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Almost sure central limit theorems for stochastic wave equations*

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Abstract

In this paper, we study almost sure central limit theorems for the spatial average of the solution to the stochastic wave equation in dimension $d \le 2$ over a Euclidean ball, as the radius of the ball diverges to infinity. This equation is driven by a general Gaussian multiplicative noise, which is temporally white and colored in space including the cases of the spatial covariance given by a fractional noise, a Riesz kernel, and an integrable function that satisfies Dalang's condition.

Keywords: almost sure central limit theorem; stochastic wave equation; Malliavin calculus; Poincaré-type inequality.
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1 Introduction

We consider the following stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d, \\ u(0,x) = 1, \quad x \in \mathbb{R}^d, \\ \dfrac{\partial u}{\partial t}(0,x) = 0, \quad x\in \mathbb{R}^d, \end{cases} \tag{1.1}$$

for fixed $d \in \{1,2\}$, where Δ is the Laplacian operator in space variables and $\dot{W}$ is a centered Gaussian noise with covariance given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\gamma(x-y), \tag{1.2}$$

where γ is a (generalized) function that satisfies one of the following three conditions:

Hypothesis 1.1. $(d=1)$ *The Gaussian noise behaves as a fractional noise in space with Hurst parameter $H \in [1/2,1)$, that is, $\gamma(x) = |x|^{2H-2}$ for $H \in (1/2,1)$, and $\gamma(x) = \delta_0(x)$ for $H = 1/2$.*

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Hypothesis 1.2. $(d=2)$ *The Gaussian noise has a spatial covariance described by the Riesz kernel, that is,* $\gamma(x)=|x|^{-\beta}$, $\beta\in(0,2)$.

Hypothesis 1.3. $(d=1,2)$ γ *is a tempered nonnegative and nonnegative definite function, whose Fourier transform* μ *satisfies Dalang's condition:*

$$\int_{\mathbb{R}^d}\frac{\mu(\mathrm{d}z)}{1+|z|^2}<\infty, \tag{1.3}$$

where $|\cdot|$ *denotes the Euclidean norm on* $\mathbb{R}^d$*. Suppose also that* γ *satisfies* $\gamma\in L^1(\mathbb{R})$ *if* $d=1$ *and* $\gamma\in L^1\left(\mathbb{R}^2\right)\cap L^\ell\left(\mathbb{R}^2\right)$ *for some* $\ell>1$ *if* $d=2$.

Throughout the paper, we assume that $\sigma:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L\in(0,\infty)$ such that $\sigma(1)\neq 0$ to avoid triviality.

It is well-known (see Dalang [10]) that we can interpret the solution to (1.1) in the following mild form:

$$u(t,x)=1+\int_0^t\int_{\mathbb{R}^d}G_{t-s}(x-y)\sigma(u(s,y))W(\mathrm{d}s,\mathrm{d}y), \tag{1.4}$$

where the above stochastic integral is understood in the sense of Dalang-Walsh and $G_{t-s}(x-y)$ denotes the fundamental solution to the corresponding deterministic wave equation, that is,

$$G_t(x):=\begin{cases}\frac{1}{2}\mathbf{1}_{\{|x|<t\}}, & \text{if } d=1;\\ \frac{1}{2\pi\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}, & \text{if } d=2.\end{cases} \tag{1.5}$$

There have been extensive achievements devoted to limit theorems for spatial averages of stochastic partial differential equations (SPDEs for short). Huang et al. [16] were the first to consider the spatial integral of the solution to the nonlinear stochastic heat equation (SHE for short) driven by space-time white noise on $\mathbb{R}_+\times\mathbb{R}$ and they provided a quantitative central limit theorem (CLT for short) and a functional CLT.

Soon after, Delgado-Vences et al. [11] obtained a quantitative CLT and a functional CLT for spatial average of the one-dimensional stochastic wave equation (SWE for short) driven by a Gaussian multiplicative noise, which is white in time and behaves as a fractional noise in space with Hurst parameter $H\in[1/2,1)$. Bolaños Guerrero et al. [6] considered the case that $d=2$ and γ is given by a Riesz kernel, i.e., $\gamma(x)=|x|^{-\beta}$, $\beta\in(0,2)$. Fix $d\in\{1,2\}$, Nualart and Zheng [22] studied the d-dimensional SWE when the covariance kernel γ is integrable and satisfies Dalang's condition. Nualart and Zheng [21] investigated the spatial ergodicity for a class of SWEs with spatial dimension less than or equal to 3. After that, Ebina [12] considered the Gaussian fluctuations for nonlinear SWEs in dimension three, which was left open in Nualart and Zheng [21]. Balan et al. [3] studied the hyperbolic Anderson model driven by a space-time colored Gaussian homogeneous noise with spatial dimension $d\in\{1,2\}$. We refer to Balan et al. [2], Gu and Huang [13] and Huang and Khoshnevisan [15] for several other investigations on SPDEs.

