Empirically, if condition (14.10) is satisfied, then condition (14.9) is often satisfied automatically.

Consider an equireplicate **BB** design N with l kinds of block sizes. By permuting the blocks, we can write N as

$$N = [N_1 : N_2 : \cdots : N_l],$$

where the blocks of N_p are of size k_p $(p=1, 2,\dots, l)$. Denote by $\lambda_{ij}^{(p)}$ the number of blocks containing the ith and jth treatments together in N_p and $\lambda_{ii}^{(p)} = r_i^{(p)}$ for $p=1, 2,\dots, l$. Then from (14.9) and (14.10) there exists an equireplicate **BB** design N, if and only if

$$\frac{\lambda_{ij}^{(1)}}{k_1} + \frac{\lambda_{ij}^{(2)}}{k_2} + \cdots + \frac{\lambda_{ij}^{(l)}}{k_l} = \text{constant},$$

and

$$r = r_i^{(1)} + r_i^{(2)} + \cdots + r_i^{(l)}$$

for all $i, j=1, 2,\dots, v$. Furthermore, in a similar way, there exists a **BB** design N with l kinds of block sizes, if and only if

$$r_i^{(1)}\left(1-\frac{1}{k_1}\right) + r_i^{(2)}\left(1-\frac{1}{k_2}\right) + \cdots + r_i^{(l)}\left(1-\frac{1}{k_l}\right)$$
$$= \text{constant},$$

for all $i=1, 2,\dots, v$, and

$$\frac{\lambda_{ij}^{(1)}}{k_1} + \frac{\lambda_{ij}^{(2)}}{k_2} + \cdots + \frac{\lambda_{ij}^{(l)}}{k_l} = \text{constant}$$

for all i, j $(i \neq j) = 1, 2, \ldots, v$. These necessary and sufficient conditions are useful when we want to construct **BB** designs by trial and error.

15. Construction of PBB designs

Similarly to Section 14, we can consider the construction of **PBB** designs.

THEOREM 15.1. *If there are l* **PBB** *design N_i with the same association scheme for $i = 1, 2, \ldots, l$, then juxtaposition of its designs*

$$N = [N_1 : N_2 : \cdots : N_l]$$

is a **PBB** *design with the same association scheme as N_i.*

Proof is given by (12.5) and the fact that the C-matrix of N is equal to the sum of C-matrices of N_i, $i = 1, 2, \ldots, l$. Simple examples are given when N_i $(i = 1, 2, \ldots, l)$ are **PBIB** designs, based on the same association scheme, with a common treatment number and different block sizes. Furthermore, in this case cyclic solutions of **PBB** designs N are obtainable through more than one initial block (i.e., difference set). For example, juxtaposition $[N_1 : N_1^*]$ of a **PBIB** design N_1, with parameters v, b, r, k, λ_i and $v \neq 2k$, generated by some initial blocks and its complement N_1^*, since an association scheme of the complementary **PBIB** design is the same as that of the original **PBIB** design.

Note that symmetrical unequal-block arrangements with two unequal block sizes, introduced by Kishen [32] and their constructions and analysis have been thoroughly investigated by Raghavarao [44], are essentially included in the type of Theorem 15.1. Further, note [44] that no symmetrical unequal block arrangement with two unequal block sizes is balanced.

COROLLARY 15.2. *If there exists an association scheme of m associate classes with association matrices $A_0, A_1, \ldots, A_m$, then*

$$[A_i : A_j]$$

is a **PBB** *design with the same association scheme.*

Proof is included in that of Corollary 14.6.

REMARK. Considering a mixed type of a linear combination and juxtaposition of association matrices, Kageyama [29] gave a remark on the construction of **PBIB** designs under some restrictions. The approach used in them may lead to some **PBB** designs under conditions, but they are omitted here.

COROLLARY 15.3. *If there exists a* **PBB** *design N based on an association scheme of m associate classes, then*

$$[N: I_v] \quad \text{and} \quad [N: E_{v\times 1}]$$

are **PBB** *designs based on the same association scheme.*

For, since the C-matrix of $E_{v\times 1}$ is

$$I_v - \frac{1}{v} E_{v\times 1} E_{1\times v} = A_1^\# + A_2^\# + \cdots + A_m^\#,$$

from Theorem 15.1 we have the required result.

Note that if N is a disconnected **PBB** design, then from Corollary 15.3 we can have a connected **PBB** design $[N: E_{v\times 1}]$ and hence we may treat disconnected **PBB** designs.

COROLLARY 15.4. *When N and N_1 are* **PBB** *designs with the same association scheme in $N=[N_1: N_2]$, N_2 is a* **PBB** *design with the same association scheme provided N_2 is connected.*

When there are two **PBB** designs N_i with $D_{r^{(i)}} = \mathrm{diag}\{r_1^{(i)}, r_2^{(i)}, \ldots, r_{v_i}^{(i)}\}$ and $D_{k^{(i)}} = \mathrm{diag}\{k_1^{(i)}, k_2^{(i)}, \ldots, k_{b_i}^{(i)}\}$ for $i=1, 2$, the C-matrix of $N=N_1 \otimes N_2$ is as follows:

$$\begin{aligned} C &= D_{r^{(1)}} \otimes D_{r^{(2)}} - (N_1 \otimes N_2)(D_{k^{(1)}} \otimes D_{k^{(2)}})^{-1}(N_1 \otimes N_2)' \\ &= D_{r^{(1)}} \otimes D_{r^{(2)}} - N_1 D_{k^{(1)}}^{-1} N_1' \otimes N_2 D_{k^{(2)}}^{-1} N_2' \\ &= D_{r^{(1)}} \otimes D_{r^{(2)}} - (D_{r^{(1)}} - C_1) \otimes (D_{r^{(2)}} - C_2) \end{aligned}$$

$$(15.1) \qquad = D_{r^{(1)}} \otimes C_2 + C_1 \otimes D_{r^{(2)}} - C_1 \otimes C_2,$$

where C_i are C-matrices of N_i, $i=1, 2$. Then we have

THEOREM 15.5. *If there are equireplicate* **PBB** *designs N_i $(i=1,2)$ with parameters $v^{(i)}$, $b^{(i)}$, $r^{(i)}$, $k_j^{(i)}$ $(j=1, 2,\ldots, b^{(i)})$ having association schemes of s and t associate classes, respectively, then*

$$N = N_1 \otimes N_2 \qquad (or\ N_2 \otimes N_1)$$

is an equireplicate **PBB** *design with at most $st+s+t$ associate classes.*

PROOF. Denote the association matrices and the corresponding mutually orthogonal idempotents of association schemes of s and t associate classes, respectively, by $B_0, B_1,\ldots, B_s$; $B_0^\#, B_1^\#,\ldots, B_s^\#$ and $A_0, A_1,\ldots, A_t$; $A_0^\#, A_1^\#,\ldots, A_t^\#$. Since we can now write C-matrices of N_i $(i=1, 2)$ as

$$\begin{aligned} C_1 &= \rho_1^{(1)} B_1^\# + \rho_2^{(1)} B_2^\# + \cdots + \rho_s^{(1)} B_s^\#, \\ C_2 &= \rho_1^{(2)} A_1^\# + \rho_2^{(2)} A_2^\# + \cdots + \rho_t^{(2)} A_t^\#, \end{aligned}$$

from (15.1) the C-matrix of N is

$$\begin{aligned}
C &= r^{(1)}(B_0^{\#}+B_1^{\#}+\cdots+B_s^{\#}) \otimes (\rho_1^{(2)}A_1^{\#}+\cdots+\rho_t^{(2)}A_t^{\#}) \\
&\quad + r^{(2)}(\rho_1^{(1)}B_1^{\#}+\cdots+\rho_s^{(1)}B_s^{\#}) \otimes (A_0^{\#}+A_1^{\#}+\cdots+A_t^{\#}) \\
&\quad - (\rho_1^{(1)}B_1^{\#}+\cdots+\rho_s^{(1)}B_s^{\#}) \otimes (\rho_1^{(2)}A_1^{\#}+\cdots+\rho_t^{(2)}A_t^{\#}) \\
&= r^{(1)}\sum_{j=1}^{t}\rho_j^{(2)}(B_0^{\#}\otimes A_j^{\#}) + r^{(2)}\sum_{i=1}^{s}\rho_i^{(1)}(B_i^{\#}\otimes A_0^{\#}) \\
&\quad + \sum_{i=1}^{s}\sum_{j=1}^{t}(r^{(1)}\rho_j^{(2)}+r^{(2)}\rho_i^{(1)}-\rho_i^{(1)}\rho_j^{(2)})(B_i^{\#}\otimes A_j^{\#}) .
\end{aligned}$$

Furthermore, it is easily shown that the association matrices of design N are given by $B_i \otimes A_j$ for $i=0, 1,\ldots, s$; $j=0, 1,\ldots, t$ and that

$$\begin{aligned}
(B_{i_1}^{\#}\otimes A_{j_1}^{\#})(B_{i_2}^{\#}\otimes A_{j_2}^{\#}) &= B_{i_1}^{\#}\otimes A_{j_1}^{\#}, \qquad \text{if } i_1=i_2 \text{ and } j_1=j_2, \\
&= 0_{v^{(1)}v^{(2)}\times v^{(1)}v^{(2)}}, \text{ otherwise} .
\end{aligned}$$

Therefore, definition (12.5) implies the result. The case of $N_2 \otimes N_1$ is also shown similarly.

