Banach J. Math. Anal. 9 (2015), no. 1, 137–143
http://doi.org/10.15352/bjma/09-1-11
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

MULTIPLIERS IN PERFECT LOCALLY m-CONVEX ALGEBRAS

MARINA HARALAMPIDOU[1*], LOURDES PALACIOS[2] AND CARLOS SIGNORET[2]

Communicated by M. Abel

Abstract. In this paper we describe the multiplier algebra of a perfect complete locally m-convex algebra with an approximate identity and with complete Arens-Michael normed factors.

1. Introduction and Preliminaries

Multipliers are important in various areas of mathematics where an algebra structure appears (see e.g [1]; for (non-normed) topological algebras cf. e.g. [4]).

The algebras considered throughout are taken over the field of complexes $\mathbb{C}$. Denote by $L(E)$ the algebra of all linear operators on an algebra E.

Definition 1.1. A mapping $T : E \to E$ is called a *left (right) multiplier* on E if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in E$; it is called a *two-sided multiplier* on E if it is both a left and a right multiplier.

It is known that *if E is a proper algebra,* namely $xE = \{0\}$ implies $x = 0$ or $Ex = \{0\}$ implies $x = 0$, where 0 denotes the zero element of E, *then any two-sided multiplier on E is automatically a linear mapping* [6, p. 20].

In the sequel, a two-sided multiplier will be called in short, a *multiplier*. We denote by $M_l(E)$ the set of all left multipliers on E, by $M_r(E)$ the set of all right multipliers on E and by $M(E)$ that of all multipliers on E. Note that, by definition, $M(E) = M_l(E) \cap M_r(E)$.

Date: Received: Mar. 1, 2014; Revised: Apr. 1, 2014; Accepted: Apr. 10, 2014.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 46H05; Secondary 46H10, 46K05.

Key words and phrases. Proper algebra, Arens-Michael decomposition, multiplier algebra, perfect projective system, perfect locally m-convex algebra.

Obviously $M(E)$ *is a subalgebra of* $L(E)$ *in case the algebra is proper.* The same holds for $M_r(E)$ and $M_l(E)$. Now, for $x \in E$, the operator l_x on E given by $l_x(y) = xy\,, y \in E$, is, due to the associativity of E, a left multiplier. Similarly, we can also define the right multiplier with respect to $x \in E$, say r_x.

It is known that if E is a proper algebra, then the mapping

$$L : E \to M_l(E) \text{ given by } x \mapsto l_x$$

defines an algebra monomorphism which identifies E with a subalgebra of $M_l(E)$. Moreover, E is a left ideal of the algebra $M_l(E)$. A similar result is also valid for right multipliers. For multipliers, the algebra E can be identified with a two-sided ideal in $M(E)$ ([3, p. 1933, Proposition 2.2 and p. 1934, Corollary 2.3]).

Definition 1.2. An *approximate identity* in a topological algebra E is a net $(e_\delta)_{\delta\in\Delta}$ such that for each $x \in E$ we have

$$(x - xe_\delta) \underset{\delta}{\to} 0 \text{ and } (x - e_\delta x) \underset{\delta}{\to} 0 \text{ for all } x \in E.$$

Note that an algebra with an approximate identity is proper. In this paper we describe the multiplier algebra $M(E)$ in the case where E is a certain complete locally m-convex algebra with an approximate identity.

For the sake of completeness, we recall what we mean by the "Arens-Michael decomposition" ([7, p. 88, Theorem 3.1]).

