SOME EXAMPLES OF FREE INVOLUTIONS ON HOMOTOPY $S^l \times S^l$'S

BY

Robert Wells[1]

1. Introduction

Suppose M is a closed s-parallelizable manifold and G is a finite group acting freely and differentiably on M. A natural question to ask is: What is the (reduced) normal bundle $v(M/G)$ of the quotient M/G?

If $r : G \to GL(n; R)$ is any representation we may form the vector bundle $M \times_r R^n \to M/G$ associated to the principal fibration

$$M \xrightarrow{\pi} M/G.$$

Call the reduced class of this bundle $\xi(r)$. One may then conjecture an answer to the above question, namely

Answer 1. $v(M/G) = \xi(r)$ some r.

However, it is not hard to see that this answer has to be expanded at least to

Answer 2. $v(M/G) = \xi(r) + \eta$ some r and some reduced fiber homotopically trivial η, such that $\pi^*\eta$ is trivial.

That is, one hopes that the normal bundle is 'essentially' $\xi(r)$ for some r. The purpose of this paper is to show that Answer 2 is also wrong, even if $G = Z_2$, the action is orientation preserving and M is highly connected. In fact

If $l \equiv 0$ (8) and $l \geq 8$, then there is a free orientation preserving action of Z_2 on M, a differentiable manifold homotopy equivalent to $S^l \times S^l$ such that $v(M/Z_2) - \xi(r)$ is stably fiber homotopically trivial for *no* r.

It follows from Wall's classification of $(l - 1)$-connected manifolds [2] that M is s-parallelizable, so that M/Z_2 is a counterexample to Answer 2, and Answer 1 too for that matter. It would be interesting to know just what is the right answer.

2. A map

Suppose we have a free orientation preserving action of Z_2 on M a differentiable manifold homotopy equivalent to $S^l \times S^l$. In [3], it is shown that if l is even then $M/Z_2 = E(\gamma) \cup_\psi E(\gamma)$ where γ is an l-dimensional vector bundle over P_l, with twisted Euler class equal to 1 or 0, and $\psi : S(\gamma) \to S(\gamma)$ is a diffeomorphism.

To distinguish the two terms in expressions like $X \cup_f X$, recall the definition

Received February 17, 1969.

[1] The author was partially supported by a National Science Foundation grant during the preparation of this paper.

of such a space: It is $X \times 0 \cup X \times 1$ divided by the smallest equivalence relation containing $(f(x), 0) \neq (x, 1)$ for all $x \in \operatorname{dom} f$, and given the quotient topology.

We obtain an embedding $P_l \subset M/Z_2$ by

$$P_l \xrightarrow{\text{0 section}} E(\gamma) \times 0.$$

Let $g : S^l \to M$ be the cover of $P_l \subset M/Z_2$. Then we know from [3] that $g_*[S^l] = (1, -1)$ or $(1, 0)$ with respect to a symplectic basis of $H_l(M) = Z + Z$. Thus in either case an embedding $f : S^l \to M$ representing $(0, 1)$ will have algebraic intersection equal to 1 with g. Since

$$f_*[S^l] \cdot f_*[S^l] = f_*[S^l] \cdot \rho_* f_*[S^l] = 0,$$

where $\rho \in Z_2$ is the non-trivial element, we may assume $\pi \circ f$ is an embedding with trivial normal bundle [3]. We may suppose as well that f is transverse regular along g. Suppose x, y are two intersections of g with f of opposite signs. We may pick an arc α in $g(S^l)$ from x to y which misses the rest of

$$g(S^l) \cap [f(s^l) \cup \rho f(S^l)]$$

and misses $\rho\alpha$ as well. We may pick an arc β in $f(S^l)$ from y to x which misses the rest of $g(S^l) \cap f(S^l)$. Notice that $f(S^l) \cap \rho f(S^l) = \emptyset$. Then we may find $\gamma : D^2 \to M$ with boundary $\alpha + \beta$ such that $\pi \circ \gamma$ is an embedding normal at the boundary to P_l and $f(S^l)$, such that

$$\gamma(D^2) \cap \rho f(S^l) = \emptyset \quad \text{and} \quad \pi \circ \gamma\ (\operatorname{int} D^2) \cap (P_l \cup \pi f(S^l)) = \emptyset.$$