COROLLARY 15.6. *If there exists an equireplicate* **PBB** *design N based on an association scheme with m associate classes of v treatments, then for a positive integer $l \geqq 1$,*

$$E_{v\times l}\otimes N \qquad (or\ N\otimes E_{v\times l})$$

is an equireplicate **PBB** *design with at most m^2+2m associate classes.*

Pairwise balanced designs introduced by Bose and Shrikhande may lead to **PBB** designs. For example, if there exists a **PBIB** design N with parameters $v=mn$, b, r, k, $\lambda_1=0$ and $\lambda_2=1$, based on an N_2 type association scheme of $v=mn$ treatments, then, by adding m new sets corresponding to the groups of the association scheme, we obtain a pairwise balanced design of index unity provided $k\neq n$, i.e.,

$$[N : I_m \otimes E_{n\times 1}]$$

which is a special case of Theorem 15.1 and hence this design is a **PBB** design.

It is useful to note that the complement of a **PBB** design is generally not a **PBB** design, though an association scheme remains invariant by the complement. However, the complement of a **PBB** design may become a **PBB** design. For example, the **PBB** design of Example 17.1 which will be given in Section 17 has this property.

Some examples of a **PBB** design are seen in [18; 19; 20]. In particular, Ishii [18] has given a numerical example with an analysis of a **PBB** design based on a rectangular lattice type association scheme with association matrices A_0, A_1, A_2, A_3 and A_4. That is the case in which $s=4$ in Section 10, and hence $v=12$, whose incidence matrix is given by

	1	2	3	4	5	6	7	8	9	10	11	r_i
1	1	1	1	0	0	0	0	0	0	0	0	3
2	1	0	0	1	1	0	0	0	0	0	0	3
3	1	0	0	0	0	1	1	1	0	0	0	4
4	1	1	1	0	0	0	0	0	0	0	0	3
5	1	0	0	0	0	0	1	0	1	1	0	4
6	1	0	0	1	1	0	0	0	0	0	0	3
7	1	0	0	1	1	0	0	0	0	0	0	3
8	1	0	0	0	0	0	0	1	1	0	1	4
9	1	1	1	0	0	0	0	0	0	0	0	3
10	1	0	0	0	0	1	0	0	0	1	1	4
11	1	0	0	1	1	0	0	0	0	0	0	3
12	1	1	1	0	0	0	0	0	0	0	0	3
k_j	12	4	4	4	4	2	2	2	2	2	2	40

$$C = \frac{29}{12} A_0 - \frac{1}{12}(A_1 + A_2) - \frac{7}{12}(A_3 + A_4)$$

$$= 3A_1^{\#} + 3A_2^{\#} + A_3^{\#} + 3A_4^{\#},$$

where

$$A_0^{\#} = \frac{1}{12} G_{12}, \quad A_1^{\#} = \frac{1}{4}(A_0 - A_3 + A_4),$$

$$A_2^{\#} = \frac{1}{8}(2A_0 - A_1 + A_2 - 2A_4),$$

$$A_3^{\#} = \frac{1}{12}(2A_0 - A_1 - A_2 + 2A_3 + 2A_4),$$

$$A_4^{\#} = \frac{1}{8}(2A_0 + A_1 - A_2 - 2A_4).$$

As other simple examples we present

EXAMPLE 15.1. A **PBB** design based on an N_3 type association scheme of $v=s_1s_2s_3$ treatments defined in Section 7.

$$A_0 = I_v, \quad A_1 = I_{s_1s_2} \otimes (G_{s_3} - I_{s_3}), \quad A_2 = I_{s_1} \otimes (G_{s_2} - I_{s_2}) \otimes G_{s_3},$$

$$A_3 = (G_{s_1} - I_{s_1}) \otimes G_{s_2s_3},$$

$$A_0^\# = \frac{1}{v} G_v, \quad A_1^\# = \frac{1}{v}\{(s_1-1)(A_0+A_1+A_2) - A_3\},$$

$$A_2^\# = \frac{s_2-1}{s_2s_3} A_0 + \frac{s_2-1}{s_2s_3} A_1 - \frac{1}{s_2s_3} A_2,$$

$$A_3^\# = \frac{s_3-1}{s_3} A_0 - \frac{1}{s_3} A_1.$$

Consider a design whose incidence matrix N is given by

$$N = [I_{s_1} \otimes E_{s_2s_3\times 1} : E_{v\times 1}].$$

Then

$$C = 2I_v - N \operatorname{diag}\Big\{\underbrace{\frac{1}{s_2s_3}, \ldots, \frac{1}{s_2s_3}}_{s_1 \text{ times}}, \quad \frac{1}{v}\Big\} N'$$

$$= \left(2 - \frac{s_1+1}{v}\right) A_0 - \frac{s_1+1}{v}(A_1+A_2) - \frac{1}{v} A_3$$

$$= A_1^\# + 2(A_2^\# + A_3^\#).$$

Thus, the design N is an equireplicate **PBB** design with unequal block sizes.

EXAMPLE 15.2. A **PBB** design based on an F_3 type association scheme of $v=v_1v_2v_3$ treatments provided $v_1=2$, defined in Section 9.

$$A_{000} = I_v, \quad A_{001} = I_{v_1v_2} \otimes (G_{v_3} - I_{v_3}),$$

$$A_{010} = I_{v_1} \otimes (G_{v_2} - I_{v_2}) \otimes I_{v_3},$$

$$A_{011} = I_{v_1} \otimes (G_{v_2} - I_{v_2}) \otimes (G_{v_3} - I_{v_3}),$$

$$A_{100} = (G_{v_1} - I_{v_1}) \otimes I_{v_2v_3},$$

$$A_{101} = (G_{v_1} - I_{v_1}) \otimes I_{v_2} \otimes (G_{v_3} - I_{v_3}),$$

$$A_{110} = (G_{v_1} - I_{v_1}) \otimes (G_{v_2} - I_{v_2}) \otimes I_{v_3},$$

$$A_{111} = (G_{v_1}-I_{v_1}) \otimes (G_{v_2}-I_{v_2}) \otimes (G_{v_3}-I_{v_3}),$$

$$A^{\#}_{000} = \frac{1}{v} G_v,$$

$$A^{\#}_{001} = \frac{1}{v}\{(v_3-1)(A_{000}+A_{010}+A_{100}+A_{110})-A_{001}-A_{011} -A_{101}-A_{111}\},$$

$$A^{\#}_{010} = \frac{1}{v}\{(v_2-1)(A_{000}+A_{001}+A_{100}+A_{101})-A_{010}-A_{011} -A_{110}-A_{111}\},$$

$$A^{\#}_{011} = \frac{1}{v}\{(v_2-1)(v_3-1)(A_{000}+A_{100})-(v_2-1)(A_{001}+A_{101}) -(v_3-1)(A_{010}+A_{110})+A_{011}+A_{111}\},$$

$$A^{\#}_{100} = \frac{1}{v}\{(v_1-1)(A_{000}+A_{001}+A_{010}+A_{011})-A_{100}-A_{101} -A_{110}-A_{111}\},$$

$$A^{\#}_{101} = \frac{1}{v}\{(v_1-1)(v_3-1)(A_{000}+A_{010})-(v_1-1)(A_{001}+A_{011}) -(v_3-1)(A_{100}+A_{110})+A_{101}+A_{111}\},$$

$$A^{\#}_{110} = \frac{1}{v}\{(v_1-1)(v_2-1)(A_{000}+A_{001})-(v_1-1)(A_{010}+A_{011}) -(v_2-1)(A_{100}+A_{101})+A_{110}+A_{111}\},$$

$$A^{\#}_{111} = \frac{1}{v}\{(v_1-1)(v_2-1)(v_3-1)A_{000}-(v_1-1)(v_2-1)A_{001} -(v_1-1)(v_3-1)A_{010}+(v_1-1)A_{011}-(v_2-1)(v_3-1)A_{100} +(v_2-1)A_{101}+(v_3-1)A_{110}-A_{111}\}.$$

