ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 4, Winter 1991

# ON THE HURWITZ ZETA-FUNCTION

BY

ZHANG WENPENG[1]

## 1. Introduction

For $0 < \alpha \leq 1$, let $\zeta(s, \alpha)$ be the Hurwitz zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad \text{for } \operatorname{Re}(s) > 1$$

and its analytic extension,

$$\zeta_x(s, \alpha) = \zeta(s, \alpha) - \sum_{0 \leq n \leq x - \alpha} \frac{1}{(n + \alpha)^s}.$$

J.F. Koksma and C.G. Lekkerkerker [1] first studied the mean square value

$$f(s) = \int_0^1 |\zeta_1(s, \alpha)|^2 \, d\alpha,$$

and obtained the following results:

THEOREM A. *If* $1/2 < \sigma \leq 1$ *and* $|t| \geq 3$, *then*

(I) $$f\left(\frac{1}{2} + it\right) \leq 64 \ln|t|,$$

(II) $$\left| f(\sigma + it) - \frac{1}{2\sigma - 1} \right| \leq |t|^{1-2\sigma}\left(32 \ln|t| + \frac{1}{2\sigma - 1}\right).$$

Secondly, V.V. Rane [2] gave a general conclusion, namely:

---

Received January 26, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 10H10; Secondary 11M35.

[1]Project Supported by the National Natural Science Foundation of China.

THEOREM B. *For $x \geq 1$, $t \geq 2$, let $2\pi xy = t$,*

$$K(s) = \left(\frac{2\pi}{i}\right)^{s-1} \Gamma(1-s), \quad s = \sigma + it;$$

*Then*

$$\int_0^1 |\zeta_x(s, \alpha)|^2 \, d\alpha = |k(s)|^2 \sum_{n \leq y} \frac{1}{n^{2(1-\sigma)}} + O(t^{-1} x^{2(1-\sigma)}) + O(x^{-2\sigma})$$

*and from this, it is easy to deduce that*

$$\int_0^1 \left|\zeta_1\left(\frac{1}{2} + it, \alpha\right)\right|^2 d\alpha = \ln t + O(1) \tag{1}$$

In this paper, as an improvement of (1), we shall give a sharper asymptotic formula for $f(\frac{1}{2} + it)$. In fact we shall establish the following two theorems:

THEOREM 1. *For $t \geq 2$, we have the asymptotic formula*

$$f\left(\frac{1}{2} + it\right) = \ln\left(\frac{t}{2\pi}\right) + \gamma + O(t^{-7/36} \ln^{25/18} t)$$

*where $\gamma$ is Euler's constant.*

THEOREM 2. *For $0 < \sigma < 1$, $\sigma \neq 1/2$, $t \geq 2$, we have*

$$f(\sigma + it) = \left(\frac{t}{2\pi}\right)^{1-2\sigma} \zeta(2 - 2\sigma) + \frac{1}{2\sigma - 1} + O(R(\sigma, t)),$$

*where $\zeta(s)$ is the Riemann zeta-function, and*

$$R(\sigma, t) = \begin{cases} t^{-\sigma(3+\sigma)/2(5-\sigma)} (\ln t)^{3/2} & \text{if } \frac{1}{2} < \sigma < 1, \\ t^{(5+5\sigma^2-16\sigma)/2(5-\sigma)} (\ln t)^{3/2} & \text{if } 0 < \sigma < \frac{1}{2}. \end{cases}$$

Let

$$\Delta(t) = f\left(\frac{1}{2} + it\right) - \ln\left(\frac{t}{2\pi}\right) - \gamma$$

and

$$\beta = \sup(b \colon |\Delta(t)| \ll t^{-b}, 2 \leq t < +\infty).$$

From Theorem 1 we know that $\beta \geq 7/36$. We conjecture that

$$\beta = \frac{1}{4}.$$

This is an open problem in analytic number theory.

## 2. Proof of the theorems

In this section, we shall complete the proof of the theorems. For the sake of simplicity we do not repeat the commonly used conclusions, which can be found in the references. First by the approximate equation of Hurwitz zeta-function we may get

$$\begin{aligned}\zeta_1(\sigma + it, \alpha) = \sum_{1 \leq n \leq x} \frac{1}{(n+\alpha)^{\sigma+it}} + \left(\frac{2\pi}{t}\right)^{\sigma - 1/2 + it} \\ \times e^{i(\pi/4+t)} \sum_{1 \leq m \leq y} \frac{e^{-2\pi i m \alpha}}{m^{1-\sigma-it}} + O(x^{-\sigma} \ln t),\end{aligned} \tag{2}$$

where $1 \leq x \leq y$, and $2\pi xy = t$.