Let $(E, (p_\alpha)_{\alpha\in\Lambda})$ be a complete locally m-convex algebra and

$$\rho_\alpha : E \to E/\ker(p_\alpha) \equiv E_\alpha \text{ defined by } \rho_\alpha(x) = x + \ker(p_\alpha) \equiv x_\alpha, \alpha \in \Lambda$$

the respective quotient maps. Then $\dot{p}_\alpha(x_\alpha) := p_\alpha(x)$, $x \in E$, $\alpha \in \Lambda$ defines on E_α an algebra norm, so that E_α is a normed algebra and the morphisms ρ_α, $\alpha \in \Lambda$ are continuous. $\tilde{E}_\alpha$, $\alpha \in \Lambda$ denotes the completion of E_α (with respect to $\dot{p}_\alpha$). Λ is endowed with a partial order by putting $\alpha \le \beta$ if and only if $p_\alpha(x) \le p_\beta(x)$ for every $x \in E$. Thus, $\ker(p_\beta) \subseteq \ker(p_\alpha)$ and hence the continuous (onto) morphism $f_{\alpha\beta} : E_\beta \to E_\alpha : x_\beta \mapsto f_{\alpha\beta}(x_\beta) = x_\alpha,\ \alpha \le \beta$ is defined. Moreover, $f_{\alpha\beta}$ is extended to a continuous morphism $\bar{f}_{\alpha\beta} : \tilde{E}_\beta \to \tilde{E}_\alpha,\ \alpha \le \beta$. Thus, $(E_\alpha, f_{\alpha\beta})$, $(\tilde{E}_\alpha, \bar{f}_{\alpha\beta})$, $\alpha, \beta \in \Lambda$ with $\alpha \le \beta$ are projective systems of normed (resp. Banach) algebras, so that $E \cong \varprojlim E_\alpha \cong \varprojlim \tilde{E}_\alpha$ (Arens-Michael decomposition) within topological algebra isomorphisms.

In [3, p. 1934, Theorem 3.1], it is shown that, in a special case, the algebra $M(E)$ is a subalgebra of $\mathcal{L}(E)$, the algebra of all continuous linear operators on E; for completeness, we refer it here.

Theorem 1.3. *Let* $(E, (p_\alpha)_{\alpha\in\Lambda})$ *be a complete locally* m*-convex algebra with an approximate identity* $(e_\delta)_{\delta\in\Delta}$. *Suppose that each factor* $E_\alpha = E/\ker p_\alpha$ *in the Arens-Michael decomposition of* E *is complete. Then each multiplier* T *of* E *is continuous, viz.* $M(E)$ *is a subalgebra of* $\mathcal{L}(E)$.

2. Perfectness and Multipliers in locally m-convex Algebras

To proceed, we use the notion of a perfect projective system as it appeared in [2, p. 199, Definition 2.7]. To fix notation, we repeat it.

Definition 2.1. A projective system $\{(E_\alpha, f_{\alpha\beta})\}_{\alpha\in\Lambda}$ of topological algebras is called *perfect* , if the restrictions to the projective limit algebra

$$E = \varprojlim E_\alpha = \{(x_\alpha) \in \prod_{\alpha\in\Lambda} E_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha, \text{ if } \alpha \leq \beta \in \Lambda\}$$

of the canonical projections $\pi_\alpha : \prod_{\alpha\in\Lambda} E_\alpha \to E_\alpha$, $\alpha \in \Lambda$, namely, the (continuous algebra) morphisms

$$f_\alpha = \pi_\alpha \mid_{E=\varprojlim E_\alpha} : E \to E_\alpha, \quad \alpha \in \Lambda,$$

are onto maps. The resulting projective limit algebra $E = \varprojlim E_\alpha$ is then called a *perfect (topological) algebra.*

Definition 2.2. In the sequel, by the term *perfect locally m-convex algebra* we mean a locally m-convex algebra $(E, (p_\alpha)_{\alpha\in\Lambda})$ for which the respective Arens-Michael projective system $\{(E_\alpha, f_{\alpha\beta})\}_{\alpha\in\Lambda}$ is perfect.

Every Fréchet locally m-convex algebra $(E, (p_n)_{n\in\mathbb{N}})$ gives a perfect projective system of normed algebras, and thus it is a perfect algebra (see [2], and [5]).

Example 2.3. Let E be a non-complete normed algebra. Take $E = E_\alpha$ for each $\alpha \in \Lambda$ and, for $\alpha \leq \beta$, let $f_{\alpha\beta} : E_\beta \to E_\alpha$ be the identity map. Then $\Delta = \varprojlim E_\alpha$, the diagonal algebra, is a perfect locally m-convex algebra, but Δ is not complete.