Consider a design whose incidence matrix N is given by

$$N = [I_{v_1} \otimes E_{v_2v_3\times 1} : E_{v_1\times 1} \otimes I_{v_2v_3}].$$

$$C = 2I_v - N\,\mathrm{diag}\Big\{\underbrace{\frac{1}{v_2v_3},\dots,\frac{1}{v_2v_3}}_{v_1\text{ times}},\ \underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{v_2v_3\text{ times}}\Big\}\,N'$$

$$= \left(\frac{3}{2}-\frac{1}{v_2v_3}\right)A_{000}-\frac{1}{v_2v_3}(A_{001}+A_{010}+A_{011})-\frac{1}{2}A_{100}\,.$$

From (12.7) we have

$$\left(\sum_{i=1}^{v} r_i-b\right)/v=(3/2-1/v_2v_3)$$

which leads to $v_1=2$. In this case

$$C = A^{\#}_{001}+A^{\#}_{010}+A^{\#}_{011}+A^{\#}_{100}+2(A^{\#}_{101}+A^{\#}_{110}+A^{\#}_{111})\,.$$

Thus, the design N is an equireplicate **PBB** design with unequal block sizes provided $v_1=2$.

EXAMPLE 15.3. A **PBB** design with parameters $v=5$, $b=6$, $r=3$, $k_j=2$ or 5 based on the cyclic type association scheme with two associate classes of five treatments, whose incidence matrix is given by

	1	2	3	4	5	6	r_i
1	1	1	0	0	1	0	3
2	1	0	1	0	0	1	3
3	1	1	0	1	0	0	3
4	1	0	1	0	1	0	3
5	1	0	0	1	0	1	3
k_j	5	2	2	2	2	2	15

For this cyclic type association we have, for example,

$$A_0 = I_5,\quad A_1 = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix},$$

$$A^{\#}_0 = \frac{1}{5}(A_0+A_1+A_2)\,,$$

$$A^{\#}_1 = \frac{2}{5}A_0+\frac{1}{2}\left(\frac{1}{\sqrt{5}}-\frac{1}{5}\right)A_1-\frac{1}{2}\left(\frac{1}{\sqrt{5}}+\frac{1}{5}\right)A_2\,,$$

$$A^{\#}_2 = \frac{2}{5}A_0-\frac{1}{2}\left(\frac{1}{\sqrt{5}}+\frac{1}{5}\right)A_1+\frac{1}{2}\left(\frac{1}{\sqrt{5}}-\frac{1}{5}\right)A_2\,.$$

Then

$$C = \frac{9}{5}A_0 - \frac{1}{5}A_1 - \frac{7}{10}A_2$$
$$= \left(\frac{9+\sqrt{5}}{4}\right)A_1^\sharp + \left(\frac{9-\sqrt{5}}{4}\right)A_2^\sharp .$$

Examples of **PBB** designs based on association schemes of the other types are easily given by Corollary 15.3 provided there are **PBIB** designs based on the association schemes of the other types. Furthermore, since we may consider a **BB** design as a **PBB** design with one associate class, if there exists a **BB** design with v treatments, then the **BB** design may be considered as a trivial example of a **PBB** design based on a certain association scheme of v treatments.

Finally, we will review the construction of a **PBB** design from a structural point of view. As indicated in Sections 1 and 12, a **PBB** design with m associate classes, whose association matrices are $A_0, A_1,\dots, A_m$, is given by an incidence matrix $N=\|n_{ij}\|$ such that

$$\begin{aligned} C &= \operatorname{diag}\{r_1,\dots, r_v\} - N\operatorname{diag}\{k_1^{-1},\dots, k_b^{-1}\}N' \\ &= a_0A_0+a_1A_1+\cdots+a_mA_m, \end{aligned}$$

(15.2)
$$\begin{aligned} & a_0+a_1n_1+\cdots+a_mn_m = 0, \\ & a_0 = (\textstyle\sum_{i=1}^{v} r_i-b)/v, \quad \text{and} \\ & a_i \leqq 0, \quad i=1, 2,\dots, m. \end{aligned}$$

As an element-wise representation, we have that

$$\frac{n_{i1}}{k_1}+\frac{n_{i2}}{k_2}+\cdots+\frac{n_{ib}}{k_b} = r_i-a_0 \tag{15.3}$$

for all $i=1, 2,\dots, v$, and

$$\frac{n_{p1}n_{q1}}{k_1}+\frac{n_{p2}n_{q2}}{k_2}+\cdots+\frac{n_{pb}n_{qb}}{k_b} = -a_i \tag{15.4}$$

for all p, q $(p\neq q)=1, 2,\dots, v$, provided the pth and qth treatments are ith associates. Conditions (15.3) and (15.4) with (15.2) are very useful when we want to construct **PBB** designs by trial and error.

16. μ-resolvability of BB designs and PBB designs

It may be known that the resolvable solutions of a **BIB** design or a **PBIB** design are useful to the analysis of incomplete block designs and to constructions of other combinatorial arrangements. In a similar sense, it is conceivable that if the concept similar to the resolvability of a **BIB** design introduced by Bose [5] is defined in an incomplete block design with unequal block sizes, then such a resolvable solution generating the block design may be useful. In this section we shall consider the only combinatorial aspects of incomplete block designs with the concept of resolvability.

DEFINITION: A block design is called $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable if the blocks can be separated into t ($\geqq 2$) sets of m_i ($\geqq 1$) blocks such that the set consisting of m_i blocks contains every treatment exactly μ_i times ($i = 1, 2, \ldots, t$).

For a $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable block design, we necessarily have

$$b = m_1 + m_2 + \cdots + m_t,$$

$$r = \mu_1 + \mu_2 + \cdots + \mu_t,$$

and hence the block design is equireplicate. When $\mu_1 = \mu_2 = \cdots = \mu_t$ ($= \mu$, say), it is called μ-resolvable for $\mu \geqq 1$ and hence $r = \mu t$ which is a necessary condition for the existence of a μ-resolvable incomplete block design. In this case, if the block design has equal block sizes, then it corresponds to the definition of μ-resolvability of a **BIB** design introduced by Shrikhande and Raghavarao [51]. A 1-resolvable block design may be simply called resolvable.

First, we will treat the resolvability of a **BB** design. Some examples are given as follows.

EXAMPLE 16.1. A 3-resolvable **BB** design with parameters $v = 5$, $b = 15$, $r = 9$, $k_j = 2$, 3 or 4, $\mu = 3$ and $m_1 = m_2 = m_3 = 5$, whose incidence matrix is given by

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	r_i
1	0	0	1	1	1	1	1	0	1	0	1	1	1	0	0	9
2	1	1	1	0	0	0	0	1	1	1	1	1	1	0	0	9
3	1	1	1	0	0	1	1	0	1	0	0	0	1	1	1	9
4	1	1	0	1	0	1	1	1	0	0	1	1	0	1	0	9
5	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	9
k_j	4	4	3	2	2	4	4	2	3	2	4	4	3	2	2	45

,

$$C = \frac{15}{2}\left(I_5 - \frac{1}{5} G_5\right).$$

EXAMPLE 16.2 (cf. [20]). A (4, 2, 2)-resolvable **BB** design with parameters $v=6$, $b=18$, $r=8$, $k_j=2$ or 4, $\mu_1=4$, $\mu_2=\mu_3=2$ and $m_1=m_2=m_3=6$, whose incidence matrix is given by

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	r_i
1	0	0	1	1	1	1	1	1	0	0	0	0	1	1	0	0	0	0	8
2	1	1	0	0	1	1	1	0	1	0	0	0	0	0	1	1	0	0	8
3	1	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	1	1	8
4	1	1	1	1	0	0	0	0	0	1	1	0	1	0	1	0	0	0	8
5	1	1	0	0	1	1	0	0	0	1	0	1	0	1	0	0	1	0	8
6	0	0	1	1	1	1	0	0	0	0	1	1	0	0	0	1	0	1	8
k_j	4	4	4	4	4	4	2	2	2	2	2	2	2	2	2	2	2	2	48

,

$$C = 6\left(I_6 - \frac{1}{6} G_6\right).$$

Combining the 2nd and 3rd sets leads to a 4-resolvable **BB** design with $m_1=6$ and $m_2=12$.