Let

$$A(\sigma + it) = \left(\frac{2\pi}{t}\right)^{\sigma - 1/2 + it} e^{i(\pi/4+t)}, \quad s = \sigma + it,$$

$$I_1(s, \alpha) = \sum_{1 \leq n \leq x} \frac{1}{(n+\alpha)^s}, \quad I_2(s, \alpha) = A(s) \sum_{1 \leq m \leq y} \frac{e^{-2\pi i m \alpha}}{m^{1-s}}.$$

Then from (2) we get

$$\begin{aligned}f(\sigma + it) &= \int_0^1 |I_1(s,\alpha)|^2 \, d\alpha + \int_0^1 |I_2(s,\alpha)|^2 \, d\alpha \\ &+ \int_0^1 \left(I_1(s,\alpha)\overline{I_2(s,\alpha)} + \overline{I_1(s,\alpha)} I_2(s,\alpha)\right) d\alpha + O(x^{-2\sigma} \ln^2 t) \\ &+ O\left(x^{-\sigma} \ln t \int_0^1 |I_1(s,\alpha) + I_2(s,\alpha)| \, d\alpha\right).\end{aligned} \tag{3}$$

Now we estimate the integrals in (3) respectively. Noticing that

$$A(s)\overline{A(s)} = \left(\frac{t}{2\pi}\right)^{1-2\sigma},$$

$$\int_0^1 e^{2\pi i n \alpha} \, d\alpha = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0 \end{cases}.$$

We may immediately get

$$\int_0^1 |I_2(s,\alpha)|^2\,d\alpha = \sum_{1\le m,n\le y} \frac{A(s)\overline{A(s)}}{m^{1-\sigma-it}n^{1-\sigma+it}} \int_0^1 e^{2\pi i(n-m)\alpha}\,d\alpha$$

$$= \left(\frac{t}{2\pi}\right)^{1-2\sigma} \sum_{n\le y} \frac{1}{n^{2-2\sigma}}. \tag{4}$$

Applying Euler's summation formula we have

$$\sum_{n\le y} \frac{1}{n} = \ln y + \gamma + O\left(\frac{1}{y}\right), \tag{5}$$

$$\sum_{n\le y} \frac{1}{n^{2-2\sigma}} = \frac{y^{2\sigma-1}}{2\sigma-1} + \zeta(2-2\sigma) + O(y^{-2(1-\sigma)}), \quad \sigma \ne \frac{1}{2}. \tag{6}$$

Now we may estimate another main term. We have

$$\int_0^1 |I_1(\sigma+it,\alpha)|^2\,d\alpha = \sum_{1\le n\le x} \int_0^1 \frac{d\alpha}{(n+\alpha)^{2\sigma}}$$

$$+ \sum_{\substack{1\le m,n\le x\\ m\ne n}} \int_0^1 \frac{d\alpha}{(m+\alpha)^{\sigma+it}(n+\alpha)^{\sigma-it}}. \tag{7}$$

If $\sigma = 1/2$, then

$$\sum_{1\le n\le x} \int_0^1 \frac{d\alpha}{n+\alpha} = \sum_{1\le n\le x} (\ln(n+1) - \ln n) = \ln x + O\left(\frac{1}{x}\right). \tag{8}$$

If $0 < \sigma < 1$ and $\sigma \ne 1/2$, then

$$\sum_{1\le n\le x} \int_0^1 \frac{d\alpha}{(n+\alpha)^{2\sigma}} = \frac{([x])^{1-2\sigma}}{1-2\sigma} + \frac{1}{2\sigma-1}$$

$$= \frac{x^{1-2\sigma}}{1-2\sigma} + \frac{1}{2\sigma-1} + O(x^{-2\sigma}). \tag{9}$$

Let

$$J = \sum_{1\le m<n\le x} \int_0^1 \frac{d\alpha}{(n+\alpha)^{\sigma+it}(m+\alpha)^{\sigma-it}}$$

and

$$F(\alpha) = t\ln\left(\frac{m+\alpha}{n+\alpha}\right), \quad G(\alpha) = \frac{1}{(m+\alpha)^{\sigma}(n+\alpha)^{\sigma}}.$$

Then

$$F'(\alpha) = \frac{t(n-m)}{(m+\alpha)(n+\alpha)}, \quad \frac{G(\alpha)}{F'(\alpha)} = \frac{(m+\alpha)^{1-\sigma}(n+\alpha)^{1-\sigma}}{t(n-m)}.$$

It is clear that $G(\alpha)/F'(\alpha)$ is increasing function, and

$$F'(\alpha)/G(\alpha) = \frac{t(n-m)}{(n+\alpha)^{1-\sigma}(m+\alpha)^{1-\sigma}} \geq \frac{t(n-m)}{2(mn)^{1-\sigma}}.$$