Since x and y have opposite signs, we may thicken $\gamma(D^2)$ to apply the Whitney procedure. It follows at once that we may thicken $\pi \circ \gamma(D^2)$ to apply the Whitney procedure, to obtain an isotopy from $\pi \circ f$ to $\pi \circ f'$ where f' has two fewer geometric intersections with g than f. Iterating the Whitney procedure as above finally gives us an embedding $f : S^l \to M$ such that $\pi \circ f$ is an embedding with trivial normal bundle meeting P_l transversally at a single point. By altering the decomposition $M/Z_2 = E(\gamma) \cup_\psi E(\gamma)$ so that $E(\gamma) \times 0$ is a suitably small tubular neighborhood of P_l, we may assume that the embedding π where

$$\pi f : D^l \cup_1 D^l \to M/Z_2$$

is just a standard inclusion of a fiber $D^l \times 0 \subset E(\gamma) \times 0$ on $D^l \times 0$, and carries $D^l \times 1$ into $E(\gamma) \times 1$. Since $v(\pi \circ f(S^l) : M/Z_2)$ is trivial, we have two isotopic copies S_1^l and $\pi \circ f(S^l)$ in M/Z_2. We may assume

$$E(\gamma) \mid P_1 \times 0 \cap S_1^l = \emptyset.$$

Let $\sigma = E(\gamma \mid P_1) \cup_1 E(\gamma \mid P_1)$ where $1 : S(\gamma \mid P_1) \to S(\gamma \mid P_1)$. We wish to find a map

$$j : \sigma \to M/Z_2 - S_1^l$$

(where $S_1^l \subset M/Z_2$ is the other copy of S^l, isotopic to $\pi \circ f(S^l)$) such that, on $E(\gamma \mid P_1) \times 0$ it is the natural embedding

$$E(\gamma \mid P_1) \times 0 \subset E(\gamma) \times 0,$$

and such that it carries $E(\gamma \mid P_1) \times 1$ into $E(\gamma) \times 1$. First, extend the natural inclusion

$$E(\gamma \mid P_0) \times 0 \subset E(\gamma \mid P_1) \times 0 \subset E(\gamma) \times 0$$

to

$$E(\gamma \mid P_0) \cup_1 E(\gamma \mid P_0) \to M$$

by

$$E(\gamma \mid P_0) \cup_1 E(\gamma \mid P_0) = D_l \cup_1 D_l = S^l \xrightarrow{\pi \circ f} M,$$

so that $E(\gamma \mid P_0) \times 1 \to E(\gamma) \times 1$. Now, σ is $S(\xi + \gamma \mid P_1)$ with γ unorientable, so that

$$\sigma = (P_1 \vee S^l) \cup_{\iota + \tau\iota} D^{l+1}$$

up to homotopy, where $\iota : S^l \to P_1 \vee S^l$ is the standard inclusion and τ represents the generator of $\pi_1(P_1 \vee S^l)$. We have also

$$\sigma = (E(\gamma \mid P_1) \cup_1 E(\gamma \mid P_0)) \cup_h D^{l+1}$$

where $1 : S(\gamma \mid P_0) \to S(\gamma \mid P_0)$ and $h : S^l \to S(\gamma \mid P_1) \cup_1 E(\gamma \mid P_0)$. Then we have the following homotopy commutative diagram

$$\begin{array}{ccccc}
S^l & \xrightarrow{\iota + \tau\iota} & P_1 \vee (S^l) & \longrightarrow & M/Z_2 - S_1^l \\
 & & \cap & & \\
\| & & E(\gamma|P_1) \cup_1 E(\gamma|P_0) & & \cup \\
 & & \cup & & \\
S^l & \xrightarrow{h} & S(\gamma|P_1) \cup_1 E(\gamma|P_0) & \xrightarrow{\psi^{-1}(\pi \circ f)} & E(\gamma) \times 1 - S_1^l \cap (E(\gamma) \times 1).
\end{array}$$