Note that in the above-mentioned limit theorems for spatial averages of SWEs, almost sure central limit theorem (ASCLT for short) is missing. Hence a natural question is whether the ASCLT holds for spatial averages of SWEs.

ASCLT has been first investigated independently by Brosamler [7] and Schatte [25]. Since then ASCLTs have attracted a significant interest in this field. We refer to Peligrad and Shao [24], Tang and Zhang [26] and Zhang et al. [27] for the classical ASCLT.

Definition 1.4. *Let* $\{Y_n,\ n\geq 1\}$ *be a sequence of independent and identically distributed random variables with zero mean, and define* $T_n=\sum_{i=1}^n Y_i$*, we say that* $\{T_n\}$ *satisfies a* *classical ASCLT if for any* $x \in \mathbb{R}$*,*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{T_k}{\sqrt{\mathrm{Var}(T_k)}} \le x \right\} = \Phi(x) \quad \text{a.s.,} \tag{1.6}$$

where $I(\cdot)$ *denotes the indicator function and* $\Phi(\cdot)$ *is the distribution function of the standard normal random variable* $\mathcal{N}(0,1)$*.*

Notice that by the conclusions in Section 2 of Peligrad and Shao [24] (see also Lacey and Philipp [17]) and Theorem 7.1 of Billingsley [5], (1.6) is equivalent to

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \rho\left(\frac{T_k}{\sqrt{\mathrm{Var}(T_k)}} \right) = \mathbb{E}\rho(\mathcal{N}(0,1)) \quad \text{a.s.,} \tag{1.7}$$

where ρ is a bounded Lipschitz function. We refer to Azmoodeh and Nourdin [1], Hörmann [14] and Zhang [28] for detailed research on ASCLT.

Li and Zhang [18] have established the ASCLT for spatial average of the nonlinear SHE with flat initial data under Dalang's condition. After that, Li and Zhang [19] derived such ASCLT for the parabolic Anderson model with delta initial condition based on the quantitative analysis of a nonnegative-definite measure. Notice that they (ibid.) proved the ASCLTs by using the property of association for parabolic SPDEs to simplify some of the expressions for the joint correlations of spatial averages of the solution. However, to the authors' best knowledge, such a property for SWEs has not been discovered so far, and we have already overcome this lack of property of association in the proofs of the main results.

In order to state the main results, let us introduce

$$\mathcal{S}_{N,t} = \int_{B_N} (u(t,x)-1)\mathrm{d}x, \qquad \text{for all } N>0, \text{ fixed } t>0, \tag{1.8}$$

where $B_N = \left\{ x \in \mathbb{R}^d : |x| \le N \right\}$.

Theorem 1.5. *Suppose that* $\{d_k\}$ *is a sequence of positive numbers satisfying the following conditions:*

(C1) $\limsup_{k\to\infty} k d_k \left(\log D_k\right)^{\alpha} / D_k < \infty$ *for some* $\alpha > 1$*, where* $D_n = \sum_{k=1}^{n} d_k$*.*

(C2) $D_n \to \infty, D_n = o\left(n^{\epsilon}\right)$*, for any* $\epsilon > 0$*.*

Assume that γ *satisfies Hypothesis 1.1, then for any* $x \in \mathbb{R}$ *and fixed* $t > 0$*,*

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{\mathcal{S}_{k,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{k,t}}} \le x \right\} = \Phi(x) \quad \text{a.s.} \tag{1.9}$$

Theorem 1.6. *Suppose that* $\{d_k\}$ *satisfies the conditions in Theorem 1.5. Assume that* γ *satisfies Hypothesis 1.2, then for any* $x \in \mathbb{R}$ *and fixed* $t > 0$*, (1.9) holds.*

Theorem 1.7. *Suppose that* $\{d_k\}$ *satisfies the conditions in Theorem 1.5. Assume that* γ *satisfies Hypothesis 1.3, then for any* $x \in \mathbb{R}$ *and fixed* $t > 0$*, (1.9) holds.*