Concerning the construction of these block designs, we have

THEOREM 16.1. *When there are two resolvable* **BIB** *designs* N_i $(i=1, 2)$ *with a common treatment number and different block sizes, the matrix*

$$[N_1 : N_2]$$

is a resolvable **BB** *design.*

THEOREM 16.2. *When there exists a resolvable* **BIB** *design* N, *the matrices*

$$[N: I_v] \quad \textit{and} \quad [N: E_{v\times l}] \quad \textit{for} \quad l \geqq 1$$

are resolvable **BB** *designs.*

Note that a slight modification of these Theorems leads to μ-resolvable **BB** designs for $\mu \geqq 1$, and that even if **BIB** designs in Theorems 16.1 and 16.2 are replaced by **BB** designs, the two Theorems remain valid. Since there are practically μ-resolvable solutions of many **BIB** designs (cf. [24; 26; 51]), we can obtain many μ-resolvable **BB** designs for $\mu \geqq 1$. Furthermore, Theorem 14.1 leads to $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable **BB** designs.

Next, we will treat the resolvability of **PBB** designs. Some examples are given as follows.

EXAMPLE 16.3. A **PBB** design in Example 15.1 is a resolvable design with parameters $v=s_1s_2s_3$, $b=s_1+1$, $r=2$, $k_j=s_2s_3$ or $s_1s_2s_3$, $\mu=1$, $m_1=s_1$ and $m_2=1$.

EXAMPLE 16.4. A **PBB** design in Example 15.2 is a resolvable design with parameters $v=2v_2v_3$, $b=v_2v_3+2$, $r=2$, $k_j=v_2v_3$ or 2, $\mu=1$, $m_1=2$ and $m_2=v_2v_3$.

Further examples will be seen in the subsequent sections. Corresponding to Theorems 16.1, 16.2 and the remarks pointed out there, methods of constructing $(\mu_1, \mu_2, \ldots, \mu_t)$-resolvable **PBB** designs are given after a slight modification of Theorems 16.1 and 16.2 by referring to Section 15.

It should be noted that as mentioned in Section 15, if there exists a resolvable **PBIB** design with parameters $v=mn$, b, r, k, $\lambda_1=0$ and $\lambda_2=1$, based on an N_2 type association scheme, then, by adding m new sets corresponding to the groups of the association scheme, we can get a resolvable **PBB** design provided $k\neq n$. This idea may be useful to constructions of these μ-resolvable **PBB** designs.

17. Reductions for the number of associate classes for PBB designs

Discussions concerning the reductions of the number of associate classes for a **PBIB** design based on a certain association scheme have appeared in Parts I and II. In this section, we shall deal with the reductions of the number of associate classes for a **PBB** design based on an association scheme.

In a **PBB** design N with m associate classes, where

$$\begin{aligned} C &= D_r - ND_k^{-1}N' \\ &= \rho_1 A_1^\# + \rho_2 A_2^\# + \cdots + \rho_m A_m^\#, \end{aligned}$$

if some ρ_i's coincide, then C may be written as, for example,

$$C = \rho_1 A_1^\# + \cdots + \rho_l (A_l^\# + A_{l+1}^\# + \cdots + A_m^\#). \tag{17.1}$$

In this case, the **PBB** design may be considered as m_1 associate classes for a positive integer m_1 such that $l \leqq m_1 \leqq m$. However, (17.1) does not express completely a criterion to determine which associate classes should be combined. An answer about those criteria will be given.

As already shown in Section 12, when there exists a **PBB** design N with m associate classes with association matrices $A_0, A_1, \ldots, A_m$, we can write the C-matrix of N as

$$\begin{aligned} C &= D_r - ND_k^{-1}N' \\ &= \rho_1 A_1^\# + \rho_2 A_2^\# + \cdots + \rho_m A_m^\# \qquad (17.2) \\ &= a_0 A_0 + a_1 A_1 + \cdots + a_m A_m, \qquad (17.3) \end{aligned}$$

where $a_0=(\sum_{i=1}^{v} r_i-b)/v$, $a_i \leqq 0$ $(i=1,2,\dots,m)$ and $a_0+a_1n_1+\cdots+a_mn_m=0$. In this case, for the reductions of associate classes we have

CRITERION: *Combine first those associate classes such that the corresponding coefficients a_i in* (17.3) *have the same value. The subsequent procedures are all the same as those for a* **PBIB** *design with m associate classes. Thus, these a_i and coincidence numbers λ_i of a* **PBIB** *design play the almost same role for reductions. Finally, combine the mutually orthogonal idempotents $A_i^{\#}$ suitably.*

REMARK. (i) When the a_i's are all distinct, even if an association scheme itself is reducible, the **PBB** design based on the association scheme is not reducible. (ii) In (17.2), the suffices j of ρ_j in the equality relations among coefficients ρ_j may mean the numbers j of combining which mutually orthogonal idempotents $A_j^{\#}$ corresponding to combinations of associate classes. However, the suffices in equality relations among coefficients ρ_j in (17.2) do not always coincide with those of a_i in (17.3).

Some examples explain this criterion for reductions.

EXAMPLE 17.1 (cf. [19]). Consider a resolvable **PBB** design with parameters $v=6$, $b=5$, $r=2$, $k_j=2$ or 3, based on the F_2 type association scheme of $v=2\times 3$ treatments, whose incidence matrix is given by

	1	2	3	4	5	r_i
1	1	0	1	0	0	2
2	1	0	0	1	0	2
3	1	0	0	0	1	2
4	0	1	1	0	0	2
5	0	1	0	1	0	2
6	0	1	0	0	1	2
k_j	3	3	2	2	2	12

$$A_{00}=I_6,\quad A_{01}=I_2\otimes(G_3-I_3),\quad A_{10}=(G_2-I_2)\otimes I_3,$$

$$A_{11}=(G_2-I_2)\otimes(G_3-I_3),$$

$$A_{00}^{\#}=\frac{1}{6}(A_{00}+A_{01}+A_{10}+A_{11}),$$

$$A_{01}^{\#}=\frac{1}{6}(2A_{00}-A_{01}+2A_{10}-A_{11}),$$

$$A^{\#}_{10} = \frac{1}{6}(A_{00}+A_{01}-A_{10}-A_{11}),$$

$$A^{\#}_{11} = \frac{1}{6}(2A_{00}-A_{01}-2A_{10}+A_{11}).$$

Then

$$C = A^{\#}_{01}+A^{\#}_{10}+2A^{\#}_{11}$$
$$= \frac{7}{6}A_{00}-\frac{1}{3}A_{01}-\frac{1}{2}A_{10}.$$

Since $a_1=-\frac{1}{3}\neq a_2=-\frac{1}{2}$, this design is not reducible, though $\rho_1=\rho_2=1$.

EXAMPLE 17.2. Consider a resolvable **PBB** design with parameters $v=6$, $b=3$, $r=2$, $k_j=3$ or 6, based on the F_2 type association scheme of $v=2\times 3$ treatments, whose incidence matrix is given by

	1	2	3	r_i
1	1	0	1	2
2	1	0	1	2
3	1	0	1	2
4	0	1	1	2
5	0	1	1	2
6	0	1	1	2
k_j	3	3	6	12

Then

$$C = 2A^{\#}_{01}+A^{\#}_{10}+2A^{\#}_{11} \tag{17.4}$$
$$= \frac{3}{2}A_{00}-\frac{1}{2}A_{01}-\frac{1}{6}A_{10}-\frac{1}{6}A_{11}.$$

Since $a_2=a_3=-\frac{1}{6}$, by combining 2nd and 3rd associate classes, the **PBB** design based on the F_2 type association scheme is reducible to a **PBB** design based on an N_2 type association scheme by referring to Section 9. Furthermore, $\rho_1=\rho_3=2$ implies a combination $A^{\#}_{01}+A^{\#}_{11}$ of mutually orthogonal idempotents. That is, for an N_2 type association scheme of $v=2\times 3$ treatments with association matrices B_0, B_1 and B_2,

$$B_0 = I_6, \quad B_1 = I_2 \otimes (G_3 - I_3),$$

$$B_2 = (G_2 - I_2) \otimes G_3,$$

$$B_0^\# = \frac{1}{6} G_6, \quad B_1^\# = \frac{1}{6}(B_0 + B_1 - B_2),$$

$$B_2^\# = \frac{1}{3}(2B_0 - B_1).$$

Then letting $B_0 = A_{00}$, $B_1 = A_{01}$ and $B_2 = A_{10} + A_{11}$, we have

$$B_0^\# = A_{00}^\#, \quad B_1^\# = A_{10}^\#, \quad B_2^\# = A_{01}^\# + A_{11}^\#.$$

In this case (17.4) becomes

$$C = B_1^\# + 2B_2^\#$$

$$= \frac{3}{2} B_0 - \frac{1}{2} B_1 - \frac{1}{6} B_2$$

which imply that the original **PBB** design is reducible to a **PBB** design based on an N_2 type association scheme.