Let $E = (E, (p_\alpha)_{\alpha\in\Lambda})$ be a perfect complete locally m-convex algebra with an approximate identity and such that each factor E_α of its Arens-Michael decomposition is complete.

Remark 2.4. If ϕ is the isomorphism $E \longrightarrow \varprojlim E_\alpha$ given by $\phi(x) = (x_\alpha)_{\alpha\in\Lambda}$, then, for each $\alpha \in \Lambda$, $\rho_\alpha = f_\alpha \circ \phi$. Therefore, $\ker p_\alpha = \ker \rho_\alpha = \ker(f_\alpha \circ \phi)$.

Remark 2.5. By the hypothesis of perfectness, each f_β is surjective, so *each time we have an element $x_\beta \in E_\beta$, we can choose an element $\omega \in E$ such that $\omega_\beta = x_\beta$, and consequently $\omega_\alpha = f_{\alpha\beta}(x_\beta) = x_\alpha$*, whenever $\alpha \leq \beta$.

For each $\alpha \leq \beta$, we define the map $h_{\alpha\beta} : M(E_\beta) \to M(E_\alpha)$ given by

$$[h_{\alpha\beta}(T_\beta)](x_\alpha) = f_{\alpha\beta}(T_\beta(x_\beta))$$

which is well defined, according to the following lemma.

Lemma 2.6. *Let $(E, (p_\alpha)_{\alpha\in\Lambda})$ be a perfect complete locally m-convex algebra with an approximate identity $(e_\delta)_{\delta\in\Delta}$ and such that each factor E_α of its Arens-Michael decomposition is complete. Then* $\ker f_{\alpha\beta}$ *is T_β-invariant for each $T_\beta \in M(E_\beta)$, that is, $T_\beta(\ker f_{\alpha\beta}) \subseteq \ker f_{\alpha\beta}$, if $\alpha \leq \beta$, and the map $h_{\alpha\beta}$ is a well-defined continuous multiplicative linear mapping.*

Proof. Take $x_\beta \in \ker f_{\alpha\beta}$. Since E has an approximate identity $(e_\delta)_{\delta\in\Delta}$ and multipliers over Banach algebras are continuous (see [6, p. 20, Theorem 1.1.1]), then

$$f_{\alpha\beta}(T_\beta(x_\beta)) = f_{\alpha\beta}(T_\beta(\lim_\delta x_\beta e_\delta)) = f_{\alpha\beta}(\lim_\delta T_\beta(x_\beta e_\delta)) = \lim_\delta f_{\alpha\beta}(T_\beta(x_\beta e_\delta)) =$$

$$= \lim_{\delta} f_{\alpha\beta}(x_\beta T_\beta(e_\delta)) = \lim_{\delta} [f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(T_\beta(e_\delta))] = 0.$$

We claim that $h_{\alpha\beta}(T_\beta)$ is well-defined. For that, let $\alpha \leq \beta$, $x \in E$ be such that $x_\alpha = x'_\alpha$ and $T_\beta \in M(E_\beta)$; then $0 = x_\alpha - x'_\alpha = \rho_\alpha(x) - \rho_\alpha(x') = \rho_\alpha(x - x')$ and hence $0 = (f_\alpha \circ \phi)(x - x') = (f_{\alpha\beta} \circ f_\beta \circ \phi)(x - x')$, which implies that $(f_\beta \circ \phi)(x - x') \in \ker f_{\alpha\beta}$. Since $\ker f_{\alpha\beta}$ is T_β -invariant, $T_\beta((f_\beta \circ \phi)(x - x')) \in \ker f_{\alpha\beta}$ too, and therefore

$$0 = f_{\alpha\beta}(T_\beta((f_\beta \circ \phi)(x - x')) = f_{\alpha\beta}(T_\beta(\rho_\beta(x - x'))) = f_{\alpha\beta}(T_\beta(x_\beta - x'_\beta)) =$$

$$= f_{\alpha\beta}(T_\beta(x_\beta)) - f_{\alpha\beta}(T_\beta(x'_\beta)),$$

that is, $f_{\alpha\beta}(T_\beta(x_\beta)) = f_{\alpha\beta}(T_\beta(x'_\beta))$. This proves the claim.