We also have the commutative diagram

$$\begin{array}{ccccc}
M/Z_2 - \pi \circ f(S^l) & \leftarrow & M - f(S^l) - \rho \circ f(S^l) & = & R \times S^{l-1} \times S^l \\
\uparrow & & \cup & & \cup \\
E(\gamma) \times 1 - \pi \circ f(S^l) \cap (E(\gamma) \times 1) & \leftarrow & S^l \times D^l - f(D^l) - \rho \circ f(D^l) & & \\
 & & & = & R \times S^{l-1} \times D^l
\end{array}$$

so that

$$\pi_l(E(\gamma) \times 1 - \pi \circ f(S^l) \cap (E(\gamma) \times 1) \to \pi_l(M/Z_2 - \pi \circ f(S^l))$$

is a monomorphism. It follows that

$$\pi_l(E(\gamma) \times 1 - S^l \cap (E(\gamma) \times 1)) \to \pi_l(M/Z_2 - S_1^l)$$

is a monomorphism.

But $P_1 \vee S^l \to M - S_1^l$ carries ι to ι' represented by $\pi \circ f$, and it carries $\tau\iota$ onto the element $\pi \circ f \circ (-1)$ where $-1 : S^l \to S^l$ is a linear map with matrix

$$\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

That is, $\tau\iota \to -\iota'$, so $\iota + \tau\iota \to 0$. Then it follows from the monomorphism above, and the first diagram, that h extends to

$$D^{l+1} \to E(\gamma) \times 1 - S_1^l \cap (E(\gamma) \times 1),$$

so that the map

$$E(\gamma \mid P_1) \cup_1 E(\gamma \mid P_0) \xrightarrow{1 \cup \pi \circ f} M/Z_2 - S_1^l$$

extends to a map

$$\varphi : \sigma = E(\gamma \mid P_1) \cup_1 E(\gamma \mid P_1) \to M/Z_2 - S_1^l .$$

Now, $S_1^l \subset M$ and $\pi \circ f(S^l) \subset M$ are isotopic, so we may interchange them and finally obtain

$$\varphi : \sigma \to M/Z_2 - \pi \circ f(S^l)$$

such that

(1) $S^l \subset \sigma \xrightarrow{\varphi} M/Z_2$ is an embedding isotopic to $\pi \circ f$,
(2) $\varphi \mid E(\gamma \mid P_1) \times 0 = \text{incl}\ (E(\gamma \mid P_1) \times 0 \subset E(\gamma) \times 0)$,
(3) $\varphi : E(\gamma \mid P_1) \times 1 \to E(\gamma) \times 1$.

3. The normal bundle

With M a homotopy $S^l \times S^l$ as above, with involution ρ and l even, we now now seek $v(M/Z_2)$. By collapsing $E(\gamma) \times 1$ to a point, we obtain

$$M/Z_2 \xrightarrow{p} T(\gamma),$$

the Thom space of γ. The sequence

$$P_l \xrightarrow{h} M/Z_2 \xrightarrow{p} T(\gamma)$$

is a cofibration, where h is the inclusion of P_l in $E(\gamma) \times 1$. The composition

$$P_l \xrightarrow{h} M/Z_2 \to P$$

is homotopic to the standard inclusion. Also, $\widetilde{KO}^{-1}(P) \to \widetilde{KO}^{-1}(P_l) \to 0$ is an epimorphism, so it follows that

$$0 \to \widetilde{KO}(T(\gamma)) \xrightarrow{p^*} \widetilde{KO}(M/Z_2) \to \widetilde{KO}(P_l) \to 0$$

is exact. It follows that the reduced stable normal bundle of M/Z_2 is $k\xi + p^*\alpha$ where ξ is the reduced canonical line bundle, α is uniquely determined in $\widetilde{KO}(T(\gamma))$. Since ρ preserves orientation, k is even. The bundle γ depends only on k and l, and we have [3], when $l \equiv 0$ (8):

$$\begin{aligned} \widetilde{KO}(T(\gamma)) &= Z, & k &\equiv 2, 6 \ (8) \\ &= Z + Z_2, & k &\equiv 0, 4 \ (8) \end{aligned}$$