REMARK. For a resolvable **PBB** design based on the F_2 type association scheme of $v = 2 \times 3$ treatments whose incidence matrix is given by

	1	2	3	4	r_i
1	1	0	0	1	2
2	0	1	0	1	2
3	0	0	1	1	2
4	1	0	0	1	2
5	0	1	0	1	2
6	0	0	1	1	2
k_j	2	2	2	6	12

;

$$C = \frac{4}{3} A_{00} - \frac{1}{6} A_{01} - \frac{2}{3} A_{10} - \frac{1}{6} A_{11}$$

$$= A_{01}^\# + 2A_{10}^\# + 2A_{11}^\#,$$

since it follows that $a_1 = a_3 = -\frac{1}{6}$ and $\rho_2 = \rho_3 = 2$, we can make an argument similar to Example 17.2.

EXAMPLE 17.3. Similarly to Example 15.1, consider a **PBB** design based on the N_3 type association scheme of $v = s_1 s_2 s_3$ treatments whose incidence matrix is given by

$$[I_{s_1} \otimes E_{s_2 s_3 \times 1} : E_{v \times 1}] .$$

Then

$$C = A_1^{\#} + 2A_2^{\#} + 2A_3^{\#}$$

$$= \left(2 - \frac{s_1+1}{s_1 s_2 s_3}\right) A_0 - \frac{s_1+1}{s_1 s_2 s_3} (A_1 + A_2) - \frac{1}{s_1 s_2 s_3} A_3 .$$

Let $s_1 = u_1$ and $s_2 s_3 = u_2$. From $a_1 = a_2 = -(s_1+1)/v$ and an argument in Section 7, by combining 1st and 2nd associate classes the **PBB** design based on the N_3 type association scheme is reducible to a **PBB** design based on an N_2 type association scheme of $v = s_1 s_2 s_3 = u_1 u_2$ treatments with association matrices B_0, B_1 and B_2. Furthermore, $\rho_2 = \rho_3 = 2$ implies a combination $A_2^{\#} + A_3^{\#}$ of mutually orthogonal idempotents. That is, the correspondence is as follows:

$$B_0 = A_0, \quad B_1 = I_{u_1} \otimes (G_{u_2} - I_{u_2}) = A_1 + A_2, \quad B_2 = A_3,$$

$$B_0^{\#} = A_0^{\#}, \quad B_1^{\#} = \left(I_{u_1} - \frac{1}{u_1} G_{u_1}\right) \otimes \frac{1}{u_2} G_{u_2} = A_1^{\#} \quad \text{and}$$

$$B_2^{\#} = I_{u_1} \otimes \left(I_{u_2} - \frac{1}{u_2} G_{u_2}\right) = A_2^{\#} + A_3^{\#} .$$

EXAMPLE 17.4. Similarly to Example 15.2, consider a **PBB** design based on the F_3 type association scheme of $v = 2v_2 v_3$ treatments whose incidence matrix is given by

$$[I_2 \otimes E_{v_2 v_3 \times 1} : E_{2 \times 1} \otimes I_{v_2 v_3}] .$$

Then

$$C = A_{001}^{\#} + A_{010}^{\#} + A_{011}^{\#} + A_{100}^{\#} + 2(A_{101}^{\#} + A_{110}^{\#} + A_{111}^{\#})$$

$$= \left(\frac{3}{2} - \frac{1}{v_2 v_3}\right) A_{000} - \frac{1}{v_2 v_3} (A_{001} + A_{010} + A_{011}) - \frac{1}{2} A_{100} .$$

Let $u = v_2 v_3$. From an argument in Section 6 and $a_1 = a_2 = a_3 = -1/v_2 v_3$, $a_5 = a_6 = a_7 = 0$, by combining 1st, 2nd and 3rd associate classes, and combining 5th, 6th and 7th associate classes, the **PBB** design based on the F_3 type association scheme is reducible to a **PBB** design based on an F_2 type association scheme of $v = 2u$ treatments. Furthermore, $\rho_1 = \rho_2 = \rho_3 = 1$ and $\rho_5 = \rho_6 = \rho_7 = 2$ imply combinations $A_{001}^{\#} + A_{010}^{\#} + A_{011}^{\#}$ and $A_{101}^{\#} + A_{110}^{\#} + A_{111}^{\#}$ of mutually orthogonal idempotents.

Finally, note that from sub-Section 10.3 and a **PBB** design based on the rectangular lattice type association scheme of 4×3 treatments given in Section 15, we can make an argument similar to the above Examples.

18. Inequalities for incomplete block designs

Since a **BB** design with the equal block size is a **BIB** design (Theorem 13.1), we shall deal with a **BB** design with unequal block sizes in this section. An inequality, $b \geqq v$, obtained for a **BIB** design by Fisher [15] holds for an equireplicate **BB** design with unequal block sizes. This result was first presented by Atiqullah [3] and a simple proof was given by Raghavarao [43]. Bhaskararao [4] proved that the equality sign in $b \geqq v$ holds when and only when the design is a symmetrical **BIB** design. Raghavarao (cf. [43; 45]) also showed that for a μ-resolvable equireplicate **BB** design an inequality $b \geqq v+t-1$ holds, which is also given for a μ-resolvable **BIB** design by Kageyama [26]. The last inequality above is an important necessary condition for the existence of a μ-resolvable **BB** design.

If the restriction of an equi-replication in a **BB** design is violated, then these arguments are not valid as will now be shown by the following examples of unequal-replicate **BB** designs.

EXAMPLE 18.1. A **BB** design with parameters $v=5$, $b=6$, $r_i=3$ or 4, $k_j=2$ or 4 given in Example 14.2. In this design, $v=5<b=6$.

EXAMPLE 18.2. A **BB** design with parameters $v=3$, $b=3$, $r_i=1$ or 2, $k_j=1$ or 3, whose incidence matrix is given by

	1	2	3	r_i
1	1	0	0	1
2	1	1	0	2
3	1	0	1	2
k_j	3	1	1	5

$$; \quad v = 3 = b = 3, \quad C = I_3 - \frac{1}{3} G_3 .$$

EXAMPLE 18.3. A **BB** design with parameters $v=4$, $b=3$, $r_i=1$ or 2, $k_j=1$ or 4, whose incidence matrix is given by

	1	2	3	r_i
1	1	0	0	1
2	1	0	0	1
3	1	1	0	2
4	1	0	1	2
k_j	4	1	1	6

$$; \quad v = 4 > b = 3 .$$

$$C = I_4 - \frac{1}{4} G_4 .$$

On the other hand, for a **PBB** design the inequality $b \geqq v$ does not necessarily hold as will be seen from the following examples.

EXAMPLE 18.4. A resolvable equireplicate **PBB** design with parameters $v=6$, $b=5$, $r=2$, $k_j=2$ or 3 given in Example 17.1 does not satisfy the inequality $b \geqq v$.

EXAMPLE 18.5. An unequal-replicate **PBB** design with parameters $v=6$, $b=5$, $r_i=2$ or 3, $k_j=1$, 3 or 6, based on the N_2 type association scheme of $v=2\times 3$ treatments, whose incidence matrix is given by

	1	2	3	4	5	r_i
1	1	0	1	0	1	3
2	1	0	0	1	1	3
3	1	0	0	0	1	2
4	0	1	0	0	1	2
5	0	1	0	0	1	2
6	0	1	0	0	1	2
k_j	3	3	1	1	6	14

; $v=6>b=5$.

$$C = B_1^{\#} + 2B_2^{\#},$$

where $B_i^{\#}$ $(i=1,2)$ are mutually orthogonal idempotents described in Example 17.2 for an N_2 type association scheme.