Moreover, $h_{\alpha\beta}(T_\beta)$ is actually a multiplier on E_α. For, let x_α and y_α be two elements in E_α. Then

$$[h_{\alpha\beta}(T_\beta)]\,(x_\alpha y_\alpha) = f_{\alpha\beta}(T_\beta(x_\beta y_\beta)) = f_{\alpha\beta}(x_\beta T_\beta(y_\beta)) = f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(T_\beta(y_\beta)) =$$

$x_\alpha(f_{\alpha\beta}(T_\beta(y_\beta)) = x_\alpha\,[h_{a\beta}(T_\beta)]\,(y_\alpha)$ and so, $h_{\alpha\beta}(T_\beta)$ is a right multiplier. In a similar way, one can prove that $h_{\alpha\beta}(T_\beta)$ is a left multiplier.

It is easily seen that $h_{\alpha\beta}$ is a linear mapping. Moreover, $h_{\alpha\beta}$ is multiplicative. For that, take T_β, $S_\beta \in M(E_\beta)$. We have

$$[h_{\alpha\beta}(T_\beta \circ S_\beta)](x_\alpha) = f_{\alpha\beta}((T_\beta \circ S_\beta)(x_\beta)) = f_{\alpha\beta}(T_\beta(S_\beta(x_\beta))). \tag{2.1}$$

On the other hand, since the system is perfect, we can choose $\omega \in E$ (equivalently $(\omega_\alpha)_{\alpha\in\Lambda} \in \varprojlim E_\alpha$) such that $f_{\alpha\beta}(S_\beta(x_\beta)) = \omega_\alpha$; note that $f_{\alpha\beta}(\omega_\beta) = \omega_\alpha$ too. Then $S_\beta(x_\beta) - \omega_\beta \in \ker f_{\alpha\beta}$. But, since $\ker f_{\alpha\beta}$ is T_β-invariant, we have $T_\beta(S_\beta(x_\beta) - \omega_\beta) \in \ker f_{\alpha\beta}$, and thus $f_{\alpha\beta}(T_\beta(S_\beta(x_\beta))) = f_{\alpha\beta}(T_\beta(\omega_\beta))$. Besides,

$$f_{\alpha\beta}(T_\beta(S_\beta(x_\beta))) = f_{\alpha\beta}(T_\beta(\omega_\beta)) = h_{\alpha\beta}(T_\beta)(\omega_\alpha) = h_{\alpha\beta}(T_\beta)(f_{\alpha\beta}(S_\beta(x_\beta))) =$$

$$= h_{\alpha\beta}(T_\beta)((h_{\alpha\beta}(S_\beta))(x_\alpha)) = (h_{\alpha\beta}(T_\beta) \circ h_{a\beta}(S_\beta))(x_\alpha).$$

The last, in connection with (2.1) gives the multiplicativity of $h_{\alpha\beta}$.

Next, we prove that $h_{\alpha\beta}$ is continuous. Since $f_{\alpha\beta} : E_\beta \to E_\alpha$ is a continuous mapping between normed algebras, there exists a constant $K > 0$ such that $\dot{p}_\alpha(f_{\alpha\beta}(y_\beta)) \leq K\ \dot{p}_\beta(y_\beta)$ for each $y_\beta \in E_\beta$. In particular,

$$\dot{p}_\alpha(f_{\alpha\beta}(T_\beta(x_\beta))) \leq K\ \dot{p}_\beta(T_\beta(x_\beta)) \text{ for each } x_\beta \in E_\beta. \tag{2.2}$$