Remark 1.8. Let $c_n > 0$ with $c_n \uparrow \infty$ and $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1$. Suppose also $\frac{k}{l} \le \left(\frac{c_k}{c_l}\right)^r$ for some constant $r > 0$ and $k < l$. Denote

$$d_k = \log \frac{c_{k+1}}{c_k} \exp\left(\log^{\beta} c_k\right), \quad D_n = \sum_{k=1}^{n} d_k, \quad 0 \le \beta < 1/2.$$

Then under the assumptions of Theorem 1.5, (1.9) holds.

Remark 1.9. If the assumption (C1) of Theorem 1.5 is satisfied for some sequence $\{D_n\}$, then it is also satisfied for any other sequence $D_n^* = \Psi(D_n)$ provided $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable, $\Psi'(x) = O(\Psi(x)/x)$ and $\log \Psi'(x)$ is uniformly continuous on (C, ∞) for some $C > 0$. Typical examples are $\Psi(x) = x^r, \Psi(x) = (\log x)^r, \Psi(x) = (\log\log x)^r$ with some suitable $r > 0$.

Remark 1.10. Notice that $d_k = l(k)/k$, where $l(x)$ is slowly varying at infinity and $D_n \to \infty$, satisfies the conditions (C1) and (C2). So representative examples including $d_k = 1/k$; $d_k = \log^\theta k/k, \theta > -1$ (see Peligrad and Révész [23]); $d_k = \exp(\log^r k)/k, 0 \leq r < 1/2, 1 < \alpha < (1-r)/r$ (see Berkes and Csáki [4]).

The rest of the paper is organized as follows. In Section 2, we recall the basic elements of Malliavin calculus. We also present some CLTs for the SWEs. And our main results will be proved in Section 3. Throughout the paper, C represents a positive constant although its value may change from one appearance to the next. And for every $Z \in L^k(\Omega)$, we write $\|Z\|_k$ instead of $\{\mathbb{E}(|Z|^k)\}^{1/k}$. $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$.

2 Preliminaries

2.1 Malliavin calculus

In this subsection, we present some basic elements of Malliavin calculus and recall the Poincaré-type inequality, which is fundamental in the proof of our results.

Define $\mathfrak{H}$ as the completion of $C_c\left(\mathbb{R}_+ \times \mathbb{R}^d\right)$ under the inner product

$$\langle f, g\rangle_{\mathfrak{H}} = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} f(s,y)g(s,z)\gamma(y-z)\mathrm{d}y\mathrm{d}z\mathrm{d}s = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^d} \mathscr{F}f(s,\xi)\mathscr{F}g(s,-\xi)\mu(\mathrm{d}\xi)\right)\mathrm{d}s,$$

where $\mathscr{F}f(s,\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(s,x)\mathrm{d}x$. The Gaussian random field $\{W(h)\}_{h\in\mathfrak{H}}$, formed by the Wiener integrals

$$W(h) = \int_0^\infty \int_{\mathbb{R}^d} h(t,x)W(\mathrm{d}t,\mathrm{d}x),$$

defines an isonormal Gaussian process on the Hilbert space $\mathfrak{H}$ such that $\mathbb{E}[W(\phi)W(\psi)] = \langle \phi, \psi\rangle_{\mathfrak{H}}$ for any $\phi, \psi \in \mathfrak{H}$. Notice that the noise W is white in time, hence, a martingale structure naturally appears. Define $\mathcal{F}_t$ to be the σ-algebra generated by $\mathbb{P}$-negligible sets and $\left\{W(\phi) : \phi \in C\left(\mathbb{R}_+ \times \mathbb{R}^d\right) \text{ has compact support contained in } [0,t]\times\mathbb{R}^d\right\}$, so we have a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathbb{R}_+\}$.

Denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathcal{S}$ be the space of smooth and cylindrical random variables of the form

$$F = f\left(W(h_1), \ldots, W(h_n)\right),$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}, 1 \leq i \leq n$. Then the Malliavin derivative $\mathbf{D}F$ is the $\mathfrak{H}$-valued random variable defined by

$$\mathbf{D}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(W(h_1), \ldots, W(h_n)\right) h_i. \tag{2.1}$$

The derivative operator $\mathbf{D}$ is a closable operator from $L^p(\Omega)$ to $L^p(\Omega;\mathfrak{H})$ for any $p \geq 1$. The Gaussian Sobolev space $\mathbb{D}^{1,p}$ is defined by the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}|F|^p + \mathbb{E}\|\mathbf{D}F\|_{\mathfrak{H}}^p\right)^{1/p}.$$

When $F \in \mathbb{D}^{1,p}$ and $\mathbf{D}F$ is a random function valued in $\mathfrak{H}$, we write this function as $\mathbf{D}_{t,x}F$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. For instance, if $F = f\left(W\left(\varphi_1\right), \dots, W\left(\varphi_n\right)\right)$ for some $f \in C_{\mathrm{p}}^{\infty}\left(\mathbb{R}^n\right)$ and $\varphi_i \in C_0^{\infty}\left(\mathbb{R}_+ \times \mathbb{R}^d\right), i = 1, 2, \dots, n$, then $\mathbf{D}F$ is a random function and

$$\mathbf{D}_{t,x}F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W\left(\varphi_1\right), \dots, W\left(\varphi_n\right)\right)\varphi_i(t,x). \tag{2.2}$$

The divergence operator δ is the adjoint of the derivative operator $\mathbf{D}$ given by the duality formula

$$\mathbb{E}(\delta(v)F) = \mathbb{E}\left(\langle v, \mathbf{D}F\rangle_{\mathfrak{H}}\right)$$

for any $F \in \mathbb{D}^{1,2}$, and every $v \in \mathrm{Dom}\ \delta$, where $\mathrm{Dom}\ \delta$ is the domain of δ in $L^2(\Omega; \mathfrak{H})$. The operator δ is also called the Skorohod integral, that is,

$$\delta(F) = \int_0^{\infty} \int_{\mathbb{R}^d} F(t,x) W(\mathrm{d}t, \mathrm{d}x), \tag{2.3}$$

when F is a predictable and square-integrable random field. We refer to Nualart [20] for a detailed account on the Malliavin calculus.

As a consequence, the mild solution (1.4) can also be written as

$$u(t,x) = 1 + \delta\left(G_{t-\cdot}(x-*)u(\cdot,*)\right). \tag{2.4}$$

It is known that the solution $u(t,x)$ of (1.1) belongs to $\mathbb{D}^{1,p}$ for any $p \geq 2$ and any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. The derivative can be written as

$$\begin{aligned} \mathbf{D}_{s,y}u(t,x) =& G_{t-s}(x-y)\sigma(u(s,y)) \\ &+ \int_s^t \int_{\mathbb{R}^d} G_{t-r}(x-z)\Sigma(r,z)\mathbf{D}_{s,y}u(r,z)W(\mathrm{d}r, \mathrm{d}z), \end{aligned} \tag{2.5}$$

for $t \geq s$, where $\Sigma(r,z)$ is an adapted process, bounded by the Lipschitz constant of σ. If σ is continuously differentiable, then $\Sigma(r,z) = \sigma'(u(r,z))$. According to Lemma 2.2 in Delgado-Vences et al. [11], Theorem 1.3 in Bolaños Guerrero et al. [6] and Theorem 1.4 in Nualart and Zheng [22], we can get the following Lemma.

Lemma 2.1. *Assume that γ satisfies Hypothesis 1.1 or Hypothesis 1.2 or Hypothesis 1.3. For any $k \in [2,\infty)$ and any $t > 0$, the following estimate holds for almost every $(s,y) \in [0,t] \times \mathbb{R}^d$:*

$$\left\|\mathbf{D}_{s,y}u(t,x)\right\|_k \leq C G_{t-s}(x-y). \tag{2.6}$$

We recall the following Clark-Ocone formula (see Proposition 6.3 in Chen et al. [8]):

$$F = \mathbb{E}(F) + \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbb{E}(\mathbf{D}_{r,z}F \mid \mathcal{F}_r) W(\mathrm{d}r, \mathrm{d}z) \quad \text{a.s.} \tag{2.7}$$

valid for every random variable F in the Gaussian Sobolev space $\mathbb{D}^{1,2}$. As a consequence of Clark-Ocone formula, we can derive the following Poincaré-type inequality:

$$|\operatorname{Cov}(F,G)| \leq \int_0^{\infty} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}y' \left\|\mathbf{D}_{s,y}F\right\|_2 \left\|\mathbf{D}_{s,y'}G\right\|_2 \gamma(y-y'), \tag{2.8}$$

for all $F, G \in \mathbb{D}^{1,2}$.