Since index $(M/Z_2) = 0$, α cannot have infinite order, so $\alpha = 0$ or possibly the element of order 2.

Now we have to investigate α more closely. Recall that

$$\gamma + \varepsilon^t = (2^{\varphi(l)} - l - 1 - k)\xi$$

where $t = 2^{\varphi(l)} - 2l - 1 - k$, so $S^t T(\gamma) = P_{t+2l}/P_{t+l-1}$. Also,

$$S^t T(\gamma \mid P_1) = P_{t+l+1}/P_{t+l-1}$$

and $S^t T(\gamma \mid P_1) \subset S^t T(\gamma)$ is the natural inclusion

$$P_{t+l+1}/P_{t+l-1} \subset P_{t+2l}/P_{t+l-1}.$$

Assume $l \equiv k \equiv 0$ (8). Then $t \equiv -1$ (8) so that

$$\begin{array}{ccccc} \widetilde{KO}(T(\gamma)) & = & \widetilde{KO}^t(S^t(T(\gamma))) & = & \widetilde{KO}^{-1}(P_{t+2l}/P_{t+l-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{KO}(T(\gamma \mid P_1)) & = & \widetilde{KO}^t(S^t T(\gamma \mid P_1)) & = & \widetilde{KO}^{-1}(P_{t+l+1}/P_{t+l-1}) \end{array}$$

Now, $\widetilde{KO}^{-2}(P_{t+l-1}) = Z_2 + Z_2$ and the image of

$$\widetilde{KO}^{-2}(P_{t+2l}) \to \widetilde{KO}^{-2}(P_{t+l-1}) \quad \text{and} \quad \widetilde{KO}^{-2}(P_{t+l+1}) \to \widetilde{KO}^{-2}(P_{t+l-1})$$

are the same subgroup Z_2 of $\widetilde{KO}^{-2}(P_{t+l-1})$. Let $\beta \in \widetilde{KO}^{-2}(P_{t+l-1})$ be an element not in that image. Then if δ is the coboundary

$$\widetilde{KO}^{-2}(P_{t+l-1}) \xrightarrow{\delta} KO^{-1}(P_{t+2l}/P_{t+l-1}),$$

it is straightforward to check that $\delta\beta$ is the element of order 2. But if δ' is the coboundary

$$\widetilde{KO}^{-2}(P_{t+l-1}) \to \widetilde{KO}^{-1}(P_{t+l+1}/P_{t+l-1})$$

we must have $\delta'\beta \neq 0$. Since

$$\widetilde{KO}^{-1}(P_{t+l+1}/P_{t+l-1}) = Z_2,$$

it follows that the element of order 2 in $\widetilde{KO}^{-1}(P_{t+2l}/P_{t+l-1})$ is carried onto the generator of $\widetilde{KO}^{-1}(P_{t+l+1}/P_{t+l-1})$. Thus the element of order 2 in $\widetilde{KO}(T(\gamma))$ is carried onto the generator of $\widetilde{KO}(T(\gamma \mid P_1)) = Z_2$.