Taking the supremum on the right hand of (2.2) and since $M(E_\beta)$ is a Banach algebra (see [6, p. 20, Theorem 1.1.1]), we get

$$\dot{p}_\alpha(f_{\alpha\beta}(T_\beta(x_\beta))) \leq\ K\ \dot{p}_\beta(T_\beta(x_\beta)) \leq K \sup_{\dot{p}_\beta(x_\beta)\leq 1} \{\dot{p}_\beta(T_\beta(x_\beta))\} \leq K\,\|T_\beta\|_\beta \tag{2.3}$$

for every $x_\beta \in E_\beta$ with $\dot{p}_\beta(x_\beta) \leq 1$, and where $\|\cdot\|_\beta$ is the norm in the multiplier algebra $M(E_\beta)$. Since $f_{\alpha\beta}(T_\beta(x_\beta)) = [h_{\alpha\beta}(T_\beta)](x_\alpha)$ whenever $\alpha \leq \beta$ (hence $\dot{p}_\alpha(x_\alpha) \leq \dot{p}_\beta(x_\beta)$), then $\dot{p}_\alpha([h_{\alpha\beta}(T_\beta)](x_\alpha)) \leq K\ \ \|T_\beta\|_\beta$ for every $x_\alpha \in E_\alpha$ with $\dot{p}_\alpha(x_\alpha) \leq 1$ by (2.3). Taking now the supremum in this latter relation, we have $\sup\limits_{\dot{p}_\alpha(x_\alpha)\leq 1} \dot{p}_\alpha([h_{\alpha\beta}(T_\beta)](x_\alpha)) \leq K\ \ \|T_\beta\|_\beta$. Thus $\|h_{\alpha\beta}(T_\beta)\|_\alpha \leq K\ \ \|T_\beta\|_\beta$, namely, each $h_{\alpha\beta}$ is continuous. $\square$

So far, we have the family of topological algebras $M(E_\alpha)$ and the family of multiplicative continuous linear mappings $h_{\alpha\beta} : M(E_\beta) \to M(E_\alpha)$, $\alpha \leq \beta$ in Λ. Actually, they form a projective system. In fact, if $\alpha \leq \beta \leq \gamma$, then $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$, and therefore

$$[h_{\alpha\gamma}(T_\gamma)](x_\alpha) = f_{\alpha\gamma}(T_\gamma(x_\gamma)) = (f_{\alpha\beta} \circ f_{\beta\gamma})(T_\gamma(x_\gamma)) = f_{\alpha\beta}(f_{\beta\gamma}(T_\gamma(x_\gamma)) =$$
$$= f_{\alpha\beta}([h_{\beta\gamma}(T_\gamma)](x_\beta)) = [h_{\alpha\beta}(h_{\beta\gamma}(T_\gamma))](x_\alpha) = [(h_{\alpha\beta} \circ h_{\beta\gamma})(T_\gamma)](x_\alpha)$$

for each $x_\alpha \in E_\alpha$. That is, $h_{\alpha\gamma}(T_\gamma) = (h_{\alpha\beta} \circ h_{\beta\gamma})(T_\gamma)$ for each $T_\gamma \in M(E_\gamma)$, which implies that $h_{\alpha\gamma} = h_{\alpha\beta} \circ h_{\beta\gamma}$; it is clear that $h_{\alpha\alpha} = Id_{M(E_\alpha)}$.

Thus, we have the projective system of Banach algebras $\{(M(E_\alpha), h_{\alpha\beta})\}_{\alpha\in\Lambda}$ and we can take its inverse limit, $\varprojlim M(E_\alpha)$.

Now, we prove a lemma that will be useful in the sequel.