2.2 Central limit theorems for the stochastic wave equations

Recall that the total variation distance between two random variables F and G is defined by

$$d_{\mathrm{TV}}(F,G) := \sup_{B\in\mathcal{B}(\mathbb{R})} |P(F\in B) - P(G\in B)|,$$

where $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$.

Lemma 2.2 (Delgado-Vences et al. [11])**.** *Assume that γ satisfies Hypothesis 1.1, then for all fixed $t>0$,*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{N,t}}}, \mathcal{N}(0,1)\right) \le CN^{H-1}.$$

Moreover, if $H=1/2$, $\operatorname{Var}(\mathcal{S}_{N,t}) \sim CN$ as $N\to\infty$; if $H\in(1/2,1)$, $\operatorname{Var}(\mathcal{S}_{N,t}) \sim CN^{2H}$ as $N\to\infty$.

Lemma 2.3 (Bolaños Guerrero et al. [6])**.** *Assume that γ satisfies Hypothesis 1.2, then for all fixed $t>0$,*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{N,t}}}, \mathcal{N}(0,1)\right) \le CN^{-\beta/2}.$$

Moreover, $\operatorname{Var}(\mathcal{S}_{N,t}) \sim CN^{4-\beta}$ as $N\to\infty$.

Lemma 2.4 (Nualart and Zheng [22])**.** *Assume that γ satisfies Hypothesis 1.3, then for all fixed $t>0$,*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{N,t}}}, \mathcal{N}(0,1)\right) \le CN^{-d/2}.$$

Moreover, $\operatorname{Var}(\mathcal{S}_{N,t}) \sim CN^{d}$ as $N\to\infty$.

3 Proofs

The following lemmas are useful for the proofs of main results.

Lemma 3.1 (Zhang [28])**.** *Let $\{\zeta_n, n\ge 1\}$ be a sequence of uniformly bounded random variables and $\{d_k\},\{D_n\}$ be defined as in Theorem 1.5. If there exist constants $C>0$ and $\delta>0$ and a sequence of positive numbers $\{a(k)\}$ such that $\sum_{n=1}^{\infty} a(n)<\infty$ and*

$$\mathbb{E}\,|\zeta_k\zeta_j| \le C\left((k/j)^{\delta} + a(k)\right), \quad \textit{for} \quad j/k > b_n = (\log D_n)^{\alpha/\delta},$$

then

$$\lim_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^{n} d_k\zeta_k = 0 \quad \textit{a.s.}$$

Lemma 3.2. *Assume that γ satisfies Hypothesis 1.1 or Hypothesis 1.2 or Hypothesis 1.3. Set*

$$\xi_{N,t}(s,y) = \int_{B_N} G_{t-s}(x-y)\mathrm{d}x.$$

For any $0<s<t$, there exists some constant C such that

$$\|\mathbf{D}_{s,y}\mathcal{S}_{N,t}\|_2 \le C\xi_{N,t}(s,y).$$

Proof. Recalling the definition of $\mathcal{S}_{N,t}$ and applying the stochastic Fubini theorem (see Da Prato and Zabczyk [9]), we can get

$$\begin{aligned}\mathcal{S}_{N,t} &= \int_{B_N} (u(t,x)-1)\mathrm{d}x \\ &= \int_{B_N}\int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))W(\mathrm{d}s,\mathrm{d}y)\mathrm{d}x \\ &= \int_0^t\int_{\mathbb{R}^d}\left(\int_{B_N} G_{t-s}(x-y)\sigma(u(s,y))\mathrm{d}x\right)W(\mathrm{d}s,\mathrm{d}y).\end{aligned}$$

As a consequence, taking into account (2.4), we have, for any fixed $t\geq 0, \mathcal{S}_{N,t}=\delta\left(v_N\right)$, where

$$v_N(s,y)=\mathbf{1}_{[0,t]}(s)\int_{B_N} G_{t-s}(x-y)\sigma(u(s,y))\mathrm{d}x,$$

which means $\mathcal{S}_{N,t}\in\mathbb{D}^{1,2}$. Moreover,

$$\mathbf{D}_{s,y}\mathcal{S}_{N,t}=\mathbf{1}_{[0,t]}(s)\int_{B_N}\mathbf{D}_{s,y}u(t,x)\mathrm{d}x.$$