EXAMPLE 18.6. A resolvable **PBB** design with parameters $v=6$, $b=6$, $r=3$, $k_j=2$, 3 or 6 constructed from Corollary 14.2 and Example 18.4, whose incidence matrix is given by

	1	2	3	4	5	6	r_i
1	1	1	0	1	0	0	3
2	1	1	0	0	1	0	3
3	1	1	0	0	0	1	3
4	1	0	1	1	0	0	3
5	1	0	1	0	1	0	3
6	1	0	1	0	0	1	3
k_j	6	3	3	2	2	2	18

; $v=6=b=6$.

$$C = 2A_{01}^{\#} + 2A_{10}^{\#} + 3A_{11}^{\#}.$$

Since an unequal-replicate **PBB** design with unequal block sizes satisfying $b>v$ can be easily constructed, it is omitted here. Note that for a symmetrical unequal-block arrangement of two block sizes, which is a special case of **PBB** designs as mentioned in Section 15, the inequality $b>v$ holds (cf. [44]).

In a μ-resolvable equireplicate **BB** design, when $\mu=1$ we can easily get examples of resolvable equireplicate **BB** designs satisfying $b\geqq v+r-1$ from Theorems 16.1 and 16.2. In particular, as a resolvable equireplicate **BB** design satisfying $b=v+r-1$, we have, for example, the designs constructed by adding a block consisting of all elements unity to the blocks of an affine resolvable **BIB** design (cf. [5; 26]). When $\mu\geqq 2$, the inequality $b\geqq v+t-1$ holds and we can construct μ-resolvable **BB** designs satisfying $b>v+t-1$ from remarks in Section 16. We can also construct μ-resolvable **BB** designs ($\mu\geqq 2$) satisfying $b=v+t$, but we have failed to construct a μ-resolvable **BB** design with unequal block sizes satisfying $b=v+t-1$ for a positive integer $\mu\geqq 2$.

Kageyama [22; 27] has shown that for a resolvable **BIB** design with parameters $v=mk$, b, r, k and λ, if $b>v+r-1$, then $b\geqq 2v+r-2$. For almost all the resolvable equireplicate **BB** designs which can be constructed by the methods described in Part III, if $b>v+r-1$, then $b\geqq 2v+r-2$ holds. We ,however, cannot improve the inequality $b\geqq v+r-1$. In fact, there exists the following resolvable equireplicate **BB** design with unequal block sizes:

	1	2	3	4	5	6	7	8	r_i
1	1	0	1	0	1	0	0	1	4
2	1	0	1	0	0	1	1	0	4
3	1	0	0	1	1	0	1	0	4
4	0	1	1	0	1	0	1	0	4
k_j	3	1	3	1	3	1	3	1	16

; $v=4,\ b=8,\ r=4,\ k_j=1$ or 3,

$m_1=m_2=m_3=m_4=2.$

$$C=\frac{8}{3}\left(I_4-\frac{1}{4}G_4\right).$$

As a bound of replication numbers, r_i, in an unequal-replicate **BB** design and **PBB** design with unequal block sizes, we can obtain in both designs

$$\min_{1\leqq i\leqq v} r_i \geqq \frac{\sum_{i=1}^{v} r_i - b}{v} + \frac{1}{k_{\max}} \tag{18.1}$$

and

$$r_i \geqq \frac{1}{v} + \frac{n_{i1}}{k_1} + \cdots + \frac{n_{ib}}{k_b} \tag{18.2}$$

for all $i=1, 2,\ldots, v$, where $k_{\max}=\max\limits_{1\leq i\leq b} k_i$ and $(n_{i1},\ldots, n_{ib})$ is the ith row of an incidence matrix $N=\|n_{ij}\|$ of order $v\times b$ of a design.

Examples 18.2 and 18.3 attain the equality sign in (18.1). Though the right-hand side of (18.2) depends on the suffix i, $i=1, 2,\ldots, v$, that of (18.1) does not. In this point, the bound (18.1) is more suitable than (18.2). The bound (18.2), however, may be useful to construct a design by trial and error.

REMARK. In a **PBB** design, (15.2) and (15.3) together imply (18.1) and (18.2). In a **BB** design, (12.3) and (14.9) together imply (18.1) and (18.2).

Finally, in particular, we shall consider inequalities to hold for the equireplicate **PBB** design based on an N_2 type association scheme of $v=mn$ treatments. As shown in Section 7, $v(=mn)$ treatments are divided into m groups of n elements each, such that any two treatments in the same group are 1st associates and two treatments in different groups are 2nd associates. Then it is known (cf. [40]) that

$$
\begin{aligned}
& n_1 = n-1, \quad n_2 = n(m-1), \\
& A_0 = I_v, \quad A_1 = I_m \otimes G_n - I_v, \quad A_2 = G_v - A_0 - A_1, \\
& A_0^\# = \frac{1}{v}(A_0+A_1+A_2), \ \operatorname{tr} A_0^\# = 1, \\
& A_1^\# = \frac{1}{v}\{(m-1)(A_0+A_1)-A_2\}, \ \operatorname{tr} A_1^\# = m-1, \qquad (18.3)\\
& A_2^\# = \frac{1}{n}\{(n-1)A_0-A_1\}, \ \operatorname{tr} A_2^\# = m(n-1), \\
& A_0^\# + A_1^\# + A_2^\# = I_v, \\
& z_{01} = n_1 = n-1, \quad z_{11} = n-1, \quad z_{21} = -1, \\
& z_{02} = n_2 = n(m-1), \quad z_{12} = -n, \quad z_{22} = 0.
\end{aligned}
$$

Let N be the equireplicate **PBB** design with parameters v, b, r, k_j $(j=1, 2,\ldots, b)$ based on an N_2 type association scheme of $v=mn$ treatments. Then from (12.5), (12.6) and (18.3) we can write the C-matrix of N as

$$
\begin{aligned}
C &= rI_v - ND_k^{-1}N' \\
&= \rho_1 A_1^\# + \rho_2 A_2^\# && (18.4)\\
&= \{(vr-b)/v\}A_0 + a_1A_1 + a_2A_2, && (18.5)
\end{aligned}
$$

where $D_k^{-1}=\mathrm{diag}\{k_1^{-1}, k_2^{-1},\dots, k_b^{-1}\}$ and $a_i \leqq 0$ $(i=1, 2)$. From (18.3) and (18.4) we obtain

$$(18.6) \qquad ND_k^{-1}N' = rA_0^{\#}+(r-\rho_1)A_1^{\#}+(r-\rho_2)A_2^{\#}$$

$$\left(=\frac{\rho_1}{v}G_v+(r-\rho_2)I_v+\frac{\rho_2-\rho_1}{n}(I_m\otimes G_n)\right).$$

Then its determinant is

$$(18.7) \qquad |ND_k^{-1}N'| = r(r-\rho_1)^{m-1}(r-\rho_2)^{m(n-1)}.$$

Now for $D_{\sqrt{k}}^{-1}=\mathrm{diag}\{k_1^{-\frac{1}{2}}, k_2^{-\frac{1}{2}},\dots, k_b^{-\frac{1}{2}}\}$, since

$$(18.8) \qquad ND_k^{-1}N' = (ND_{\sqrt{k}}^{-1})(ND_{\sqrt{k}}^{-1})',$$

which is a positive semi-definite matrix, we have

$$(18.9) \qquad r-\rho_i \geqq 0, \qquad i = 1, 2.$$

From (18.6), if $r-\rho_1=0$ and $r-\rho_2=0$ hold simultaneously, then the **PBB** design is reducible to a complete block design (i.e., $N=E_{v\times b}$) and vice versa. We shall therefore confine ourselves to the case in which $r-\rho_1=0$ and $r-\rho_2=0$ do not hold simultaneously. In this case, since we have from (18.8)

$$(18.10) \qquad \mathrm{rank}\, ND_k^{-1}N' = \mathrm{rank}\, ND_{\sqrt{k}}^{-1} = \mathrm{rank}\, N \leqq b,$$

we obtain from (18.3), (18.6), (18.7) and (18.9) the following

THEOREM 18.1. *For an equireplicate* **PBB** *design with parameters v, b, r, k_j $(j=1, 2,\dots, b)$ based on the N_2 type association scheme of $v=mn$ treatments having* (18.3) *and* (18.4), *it holds that*

(i) *if $r-\rho_1>0$ and $r-\rho_2=0$, then $b\geqq m$;*

(ii) *if $r-\rho_1=0$ and $r-\rho_2>0$, then $b\geqq v-m+1$;*

(iii) *if $r-\rho_1>0$ and $r-\rho_2>0$, then $b\geqq v$.*

If the design N is μ-resolvable, that is, the blocks can be separated into t sets of m_i blocks such that the set consisting of m_i blocks contains every treatment exactly μ times $(i=1, 2,\dots, t)$, then

$$\mathrm{rank}\, ND_k^{-1}N' = \mathrm{rank}\, ND_{\sqrt{k}}^{-1} = \mathrm{rank}\, N \leqq b-t+1,$$

since in N the sum of the columns corresponding to each set must give a column consisting of μ's. Thus not more than $b-t+1$ column vectors are independent.