Lemma 2.7. *Let $(E, (p_\alpha)_{\alpha\in\Lambda})$ be a locally m-convex algebra with an approximate identity $(e_\delta)_{\delta\in\Delta}$ and let $T \in M(E)$. Then, for each $\alpha \in \Lambda$, $\ker p_\alpha$ is T-invariant; that is, $T(\ker p_\alpha) \subseteq \ker p_\alpha$.*

Proof. Take $x \in \ker p_a$. Since the seminorms are continuous, for $\varepsilon > 0$, there exists an index $\delta_0 \in \Delta$ such that $p_\alpha(T(x) - T(x)e_\delta) < \varepsilon$ whenever $\delta \geq \delta_0$. We have

$$p_\alpha(T(x)) = p_\alpha(T(x - xe_{\delta_0} + xe_{\delta_0})) = p_\alpha(T(x) - T(xe_{\delta_0}) + T(xe_{\delta_0}))$$
$$\leq p_\alpha(T(x) - T(xe_{\delta_0})) + p_\alpha(T(xe_{\delta_0})) = p_\alpha(T(x) - T(x)e_{\delta_0}) + p_\alpha(xT(e_{\delta_0}))$$
$$\leq p_\alpha(T(x) - T(x)e_{\delta_0}) + p_a(x)p_\alpha(T(e_{\delta_0})) < \varepsilon.$$

Since this is true for an arbitrary $\varepsilon > 0$, we conclude that $p_\alpha(T(x)) = 0$, that is, $T(x) \in \ker p_\alpha$. □

Now we state our main Theorem.

Theorem 2.8. *Let $(E, (p_\alpha)_{\alpha\in\Lambda})$ be a complete locally m-convex algebra with an approximate identity $(e_\delta)_{\delta\in\Delta}$, such that the respective projective system is perfect and each factor $E_\alpha = E/\ker p_a$ in its Arens-Michael decomposition is complete. Then $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism.*

Proof. Take $T \in M(E)$. Due to Lemma 2.7, T induces a well-defined map $T_\alpha : E_\alpha \to E_\alpha$ such that $T_\alpha \circ \rho_\alpha = \rho_\alpha \circ T$ for each $\alpha \in \Lambda$, that is, $T_\alpha(x_\alpha) = T_\alpha(\rho_\alpha(x)) = \rho_\alpha(T(x)) = T(x)_\alpha$ for each $x \in E$. Since for $x_\alpha, y_\alpha \in E_\alpha$,

$$T_\alpha(x_\alpha y_\alpha) = \rho_a(T(xy)) = \rho_\alpha(xT(y)) = x_\alpha T(y)_\alpha = x_\alpha T_\alpha(y_\alpha),$$

T_α is a right multiplier. In a similar way it can be shown that it is a left multiplier, as well.

Note also that $(T_\alpha)_{\alpha\in\Lambda}$ is an element of $\varprojlim M(E_\alpha)$. Indeed, for $\alpha \leq \beta$ and $\rho_\alpha(x) = x_\alpha \in E_\alpha$, we have

$$[h_{\alpha\beta}(T_\beta)](\rho_\alpha(x)) = [h_{\alpha\beta}(T_\beta)](x_\alpha) = f_{\alpha\beta}(T_\beta((x_\beta)) = f_{\alpha\beta}(T_\beta(\rho_\beta(x)) =$$
$$= f_{\alpha\beta}(\rho_\beta(T(x)) = f_{\alpha\beta}((f_\beta \circ \phi)(T(x))) = ((f_{\alpha\beta} \circ f_\beta) \circ \phi)(T(x)) =$$
$$= (f_\alpha \circ \phi)(T(x)) = \rho_\alpha(T(x)) = T_\alpha(\rho_\alpha(x)).$$

Therefore $h_{\alpha\beta}(T_\beta) = T_\alpha$ if $\alpha \leq \beta$.

Now we define the map

$$\Phi : M(E) \longrightarrow \varprojlim M(E_\alpha) \text{ by } \Phi(T) = (T_\alpha)_{\alpha\in\Lambda},$$

which obviously is linear. Moreover, for $T, S \in M(E)$ and $x_\alpha \in E_\alpha$, we have

$$\rho_\alpha(\Phi(T\circ S))(x_\alpha) = (T\circ S)_\alpha(x_\alpha) = ((T\circ S)(x))_\alpha = (T(S(x))_\alpha = T_\alpha(S(x)_\alpha) =$$
$$= T_\alpha(S_\alpha(x_\alpha)) = (T_\alpha \circ S_\alpha)(x_\alpha),$$

which implies that $(T\circ S)_\alpha = T_\alpha \circ S_\alpha$, and therefore $\Phi(T\circ S) = \Phi(T)\circ\Phi(S)$, namely, Φ is multiplicative.