From this and [3, lemma 4.3] we get

$$\begin{aligned} |J| &\ll \sum_{1 \leq m < n \leq x} \frac{(mn)^{1-\sigma}}{t(n-m)} \ll x^{1-\sigma} t^{-1} \ln x \sum_{m \leq x} m^{1-\sigma} \\ &\ll t^{-1} x^{3-2\sigma} \ln t. \end{aligned} \tag{10}$$

Similarly, we can also get the estimate

$$|\bar{J}| \ll t^{-1} x^{3-2\sigma} \ln t. \tag{11}$$

Combining the estimations (7), (8), (9), (10) and (11) we obtain

$$\int_0^1 \left| I_1\left(\frac{1}{2} + it, \alpha\right) \right|^2 d\alpha = \ln x + O(x^{-1}) + O(t^{-1} x^2 \ln t). \tag{12}$$

If $0 < \sigma < 1$ and $\sigma \neq 1/2$, then

$$\begin{aligned} \int_0^1 |I_1(\sigma + it, \alpha)|^2 \, d\alpha = \frac{x^{1-2\sigma}}{1-2\sigma} + \frac{1}{2\sigma - 1} + O(x^{-2\sigma}) \\ + O(x^{3-2\sigma} t^{-1} \ln t). \end{aligned} \tag{13}$$

Now we estimate the integrals $\int_0^1 I_1(s, \alpha) \overline{I_2(s, \alpha)} \, d\alpha$.

Let $x < y/2$, $2\pi xy = t$; then we have

$$\begin{aligned} \int_0^1 I_1(s, \alpha) \overline{I_2(s, \alpha)} \, d\alpha = \overline{A(\sigma + it)} \sum_{1 \leq m \leq x} \sum_{1 \leq n \leq z} \frac{1}{n^{1-\sigma+it}} \int_0^1 \frac{e^{2\pi i n \alpha}}{(m+\alpha)^s} \, d\alpha \\ + \overline{A(\sigma + it)} \sum_{1 \leq m \leq x} \sum_{z < n \leq y} \frac{1}{n^{1-\sigma+it}} \int_0^1 \frac{e^{2\pi i n \alpha}}{(m+\alpha)^s} \, d\alpha \end{aligned} \tag{14}$$

where $z$ is a parameter with $x \leq z < y/2$.

Let $f(\alpha) = 2\pi n\alpha - t\ln(m+\alpha)$, $g(\alpha) = (m+\alpha)^{1-\sigma}$. Then

$$f'(\alpha) = 2\pi n - t/(m+\alpha).$$

Let

$$h(\alpha) = g(\alpha)/f'(\alpha) = \frac{(m+\alpha)^{2-\sigma}}{2\pi n(m+\alpha) - t};$$

since $t = 2\pi xy > 4\pi xz > 4\pi mn$, we know that

$$h'(\alpha) = \frac{(2-\sigma)(m+\alpha)^{1-\sigma}}{2\pi n(m+\alpha)-t} - \frac{2\pi n(m+\alpha)^{2-\sigma}}{(2\pi n(m+\alpha)-t)^2} < 0, \quad 0 < \alpha \le 1.$$

Thus, $h(\alpha)$ is a monotonically decreasing function and

$$|f'(\alpha)/g(\alpha)| = \frac{t - 2\pi n(m+\alpha)}{(m+\alpha)^{2-\sigma}} \ge \frac{t}{2m^{2-\sigma}}.$$

From this, integration by parts, and [3, lemma 4.3] we get

$$\begin{aligned}
&\sum_{1\le m\le x}\sum_{1\le n\le z}\frac{1}{n^{1-\sigma+it}}\int_0^1 \frac{e^{2\pi i n\alpha}}{(m+\alpha)^s}\,d\alpha \\
&= \sum_{1\le m\le x, 1\le n\le z}\frac{1}{n^{1-\sigma+it}}\left[\left.\frac{(m+\alpha)^{1-s}}{1-s}e^{2\pi i n\alpha}\right|_0^1 \right.\\
&\qquad\left. - \frac{2\pi i n}{1-s}\int_0^1 \frac{e^{2\pi i n\alpha}}{(m+\alpha)^{s-1}}\,d\alpha\right] \\
&= \frac{1}{1-s}\sum_{1\le n\le z}\frac{1}{n^{1-\sigma+it}}\sum_{1\le m\le x}\left((m+1)^{1-s} - m^{1-s}\right) \\
&\quad - \frac{2\pi i}{1-s}\sum_{1\le n\le z} n^{\sigma-it}\sum_{1\le m\le x}\int_0^1 \frac{e^{2\pi i n\alpha}}{(m+\alpha)^{s-1}}\,d\alpha \\
&\ll x^{1-\sigma}z^{\sigma}t^{-1} + z^{1+\sigma}x^{3-\sigma}t^{-2}.
\end{aligned} \tag{15}$$