Hence we have

COROLLARY 18.2. *For a μ-resolvable equireplicate* **PBB** *design* ($\mu \geqq 1$) *with parameters v, b, $r=\mu t$, k_j ($j=1,2,\dots,b$) based on the N_2 type association scheme of $v=mn$ treatments having* (18.3) *and* (18.4), *it holds that*

(i) *if $r-\rho_1>0$ and $r-\rho_2=0$, then $b \geqq m+t-1$;*

(ii) *if $r-\rho_1=0$ and $r-\rho_2>0$, then $b \geqq v-m+t$;*

(iii) *if $r-\rho_1>0$ and $r-\rho_2>0$, then $b \geqq v+t-1$.*

REMARK. From Section 12, (18.3), (18.4) and (18.5) we have

$$r = b/v-a_1(n-1)-a_2n(m-1),$$

$$r-\rho_1 = b/v-a_1(n-1)+a_2n,$$

$$r-\rho_2 = b/v+a_1,$$

which lead to

$$\rho_1 = -a_2v \quad \text{and} \quad \rho_1-\rho_2 = n(a_1-a_2).$$

Conditions (i), (ii) and (iii) in Corollary 18.2, respectively, may correspond to those of Singular, Semi-regular and Regular group divisible 2-associate **PBIB** designs (cf. [10; 45]).

Furthermore, the above argument can be easily applied to an equireplicate **PBB** design N with parameters v, b, r, k_j ($j=1,2,\dots,b$) based on an association scheme of m associate classes. By definition, we can write the C-matrix of N as

$$\begin{aligned} C &= rI_v-ND_k^{-1}N' \\ &= \rho_1A_1^\# +\rho_2A_2^\# +\cdots+\rho_mA_m^\#, \end{aligned} \tag{18.11}$$

which leads to

$$ND_k^{-1}N'=rA_0^\# +(r-\rho_1)A_1^\# +\cdots+(r-\rho_m)A_m^\#. \tag{18.12}$$

Then its determinant is

$$|ND_k^{-1}N'| = r(r-\rho_1)^{\alpha_1}\dots(r-\rho_m)^{\alpha_m},$$

where $\alpha_i=\operatorname{tr} A_i^\#$, $i=1, 2,\dots, m$. Therefore from (18.10) we have

THEOREM 18.3. *For an equireplicate* **PBB** *design with parameters v, b, r, k_j ($j=1,2,\dots,b$) based on an association scheme of m associate classes having*

(18.11), *the following inequality holds*:

$$b \geqq v - \sum_i \alpha_i$$

where $\alpha_i = \mathrm{tr}\, A_i^\#$ *and the summation extends over all the integers i satisfying* $r - \rho_i = 0$, $i = 1, 2, \ldots, m$. *Furthermore, for a* μ*-resolvable equireplicate* **PBB** *design it holds that*

$$b \geqq v + t - \sum_i \alpha_i - 1 .$$

REMARK. (i) From (18.11), if $r - \rho_i = 0$, $i = 1, 2, \ldots, m$, hold simultaneously, then the **PBB** design is reducible to a complete block design (i.e., $N = E_{v \times b}$) and vice versa. (ii) If $r \neq \rho_i$ for all $i = 1, 2, \ldots, m$, then $b \geqq v$ holds. (iii) The first inequality in Theorem 18.3 may correspond to an inequality for a **PBIB** design obtained by Yamamoto and Fujii [59].

Moreover, from (18.8) and (18.12) we have

THEOREM 18.4. *For an equireplicate* **PBB** *design with parameters v, b, r,* k_j $(j = 1, 2, \ldots, b)$ *based on an association scheme of m associate classes having* (18.11) *in which* $v > b$, *it holds that*

$$r(r - \rho_1)^{\alpha_1} \ldots (r - \rho_m)^{\alpha_m} = 0,$$

so that r is equal to one of the ρ_i*'s. Furthermore, when* $v = b$, *it is necessary that*

$$k_1 k_2 \ldots k_b\, r(r - \rho_1)^{\alpha_1} \ldots (r - \rho_m)^{\alpha_m}$$

is a perfect square.

Concerning the above arguments, as a bound on the latent roots of the C-matrix for a **PBB** design, we have

THEOREM 18.5. *For a* **PBB** *design N with parameters v, b,* r_i, k_j $(i = 1, 2, \ldots, v;\ j = 1, 2, \ldots, b)$ *based on an association scheme of m associate classes, where*

$$\begin{aligned} C &= D_r - N D_k^{-1} N' \\ &= \rho_1 A_1^\# + \rho_2 A_2^\# + \cdots + \rho_m A_m^\#, \end{aligned}$$

the following inequality holds:

$$0 < \rho_l \leqq \min_{1 \leqq i \leqq v} r_i, \qquad l = 1, 2, \ldots, m .$$

PROOF. It is known (cf. [19]) that the C-matrix of the incomplete block design is positive semi-definite. Hence we obtain $\rho_l \geqq 0$ for $l=0, 1,\dots, m$. From the definition of a **PBB** design $\rho_0=0$ and $\rho_l>0$ for $l=1, 2,\dots, m$. On the other hand, since C is a positive semi-definite matrix and $D_r^{-1}=\mathrm{diag}\{r_1^{-1}, r_2^{-1},\dots, r_v^{-1}\}$ is a positive definite matrix, we have from Corollary 2.2.1 in Anderson and Gupta [2]

$$\frac{\rho_l}{\max\limits_{1\leqq i\leqq v} r_i} \leqq \mathrm{ch}_l(CD_r^{-1}) \leqq \frac{\rho_l}{\min\limits_{1\leqq i\leqq v} r_i}\,,$$

$$l = 0, 1,\dots, v-1\,,$$

where $ch_l(CD_r^{-1})$ for any $l(0\leqq l\leqq v-1)$ are the latent roots of CD_r^{-1}, which imply

$$\frac{\rho_l}{\min\limits_{1\leqq i\leqq v} r_i} \leqq \frac{(\max\limits_{1\leqq i\leqq v} r_i)\mathrm{ch}_l(CD_r^{-1})}{\min\limits_{1\leqq i\leqq v} r_i} \leqq \frac{\max\limits_{1\leqq i\leqq v} r_i}{\min\limits_{1\leqq i\leqq v} r_i} \leqq 1\,,$$

since it is known (cf. [60]) that $0\leqq ch_l(CD_r^{-1})\leqq 1$ for $l=0, 1,\dots, v-1$. Hence we obtain

$$\rho_l \leqq \min_{1\leqq i\leqq v} r_i, \qquad l = 0, 1,\dots, m \quad (\leqq v-1)\,.$$

Thus, we have the required result.

Some Examples in this paper attain the upper bound on the latent roots ρ_l in Theorem 18.5. For an equireplicate **PBB** design with replication number r, Theorem 18.5 leads to

$$0 < \rho_l \leqq r, \qquad l = 1, 2,\dots, m, \tag{18.13}$$

which can be also derived from (18.8) and (18.12).

Note that when an equireplicate **PBB** design is a **PBIB** design, (18.13) leads to (1.9).

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References

[1] Adhikary, B. (1966). Some types of m-associate **PBIB** association schemes. Calcutta Statist. Assoc. Bull. **15** 47–74.

[2] Anderson, T. W. and Gupta, S. D. (1963). Some inequalities on characteristic roots of matrices. Biometrika **50** 522–524.

[3] Atiqullah, M. (1961). On a property of balanced designs. Biometrika **48** 215–218.

[4] Bhaskararao, M. (1966). A note on equireplicate balanced designs with $b=v$. Calcutta Statist. Assoc. Bull. **15** 43–44.