Next, we show that Φ is one to one. For that, take $T, S \in M(E)$ such that $(T_\alpha)_{\alpha\in\Lambda} = \Phi(T) = \Phi(S) = (S_\alpha)_{\alpha\in\Lambda}$; then $T_\alpha = S_\alpha$ for each $x \in E$ and for each $\alpha \in \Lambda$. Therefore $\rho_\alpha \circ T = \rho_\alpha \circ S$ for each $\alpha \in \Lambda$; then $T = S$. Moreover, Φ is an onto map. Indeed, for $(W_\alpha)_{\alpha\in\Lambda} \in \varprojlim M(E_\alpha)$ define the map

$$W : E \to E \text{ by } W(x) = \phi^{-1}((W_\alpha(x_\alpha))_{\alpha\in\Lambda}),$$

which obviously is linear. Also

$$W(xy) = \phi^{-1}((W_\alpha(xy)_\alpha)_{\alpha\in\Lambda}) = \phi^{-1}(W_\alpha(x_\alpha y_\alpha))_{\alpha\in\Lambda}) = \phi^{-1}((x_\alpha W_\alpha(y_\alpha))_{\alpha\in\Lambda}) =$$
$$= \phi^{-1}((x_\alpha)_{\alpha\in\Lambda})\phi^{-1}((W_\alpha(y_\alpha))_{\alpha\in\Lambda}) = xW(y)$$

and similarly on the other side, so W is a multiplier on E. Finally, it is clear that $\Phi(W) = (W_\alpha)_{\alpha\in\Lambda}$.

We claim that Φ is continuous. By [3, p. 1934, Theorem 3.1], $M(E)$ is a subalgebra of $\mathcal{L}(E)$, the algebra of all continuous linear operators on E, so that the topology on $M(E)$ is the operator topology. We denote by

$$g_\alpha : M(E) \to M(E_\alpha)$$

the map $g_\alpha(T) = T_\alpha$, which, by Lemma 2.7, is well defined and obviously linear. Let us denote by $h_\alpha : \varprojlim M(E_\alpha) \to M(E_\alpha)$ the canonical continuous homomorphism from the inverse limit to one of its factors. Note that $h_\alpha \circ \phi = g_\alpha$ holds for each $\alpha \in \Lambda$.

Since Φ is continuous if and only if, for each $\alpha \in \Lambda$, $f_\alpha \circ \Phi$ is continuous (see [7, p. 89, the proof of Theorem 3.1]), we have to prove that g_α is continuous (for each $\alpha \in \Lambda$). We recall that the topology of $M(E)$ can be given by the set of seminorms $(\overline{p}_\alpha)_{\alpha\in\Lambda}$ defined as $\overline{p}_\alpha(T) = \sup\limits_{p_\alpha(x)\le 1} p_\alpha(T(x))$ for each $T \in M(E)$. Further, the topology of $M(E_\alpha)$ can be given by the norm $\|\cdot\|_\alpha$ defined as $\|S\|_\alpha = \sup\limits_{\dot{p}_\alpha(x)\le 1} \dot{p}_\alpha(S(x))$ for each $S \in M(E_\alpha)$, where, as usual, $\dot{p}_\alpha$ is the induced norm in E_α given by $\dot{p}_\alpha(x_\alpha) = \dot{p}_\alpha(x + \ker p_\alpha) = p_\alpha(x)$. The topology of $\varprojlim M(E_\alpha)$ can be defined by the local base consisting of neighborhoods $V = \bigcap_{i=1}^n h_{\alpha_i}^{-1}(V_{\alpha_i})$, where V_{α_i} is a basic neighborhood in $M(E_{\alpha_i})$.