Hence, when $0<s<t$, by using basic property of Malliavin derivative, Minkowski inequality and Lemma 2.1, we can get that

$$\begin{aligned}\|\mathbf{D}_{s,y}\mathcal{S}_{N,t}\|_2 &= \left\|\int_{B_N}\mathbf{D}_{s,y}u(t,x)\mathrm{d}x\right\|_2 \\ &\leq \int_{B_N}\|\mathbf{D}_{s,y}u(t,x)\|_2\,\mathrm{d}x \\ &\leq C\int_{B_N} G_{t-s}(x-y)\mathrm{d}x = C\xi_{N,t}(s,y).\end{aligned}$$

Thus, we complete the proof. □

Lemma 3.3. *Assume that γ satisfies Hypothesis 1.1, for every bounded Lipschitz function ρ and fixed $t>0$, we have*

$$\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right|\leq C\left(\frac{i}{j}\right)^{1-H},\qquad 1\leq i<j.$$

Proof. Suppose that ρ is a Lipschitz function bounded by K and has a derivative bounded by Γ.

Case 1: $H=\frac{1}{2}$. by the Poincaré-type inequality, the chain rule (see Proposition 1.2.4 in Nualart [20]) and Lemma 3.2, we have

$$\begin{aligned}&\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right| \\ \leq&\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\left\|\mathbf{D}_{s,y}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right)\right)\right\|_2\left\|\mathbf{D}_{s,y}\left(\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right\|_2 \\ \leq&\frac{\Gamma^2}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\|\mathbf{D}_{s,y}\mathcal{S}_{i,t}\|_2\,\|\mathbf{D}_{s,y}\mathcal{S}_{j,t}\|_2 \\ \leq&\frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\xi_{i,t}(s,y)\xi_{j,t}(s,y).\end{aligned}$$

Taking into account that $2\xi_{N,t}(s,y)$ is the length of $[-N,N]\cap[y-t+s,y+t-s]$ when $d=1$, so we have that

$$\xi_{N,t}(s,y)=0,\text{ if } |y|\geq N+t-s;\text{ and } \xi_{N,t}(s,y)\leq N\wedge(t-s). \tag{3.1}$$

Notice that we can obtain $\operatorname{Var}\mathcal{S}_{N,t}\sim CN$ as $N\to\infty$ by Lemma 2.2, therefore, for sufficiently large i,j and fixed $t>0$, we can get

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\right)\right)\right|\\
\leq&\frac{C}{\sqrt{i}\sqrt{j}}\int_0^t\int_{-i-t+s}^{i+t-s}(t-s)^2\mathrm{d}y\mathrm{d}s\\
\leq&C\frac{1}{\sqrt{i}}\frac{1}{\sqrt{j}}i\leq C\left(\frac{i}{j}\right)^{\frac{1}{2}}.
\end{aligned}$$

Case 2: $H\in(\frac{1}{2},1)$. by the Poincaré-type inequality, the chain rule (see Proposition 1.2.4 in Nualart [20]), Lemma 3.2 and (3.1), we have

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\right)\right)\right|\\
\leq&\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}y'\left\|\mathbf{D}_{s,y}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{i,t}}}\right)\right)\right\|_2\left\|\mathbf{D}_{s,y'}\left(\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\right)\right)\right\|_2|y-y'|^{2H-2}\\
\leq&\frac{\Gamma^2}{\sqrt{\operatorname{Var}\mathcal{S}_{i,t}}\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}y'\left\|\mathbf{D}_{s,y}\mathcal{S}_{i,t}\right\|_2\left\|\mathbf{D}_{s,y'}\mathcal{S}_{j,t}\right\|_2|y-y'|^{2H-2} \qquad (3.2)\\
\leq&\frac{C}{\sqrt{\operatorname{Var}\mathcal{S}_{i,t}}\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}y'\xi_{i,t}(s,y)\xi_{j,t}(s,y')|y-y'|^{2H-2}\\
\leq&\frac{C}{\sqrt{\operatorname{Var}\mathcal{S}_{i,t}}\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{-i-t+s}^{i+t-s}\mathrm{d}y\int_{-j-t+s}^{j+t-s}\mathrm{d}y'(t-s)^2|y-y'|^{2H-2}.
\end{aligned}$$