Now we can see that the sequence

$$0 \to \widetilde{KO}(T(\gamma)) \xrightarrow{p^*} \widetilde{KO}(M/Z_2) \to \widetilde{KO}(P_l) \to 0$$

is split exact. For the splitting map $\widetilde{KO}(P_l) \to \widetilde{KO}(M/Z_2)$ we choose the one that carries the reduced canonical line bundle over P_l to ξ, the reduced canonical line bundle over M/Z_2. This splitting map will be well defined provided that $2^{\varphi(l)}\xi = 0$. Now, $2^{\varphi(l)}\xi = p^*(x)$ for x uniquely determined in $\widetilde{KO}(T(\gamma))$, and clearly $x = 0$ or $x = \alpha =$ the element of order 2 in $\widetilde{KO}(T(\gamma))$. Using properties (2) and (3) of the map $\varphi : \sigma \to M/Z_2$, we obtain the following commutative diagram

$$\begin{array}{ccc} \sigma & \xrightarrow{\varphi} & M/Z_2 \\ \downarrow & & \downarrow P \\ T(\gamma \mid P_1) & \subset & T(\gamma) \end{array}$$

where $P_1 \to \sigma \to T(\gamma \mid P_1)$ is a cofibration. Clearly, $\varphi^* p^* \alpha$ is the unique non-zero element of ker $(\widetilde{KO}(\sigma) \to \widetilde{KO}(P_1))$. But $\varphi^*\, 2\xi = 0$ so $\varphi^*(2^{\varphi(l)}\xi) = 0$. Thus $2^{\varphi(l)}\xi = p^*(\alpha)$ is impossible and we must have $2^{\varphi(l)}\xi = 0$, so the sequence above is split exact (even with respect to Adams operations).

Now consider what happens to $k\xi$ and $k\xi + p^*\alpha$ (where α is the element of order 2 in $\widetilde{KO}(T(\gamma))$) under the map

$$\widetilde{KO}(M/Z_2) \xrightarrow{\varphi^*} KO(\sigma).$$

Since $\widetilde{KO}^{-1}(P_1) = 0$, the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \widetilde{KO}(T(\gamma \mid P_1)) & \to & \widetilde{KO}(\sigma) & \to & \widetilde{KO}(P_1) & \to & 0 \\ & & \uparrow & & \uparrow \varphi^* & & \uparrow & & \\ 0 & \to & \widetilde{KO}(T(\gamma)) & \to & \widetilde{KO}(M/Z_2) & \to & \widetilde{KO}(P_l) & \to & 0 \end{array}$$

Then $\varphi^*\xi$ is the reduced canonical line bundle of σ, and that has order 2, so $\varphi^*(k\xi) = 0$ since k is even. But $\varphi^*(k\xi + p^*\alpha) = \mu$ is the unique non-zero element of $\ker(\widetilde{KO}(\sigma) \to \widetilde{KO}(P_1))$. Thus the two cases of $v(M/Z_2)$ are distinguished by φ^*.

Now,

$$\sigma/S^l = (S^l \times S^1)/(S^l \times *) = S^1 \vee S^{l+1},$$

and the map $\sigma/S^l \to S^{l+1}$ of degree 1 is simply

$$S^1 \to * \quad \text{and} \quad S^{l+1} \xrightarrow{1} S^{l+1}.$$

The map

$$S^1 = P_1 \subset \sigma \to \sigma/S^l = S^1 \vee S^{l+1}$$

is simply

$$S^1 \xrightarrow{1} S^1.$$

Let $j\colon S^{l+1} \to \sigma/S^l$ be the natural inclusion of S^{l+1}. Then

$$S^{l+1} \xrightarrow{j} \sigma/S^l \to S(S^l)$$

has degree 2 since $H^{l+1}(\sigma) = Z_2$, so the collapsing map

$$S^1 \vee S^{l+1} = \sigma/S^l \to S^{l+1}$$

is

$$S^1 \to * \quad \text{and} \quad S^{l+1} \to S^{l+1}.$$

Since $\tilde{K}O(S^{l+1}) = Z_2$ it follows that the following sequence is exact, and the attached triangle commutative.

$$\begin{tikzcd} \tilde{K}O(S^{l+1}) \arrow[r, "0"] & \tilde{K}O(S^1 \vee S^{l+1}) \arrow[r] \arrow[dr] & \tilde{K}O(\sigma) \arrow[r, "0"] \arrow[d] & \tilde{K}O(S^l) \\ & & \tilde{K}O(S^1) & \end{tikzcd}$$