Using the approximate equation (2) repeatedly, and the method of estimation of $J$, we may get

$$\overline{A(\sigma+it)}\sum_{1\le m\le x}\sum_{z<n\le y}\frac{1}{n^{1-\sigma+it}}\int_0^1\frac{e^{2\pi i n\alpha}}{(m+\alpha)^s}\,d\alpha$$

$$=\sum_{1\le m\le x}\sum_{x\le n\le t/2\pi z}\int_0^1\frac{d\alpha}{(m+\alpha)^s(n+\alpha)^{\bar{s}}}$$

$$+O\left(x^{-\sigma}\ln t\int_0^1|I_1(s,\alpha)|\,d\alpha\right)$$

$$\ll t^{-\sigma}x^{2-\sigma}z^{\sigma-1}\ln t+x^{-\sigma}\ln t\int_0^1|I_1(s,\alpha)|\,d\alpha \tag{16}$$

Taking the parameter $z=t^{3/4}x^{-1/2}$, and provided $zx^2\ge t$, by (14), (15) and (16) we get

$$\int_0^1 I_1(s,\alpha)\overline{I_2(s,\alpha)}\,d\alpha\ll t^{-(3+\sigma)/4}x^{(5-3\sigma)/2}\ln t$$
$$+x^{-\sigma}\ln t\int_0^1|I_1(s,\alpha)|\,d\alpha \tag{17}$$

By (4), (5), (6), (12), (13) and the Cauchy inequality we have

$$\int_0^1|I_1(s,\alpha)|\,d\alpha\le\left(\int_0^1|I_1(s,\alpha)|^2\,d\alpha\right)^{1/2}\ll(x^{1/2-\sigma}+1)\ln^{1/2}t. \tag{18}$$

$$\int_0^1|I_2(s,\alpha)|\,d\alpha\ll\left(\int_0^1|I_2(s,\alpha)|^2\,d\alpha\right)^{1/2}\ll(t^{1/2-\sigma}+1)\ln^{1/2}t. \tag{19}$$

Combining (3), (4), (5), (6), (12), (13), (17), (18) and (19) we obtain

$$f\left(\frac{1}{2}+it\right)=\ln\left(\frac{t}{2\pi}\right)+\gamma+O(t^{-7/8}x^{7/4}\ln t)$$
$$+O(t^{-1}x^2\ln t)+O(x^{-1/2}\ln^{3/12}t). \tag{20}$$

If $1/2<\sigma<1$, then

$$f(\sigma+it)=\frac{1}{2\sigma-1}+\left(\frac{t}{2\pi}\right)^{1-2\sigma}\zeta(2-2\sigma)+O(x^{-\sigma}\ln^{3/2}t)$$
$$+O(x^{3-2\alpha}t^{-1}\ln t)+O(t^{-(3+\sigma)/4}x^{(5-3\sigma)/2}\ln t). \tag{21}$$

If $0 < \sigma < 1/2$, then

$$f(\sigma + it) = \left(\frac{t}{2\pi}\right)^{1-2\sigma} \zeta(2 - 2\sigma) - \frac{1}{1 - 2\sigma} + O(t^{-1}x^{3-2\sigma} \ln t) + O(t^{-(3+\sigma)/4} x^{(5-3\sigma)/2} \ln t) + O(x^{-\sigma} t^{1/2-\sigma} \ln^{3/2} t). \quad (22)$$

Taking the parameter $x = t^{7/18} \ln^{2/9} t$, $t^{(3+\sigma)/2(5-\sigma)}$, $t^{(5-3\sigma)/2(5-\sigma)}$ in (20), (21) and (22) respectively, we may obtain the asymptotic formula

$$f\left(\frac{1}{2} + it\right) = \ln\left(\frac{t}{2\pi}\right) + \gamma + O(t^{-7/36} \ln^{25/18} t).$$

If $0 < \sigma < 1/2$, then

$$f(\sigma + it) = \left(\frac{t}{2\pi}\right)^{1-2\sigma} \zeta(2 - 2\sigma) + \frac{1}{2\sigma - 1} + O(t^{(5+5\sigma^2-16\sigma)/2(5-\sigma)} \ln^{3/2} t).$$

If $1/2 < \sigma < 1$, then

$$f(\sigma + it) = \frac{1}{2\sigma - 1} + \left(\frac{t}{2\pi}\right)^{1-2\sigma} \zeta(2 - 2\sigma) + O(t^{-\sigma(3+\sigma)/2(5-\sigma)} \ln^{3/2} t).$$

This completes the proof of the theorems.

## References

1. J.F. Koksma and C.G. Lekkerkerker, *A mean-value theorem for $\zeta(s, \alpha)$*, Proc. Ser. A 55 (1952), 446–452.
2. V.V. Rane, *On Hurwitz zeta-function*, Math. Ann., vol. 264 (1983), 147–151.
3. E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford, 1951.

Northwest University
Xian, People's Republic of China