By the fact that

$$\sup_{x\in\mathbb{R}}\int_{[-1,1]}|y-x|^{2H-2}\mathrm{d}y<\infty, \tag{3.3}$$

we can get

$$\begin{aligned}
&\int_{-i-t+s}^{i+t-s}\int_{-j-t+s}^{j+t-s}|y'-y|^{2H-2}\mathrm{d}y'\mathrm{d}y\\
=&(j+t-s)^{2H-1}\int_{-i-t+s}^{i+t-s}\int_{-1}^{1}\left|\tilde{y}-\frac{y}{j+t-s}\right|^{2H-2}\mathrm{d}\tilde{y}\mathrm{d}y\\
\leq&C(i+t-s)(j+t-s)^{2H-1}\sup_{x\in\mathbb{R}}\int_{[-1,1]}|\tilde{y}-x|^{2H-2}\mathrm{d}\tilde{y}\\
\leq&C(i+t-s)(j+t-s)^{2H-1}.
\end{aligned}$$

Notice that we can obtain $\operatorname{Var}\mathcal{S}_{N,t}\sim CN^{2H}$ as $N\to\infty$ by Lemma 2.2, therefore, for sufficiently large i, j and fixed $t > 0$, we can get

$$\begin{aligned}&\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right), \rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right| \\ \leq & \frac{C}{i^H j^H}\int_0^t (t-s)^2(i+t-s)(j+t-s)^{2H-1}\mathrm{d}s \\ \leq & C\frac{i}{i^H}\frac{j^{2H-1}}{j^H} \leq C\left(\frac{i}{j}\right)^{1-H}.\end{aligned}$$

□

Lemma 3.4. *Assume that γ satisfies Hypothesis 1.2, for every bounded Lipschitz function ρ and fixed $t > 0$, we have*

$$\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right), \rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right| \leq C\left(\frac{i}{j}\right)^{\beta/2}, \qquad 1 \leq i < j.$$

Proof. By Lemma 2.1 in Bolaños Guerrero et al. [6], for $0 < s < t$, we have

$$\xi_{N,t}(s,y) \leq (t-s)\mathbf{1}_{\{|y|\leq N+t\}}. \tag{3.4}$$

Similar to (3.2), we can get

$$\begin{aligned}&\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right), \rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right| \\ \leq & \frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\int_0^t \mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}y\int_{\mathbb{R}^2}\mathrm{d}y'\xi_{i,t}(s,y)\xi_{j,t}(s,y')|y-y'|^{-\beta} \\ \leq & \frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\int_0^t \mathrm{d}s\int_{|y|\leq i+t}\mathrm{d}y\int_{|y'|\leq j+t}\mathrm{d}y'(t-s)^2|y-y'|^{-\beta}.\end{aligned}$$

By the fact that

$$\sup_{x\in\mathbb{R}^2}\int_{B_2}|y-x|^{-\beta}\mathrm{d}y < \infty, \tag{3.5}$$

we can get

$$\begin{aligned}&\int_{|y|\leq i+t}\int_{|y'|\leq j+t}|y'-y|^{-\beta}\mathrm{d}y'\mathrm{d}y \\ = & \left(\frac{j+t}{2}\right)^{2-\beta}\int_{|y|\leq i+t}\int_{|\tilde{y}|\leq 2}\left|\tilde{y}-\frac{2y}{j+t}\right|^{-\beta}\mathrm{d}\tilde{y}\mathrm{d}y \\ \leq & C(i+t)^2(j+t)^{2-\beta}\sup_{x\in\mathbb{R}^2}\int_{B_2}|\tilde{y}-x|^{-\beta}\,\mathrm{d}\tilde{y} \\ \leq & C(i+t)^2(j+t)^{2-\beta}.\end{aligned}$$

Notice that we can obtain $\mathrm{Var}\,\mathcal{S}_{N,t} \sim CN^{4-\beta}$ as $N \to \infty$ by Lemma 2.3, therefore, for sufficiently large i, j and fixed $t > 0$, we can get

$$\begin{aligned}&\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right), \rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right| \\ \leq & \frac{C}{i^{2-\beta/2}j^{2-\beta/2}}\int_0^t (t-s)^2(i+t)^2(j+t)^{2-\beta}\mathrm{d}s \\ \leq & C\frac{i^2}{i^{2-\beta/2}}\frac{j^{2-\beta}}{j^{2-\beta/2}} \leq C\left(\frac{i}{j}\right)^{\beta/2}.\end{aligned}$$