The first 0 follows from $\tilde{K}O(S^{l+1}) = Z_2$ and the degree 2 map, and the second 0 follows from $\tilde{K}O(S^l) = Z$ and $\tilde{K}O(\sigma)$ finite. It follows that μ is the pull-back of the generator of $\tilde{K}O(S^{l+1}) = Z_2$ under the degree 1 map $\sigma \to S^{l+1}$, and in turn restricts to that generator under

$$S^{l+1} \xrightarrow{j} \sigma/S^l.$$

4. The example

Suppose $l \equiv 0\ (8)$ and $l \geq 8$. Let $\eta \to S^{l+1}$ be an l-plane bundle over S^{l+1} whose reduced stable class is non-zero in $\tilde{K}O(S^{l+1})$. Let $S(\eta)$ be the sphere bundle of η. Define an involution of σ of $S(\eta)$ by $\sigma(x) = -x$. Then

$$S(\eta)/Z_2 \xrightarrow{\tilde{\omega}} S^{l+1}$$

is a P_{l-1}-bundle. Then the reduced tangent bundle of $S(\eta)/Z_2$ is $l\xi + \tilde{\omega}^*\beta$ where ξ is the reduced canonical line bundle and $\beta \in \tilde{K}O(S^{l+1})$ is the non-zero element. It follows that

$$v(S(\eta)/Z_2) = (2^K - l)\xi + \tilde{\omega}^*\beta$$

so that $k = 2^K - l \equiv 0\ (8)$.

Let $w\colon S(\eta)/Z_2 \to P$ be the classifying map of the cover

$$S(\eta) \xrightarrow{c} S(\eta)/Z_2 .$$

Then the map

$$S(\eta)/Z_2 \xrightarrow{w \times \tilde{\omega}} P \times S^{l+1}$$

pulls back the normal bundle from $k\xi \times \beta$. Let $P_{l-1} \subset S(\eta)/Z_2$ be a fiber. The immersions $S^{l+1} \to P_{l-1}$ has trivial normal bundle. Observe that the canonical line bundle over P_{l-1} is included in $v(P_{l-1} : S(\eta)/Z_2)$. Regarding S^{l+1} as its S^0-bundle, we obtain an embedding $S^{l-1} \subset S(\eta)/Z_2$ with trivial normal bundle such that either of its covers generates $H_{l-1}(S(\eta)) = Z$. By surgering $S^{l-1} \subset S(\eta)/Z_2$ we obtain a manifold M/Z_2 together with a map

$$\zeta : M/Z_2 \to P \times S^{l+1}$$

such that $v(M/Z_2) = \zeta^*(k\xi \times \beta)$ and M is a homotopy $S^l \times S^l$. We obtain $S(\eta)/Z_2$ back again by surgering

$$S^l \subset S(\eta)/Z_2 - S^{l-1} \subset M/Z_2$$

where $S^l \subset S(\eta)/Z_2$ is a linking sphere of S^{l-1}.

Notice that the embedding $P_{l-1} \subset S(\eta)/Z_2 - S^{l-1}$ may be extended to an embedding $P_l \subset M/Z_2$ which meets $S^l \subset M/Z_2$ transversally, exactly once.

Since $v(S^l : M/Z_2)$ is trivial, it follows that $S^l \subset M/Z_2$ may be the mapping $\pi \circ f$ of Section 2.