[5] Bose, R. C. (1942) . A note on the resolvability of balanced incomplete block designs. Sankhyā **6** 105–110.

[6] Bose, R. C. (1950). *Least Squares Aspects of Analysis of Variance*. Institute of Statistics, University of North Carolina.

[7] Bose, R. C. and Clatworthy, W. H. (1955). Some classes of partially balanced designs. Ann. Math. Statist. **26** 212–232.

[8] Bose, R. C. and Mesner, D. M. (1959). On linear associative algebras corresponding to association schemes of partially balanced designs. Ann. Math. Statist. **30** 21–38.

[9] Bose, R. C. and Nair, K. R. (1939). Partially balanced incomplete block designs. Sankhyā **4** 337–372.

[10] Bose, R. C. and Shimamoto, T. (1952). Classification and analysis of partially balanced incomplete block designs with two associate classes. J. Amer. Statist. Assoc. **47** 151–184.

[11] Bose, R. C. and Shrikhande, S. S. (1960). On the construction of sets of pairwise orthogonal Latin squares and the falsity of a conjecture of Euler. Trans. Amer. Math. Soc. **95** 191–209.

[12] Chakrabarti, M. C. (1962). *Mathematics of Design and Analysis of Experiments*. Asia Publishing House, Bombay.

[13] Clatworthy, W. H. and Lewyckyj, R. J. (1968). Comments on Takeuchi's table of difference sets generating balanced incomplete block designs. Rev. Inst. Internat. Statist. **36** 12–18.

[14] Enomoto, H. (1974). Personal communication.

[15] Fisher, R. A. (1940). An examination of the different possible solutions of a problem in incomplete blocks. Ann. Eugen. **10** 52–75.

[16] Hinkelmann, K. (1964). Extended group divisible partially balanced incomplete block designs. Ann. Math. Statist. **35** 681–695.

[17] Hinkelmann, K. and Kempthorne, O. (1963). Two classes of group divisible partial diallel crosses. Biometrika **50** 281–291.

[18] Ishii, G. (1963). On the analysis of **PBB**D and **BB**D (in Japanese). Seminar Reports of Osaka Statistical Association **7** 221–232.

[19] Ishii, G. (1972). *The Design of Experiments*: *Theory of Allcations* (in Japanese). Baihūkan.

[20] Ishii, G. and Ogawa, J. (1965). On the analysis of balanced and partially balanced block designs. Osaka City Univ. Business Review **81** 1–31.

[21] John, P. W. M. (1964). Balanced designs with unequal numbers of replicates. Ann. Math. Statist. **35** 897–899.

[22] Kageyama, S. (1971). An improved inequality for balanced incomplete block designs. Ann. Math. Statist. **42** 1448–1449.

[23] Kageyama, S. (1972). On the reduction of associate classes for certain **PBIB** designs. Ann. Math. Statist. **43** 1528–1540.

[24] Kageyama, S. (1972). A survey of resolvable solutions of balanced incomplete block designs. Internat. Statist. Rev. **40** 269–273.

[25] Kageyama, S. (1972). Note on Takeuchi's table of difference sets generating balanced incomplete block designs. Internat. Statist. Rev. **40** 275–276.

[26] Kageyama, S. (1973). On μ-resolvable and affine μ-resolvable balanced incomplete block designs. Ann. Statist. **1** 195–203.

[27] Kageyama, S. (1973). On the inequality for **BIB**Ds with special parameters. Ann. Statist. **1** 204–207.

[28] Kageyama, S. (1974). Note on the reduction of associate classes for **PBIB** designs. Ann. Inst. Statist. Math. **26** 163–170.

[29] Kageyama, S. (1974). On the incomplete block designs derivable from the association schemes. J. Combinatorial Theory (A) **17** 269–272.

[30] Kageyama, S. (1974). On the reduction of associate classes for the **PBIB** design of a certain generalized type. Ann. Statist. **2**, No. 6.

[31] Kempthorne, O. (1952). *The Design and Analysis of Experiments.* Wiley, New York.

[32] Kishen, K. (1940–1941). Symmetrical unequal block arrangements. Sankhyā **5** 329–344.

[33] Kulshreshtha, A. C., Dey, A. and Saha, G. M. (1972). Balanced designs with unequal replications and unequal block sizes. Ann. Math. Statist. **43** 1342–1345.

[34] Kusumoto, K. (1965). Hypercubic designs. Wakayama Medical Reports **9** 123–132.

[35] Nair, C. R. (1964). The impossibility of certain **PBIB** designs. Calcutta Statist. Assoc. Bull. **13** 87–88.

[36] Nair, K. R. (1950). Partially balanced incomplete block designs involving only two replication. Calcutta Statist. Assoc. Bull. **3** 83–86.

[37] Nair, K. R. (1951). Rectangular lattices and partially balanced incomplete block designs. Biometrics **7** 145–154.

[38] Nair, K. R. and Rao, C. R. (1942). A note on partially balanced incomplete block designs. Science and Culture **7** 568–569.

[39] Nandi, H. K. and Adhikary, B. (1965). m-associate cyclical association schemes. *Essays in Probability and Statistics*, S. N. Roy memorial volume, Univ. of North Carolina.

[40] Ogawa, J. (1961). On a unified method of deriving necessary conditions for existence of symmetrical partially balanced incomplete block designs of certain types. Bull. Inst. Internat. Statist. **38** 43–57.

[41] Okuno, C. and Okuno, T. (1961). On the construction of a class of partially balanced incomplete block designs by calculus of blocks. Rep. Stat. Appl. Res., JUSE **8** 113–139.

[42] Raghavarao, D. (1960). A generalization of group divisible designs. Ann. Math. Statist. **31** 756–771.

[43] Raghavarao, D. (1962). On balanced unequal block designs. Biometrika **49** 561–562.

[44] Raghavarao, D. (1962). Symmetrical unequal block arrangements with two unequal block sizes. Ann. Math. Statist. **33** 620–633.

[45] Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley, New York.

[46] Raghavarao, D. and Chandrasekharao, K. (1964). Cubic designs. Ann. Math. Statist. **35** 389–397.

[47] Rao, V. R. (1958). A note on balanced designs. Ann. Math. Statist. **29** 290–294.

[48] Roy, P. M. (1962). On the properties and construction of HGD designs with m-associate classes. Calcutta Statist. Assoc. Bull. **11** 10–38.

[49] Shrikhande, S. S. (1959b). The uniqueness of the L_2 association scheme. Ann. Math. Statist. **30** 781–798.

[50] Shrikhande, S. S. (1962). On a two-parameter family of balanced incomplete block designs. Sankhyā **24** 33–40.

[51] Shrikhande, S. S. and Raghavarao, D. (1963). Affine α-resolvable incomplete block designs. *Contributions to Statistics*. Presented to Prof. P. C. Mahalanobis on the occasion of his 70th birthday. Pergamon Press, pp. 471–480.

[52] Sillitto, G. P. (1957). An extension property of a class of balanced incomplete block designs. Biometrika **44** 278–279.

[53] Sillitto, G. P. (1964). Note on Takeuchi's table of difference sets generating balanced incomplete block designs. Rev. Inst. Internat. Statist. **32** 251.

[54] Surendran, P. U. (1968). Association matrices and the Kronecker product of designs. Ann. Math. Statist. **39** 676–680.

[55] Takeuchi, K. (1962). A table of difference sets generating balanced incomplete block designs. Rev. Inst. Internat. Statist. **30** 361–366.

[56] Tharthare, S. K. (1963). Right angular designs. Ann. Math. Statist. **34** 1057–1067.

[57] Tharthare, S. K. (1965). Generalized right angular designs. Ann. Math. Statist. **36** 1535–1553.

[58] Vartak, M. N. (1955). On an application of Kronecker product of matrices to statistical designs. Ann. Math. Statist. **26** 420–438.

[59] Yamamoto, S. and Fujii, Y. (1963). Analysis of partially balanced incomplete block designs. J. Sci. Hiroshima Univ. Ser. A–I **27** 119–135.

[60] Yamamoto, S. and Fujikoshi, Y. (1968). Two-way classification designs with unequal cell frequencies. J. Sci. Hiroshima Univ. Ser. A–I **32** 357–370

[61] Yamamoto, S., Fujii, Y. and Hamada, N. (1965). Composition of some series of association algebras. J. Sci. Hiroshima Univ. Ser. A–I **29** 181–215.

[62] Yates, F. (1936). Incomplete randomized blocks. Ann. Eugen. **7** 121–140.

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