□

Lemma 3.5. *Assume that γ satisfies Hypothesis 1.3, for every bounded Lipschitz function ρ and fixed $t>0$, we have*

$$\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right|\le C\left(\frac{i}{j}\right)^{d/2},\qquad 1\le i<j.$$

Proof. According to the fact that (see (2.8) and (2.9) in Nualart and Zheng [22])

$$\xi_{N,t}(s,y)\le\int_{\mathbb{R}^d}G_{t-s}(x-y)\mathrm{d}x=t-s,\tag{3.6}$$

$$\int_{\mathbb{R}^d}\xi_{N,t}(s,y)\mathrm{d}y=\int_{B_N}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}yG_{t-s}(x-y)\le C(t-s)N^d,\tag{3.7}$$

for $0<s<t$, similar to (3.2), we can get

$$\begin{aligned}&\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right|\\
\le&\frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}y'\xi_{i,t}(s,y)\xi_{j,t}(s,y')\gamma(y-y')\\
\le&\frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}y'(t-s)\xi_{i,t}(s,y)\gamma(y-y')\\
\le&\frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\|\gamma\|_{L^1(\mathbb{R}^d)}\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y(t-s)\xi_{i,t}(s,y)\\
\le&\frac{C}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}i^d\|\gamma\|_{L^1(\mathbb{R}^d)}\int_0^t(t-s)^2\mathrm{d}s.\end{aligned}$$

Notice that we can obtain $\mathrm{Var}\,\mathcal{S}_{N,t}\sim CN^d$ as $N\to\infty$ by Lemma 2.4, therefore, for sufficiently large i,j and fixed $t>0$, we can get

$$\left|\mathrm{Cov}\left(\rho\left(\frac{\mathcal{S}_{i,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{i,t}}}\right),\rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{j,t}}}\right)\right)\right|\le C\frac{i^d}{i^{d/2}}\frac{1}{j^{d/2}}\le C\left(\frac{i}{j}\right)^{d/2}\square$$

Proof of Theorem 1.5. According to Lemma 2.2, for fixed $t>0$, we know that

$$\frac{\mathcal{S}_{k,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{k,t}}}\xrightarrow{\mathrm{d}}\mathcal{N}(0,1),\quad\text{ as } k\to\infty.\tag{3.8}$$

Recall that ρ is a bounded Lipschitz function. Then by (3.8), we have

$$\mathbb{E}\rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{k,t}}}\right)\to\mathbb{E}\rho(\mathcal{N}(0,1)),\quad\text{ as } k\to\infty.\tag{3.9}$$

On the other hand, notice that (1.9) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k\rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{k,t}}}\right)=\mathbb{E}\rho(\mathcal{N}(0,1))\quad\text{a.s.},\tag{3.10}$$

by the conclusions in Section 2 of Peligrad and Shao [24] (see also Lacey and Philipp [17]) and Theorem 7.1 of Billingsley [5]. Hence, to prove (1.9), it suffices to show

$$\frac{1}{D_n}\sum_{k=1}^n d_k\left(\rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{k,t}}}\right)-\mathbb{E}\rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{k,t}}}\right)\right)\to 0,\quad\text{a.s.}\quad n\to\infty.\tag{3.11}$$

For convenience, let $\zeta_k = \rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{k,t}}}\right) - \mathbb{E}\rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{k,t}}}\right)$, according to Lemma 3.3, we conclude that for $1 \le k < j \le n$,

$$|\mathbb{E}\zeta_k\zeta_j| = \left|\operatorname{Cov}\left(\rho\left(\frac{\mathcal{S}_{k,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{k,t}}}\right), \rho\left(\frac{\mathcal{S}_{j,t}}{\sqrt{\operatorname{Var}\mathcal{S}_{j,t}}}\right)\right)\right| \le C\left(\frac{k}{j}\right)^{1-H}. \tag{3.12}$$

Then by Lemma 3.1 with $\delta = 1 - H$ and $a(k) \equiv 0$, we can obtain (3.11), therefore the proof of (1.9) is completed. □

Proof of Theorem 1.6. By Lemma 2.3, Lemma 3.1 and Lemma 3.4, the proof is similar to that of Theorem 1.5, so we omit it here. □

Proof of Theorem 1.7. By Lemma 2.4, Lemma 3.1 and Lemma 3.5, the proof is similar to that of Theorem 1.5, so we omit it here. □

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