Remark. Let f be a cover of $S^l \subset M/Z_2$ above, and $\bar{g} : S^l \to M$ the cover of $P_l \subset M/Z_2$ above. Then with respect to a symplectic basis of $H_l(M)$, we may take $f*[S^l] = (0, 1)$. Then according to [3],

$$\bar{g}*[S^l] = (1, 0) + (2a, 2b) \quad \text{or} \quad (0, 1) + (2a, 2b) \quad \text{or} \quad (1, -1) + (2a, 2b).$$

Then the algebraic intersection $f*[S^l] \cdot \bar{g}*[S^l]$ is $1 + 2a$ or $2a$ or $1 + 2a$ respectively. Since the geometric intersection $= 1$, it follows that the middle case is excluded, and $a = 0$ in the other two cases. It follows that we may replace $P_l \subset M/Z_2$ by another $P_l \subset M/Z_2$ with cover $g : S^l \to M$ such that $g*[S^l] = (1, 0)$ or $(1, -1)$ and geometric intersection 1 with $\pi \circ f(S^l)$.

In any case, we have $\sigma \subset M/Z_2 - \pi \circ f(S^l)$ with $S^l \subset \sigma\ M/Z_2$ isotopic to $\pi \circ f(S^l)$. By performing the surgery reverse to the one above, on $\pi \circ f(S^l)$, we get $S(\eta)/Z_2$ back again, with

$$\begin{tikzcd} \sigma \arrow[d] \arrow[r, phantom, "\subset"] & S(\eta) \\ \sigma/S^l \arrow[ur] & \end{tikzcd}$$

commutative. Then we have

$$S^{l+1} \xrightarrow{j} \sigma/S^l \to S(\eta)/Z_2 \xrightarrow{\tilde{\omega}} S^{l+1}.$$

We wish to show that this composition has degree $+ 1$. We may write

$$\sigma/S^l = D^{l+1} \cup_{\,1} S^l \vee S^l \cup_{\iota+\tau\iota} D^{l+1}$$

as above. If e_1 is the $(l + 1)$ cell represented by the left D^{l+1} and e_2 is the $(l + 1)$ cell represented by the right D^{l+1}, then $\iota_* = (\tau\iota)_* = 1$ so we have that $e_2 - 2e_1$ represents the generator of $H_{l+1}(\sigma/S^l)$. Now let $S^{l-1} \subset S(\eta)/Z_2$ be the sphere that was surgered to obtain M/Z_2. Then $\sigma \cap S^{l-1} = \emptyset$ and the geometric intersection of e_1 with S^{l-1} is 1. Thus the algebraic intersection $(e_2 - 2e_1) \cdot S^{l-1}$ is -2. But the homology class represented by S^{l-1} is twice the homology class represented by a fiber P_{l-1}. Thus $(e_2 - 2e_1) \cdot P_{l-1} = -1$, so $e_2 - 2e_1$ representes the generator of $H_{l+1}(S(\eta)/Z_2)$ and consequently

$$\sigma/Z^l \to S(\eta)/Z_2 \to S^{l+1}$$

carries $e_2 - 2e_1$ to a generator of S^{l+1}. It follows that the map above has degree ± 1, and consequently that

$$v(S(\eta)/Z_2)/\sigma = \mu.$$

But then $v(M/Z_2)/\sigma = v(S(\eta)/Z_2)/\sigma = \mu$, so $v(M/Z_2) = k'\xi + \pi^*\alpha$ (where $k' = k$ or $k + 2^{\varphi(l-1)}$).

Finally, $\pi^*(\alpha)$ cannot be fiber homotopically trivial because $\pi^*(\alpha) \mid \sigma = \mu$, and it is straightforward to see that μ cannot be fiber homotopically trivial since it corresponds to the generator of $\widetilde{KO}(S^{l+1})$ under the isomorphism

$$\widetilde{KO}(S^1 \vee S^{l+1}) \cong \widetilde{KO}(\sigma).$$

Bibliography

1. C. T. C. Wall, *Surgery of non-simply connected manifolds*, Ann. of Math., vol. 84 (1966), pp. 217–276.
2. ———, *Classification of Handlebodies*, Topology, vol. 2 (1963), pp. 253–261.
3. R. Wells, *Free involutions of homotopy $S^l \times S^l$'s*, Illinois J. Math., vol. 15 (1971), pp. 160–184.

University of Chicago
Chicago, Illinois