Let $\varepsilon_i > 0$ be given and let

$$V_{\alpha_i} = \{S \in M(E_{\alpha_i}) : \|S\|_{\alpha_i} < \varepsilon_i\} \text{ and } U_{\alpha_i} = \{T \in M(E) : \overline{p}_{\alpha_i}(T) < \varepsilon_i\}.$$

We claim that

$$T \in U_{\alpha_i} \Longleftrightarrow T_{\alpha_i} \in V_{\alpha_i}. \tag{2.4}$$

Indeed,

$$T \in U_{\alpha_i} \iff \overline{p}_{\alpha_i}(T) < \varepsilon_i \iff \sup_{p_{\alpha_i}(x)\le 1} p_{\alpha_i}(T(x)) < \varepsilon_i$$
$$\iff \sup_{\dot{p}_{\alpha_i}(x_{\alpha_i})\le 1} \dot{p}_{\alpha_i}((T(x))_{\alpha_i}) < \varepsilon_i \iff \sup_{\dot{p}_{\alpha_i}(x_{\alpha_i})\le 1} \dot{p}_{\alpha_i}(T_{\alpha_i}(x_{\alpha_i})) < \varepsilon_i$$
$$\iff \|T_{\alpha_i}\|_{\alpha_i} < \varepsilon_i \iff T_{\alpha_i} \in V_{\alpha_i}.$$

Now, let V_α be a basic neighborhood of 0 in $M(E_\alpha)$, say

$$V_\alpha = \{S \in M(A_\alpha) : \|S\|_\alpha < \varepsilon\}.$$

Put $U_\alpha = g_\alpha^{-1}(V_\alpha)$. Then $U_\alpha = \{T \in M(E) : \overline{p}_\alpha(T) < \varepsilon\}$. This implies the continuity of g_α for each $\alpha \in \Lambda$. Hence Φ is continuous.

Finally, we show that Φ is an open map. Let $V = \bigcap_{i=1}^n h_{\alpha_i}^{-1}(V_{\alpha_i})$ be a basic neighborhood of 0 in $M(E)$. Take $T \in V$; then $T \in h_{\alpha_i}^{-1}(V_{\alpha_i})$ for all $i = 1, \ldots, n$. Therefore $h_{\alpha_i}(T) \in V_{\alpha_i}$ and, due to (2.4), $T_{\alpha_i} \in U_{\alpha_i}$. Then $\Phi(T) \in U = (U_\alpha)$, where $U_\alpha = U_{\alpha_i}$ for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ and $U_\alpha = M(E_\alpha)$ otherwise. This proves that Φ is an open map, and the proof is complete. $\square$

Acknowledgements. We want to thank Professor Mohamed Oudadess for the suggestion of Example 2.3 and for his fruitful comments. We are especially indebted to the referee for his pertinent remarks.

References

1. G.F. Bachelis and J.W. McCoy, *Left centralizers of an H^*-algebra*, Proc. Amer. Math. Soc. **43**(1)(1974), 106–110.
2. M. Haralampidou, *The Krull nature of locally C^* -algebras*, Function Spaces (Edwardsville, IL, 2002), 195–200, Contemp. Math. 328, Amer. Math. Soc., Providence, RI, 2003.
3. M. Haralampidou, L. Palacios and C. Signoret, *Multipliers in locally convex *-algebras*, Rocky Mountain J. Math. **43**, No. 6, 2013, 1931–1940.
4. T. Husain, *Multipliers of topological algebras*, Dissertationes Math. (Rozprawy Mat.) **285**(1989), 40 pp.
5. G. Köthe, *Topological Vector Spaces, I.* Springer-Verlag, Berlin, 1969.
6. R. Larsen, *The multiplier problem*, Lectures Notes in Math. No. 105, Springer-Verlag, Berlin, 1969.
7. A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, Amsterdam, 1986.

[1] Department of Mathematics, University of Athens, Panepistimioupolis, Athens 15784, Greece.

E-mail address: mharalam@math.uoa.gr

[2] Universidad Autónoma Metropolitana Iztapalapa, Mexico City 09340, Mexico.

E-mail address: pafa@xanum.uam.mx

E-mail address: casi@xanum.uam.mx