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Research Article

Numerical Simulation of Fractional Fornberg-Whitham Equation by Differential Transformation Method

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An approximate analytical solution of fractional Fornberg-Whitham equation was obtained with the help of the two-dimensional differential transformation method (DTM). It is indicated that the solutions obtained by the two-dimensional DTM are reliable and present an effective method for strongly nonlinear partial equations. Exact solutions can also be obtained from the known forms of the series solutions.

1. Introduction

A homogeneous nonlinear fractional Fornberg-Whitham equation [1] is considered as in the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - u_{xxt} + u_x = u u_{xxx} - u u_x + 3 u_x u_{xx}, \quad t > 0,\ 0 < \alpha \le 1, \tag{1.1}$$

with boundary conditions and

$$u(0,t) = f_1(t), \qquad u_x(0,t) = f_2(t), \tag{1.2}$$

with initial conditions

$$u(x,0) = f_3(x), \qquad u_t(x,0) = f_4(x), \tag{1.3}$$

where $u(x,t)$ is the fluid velocity, α is constant and lies in the interval $(0,1]$, t is the time and x is the spatial coordinate.

Subscripts denote the partial differentiation unless stated otherwise. Fornberg and Whitham obtained a peaked solution of the form $u(x,t) = A\exp((-1/2)|x-4t/3|)$, where A is an arbitrary constant. In recent years, considerable interest in fractional calculus used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, accustics, material science, and signal processing can be successfully modelled by linear or nonlinear fractional order differential equations [2–8].

See fractional diffusion equation with absorbent term and external force by Das and Gupta [9], fractional convection-diffusion equation with nonlinear source term by Momani and Yıldırım [10], space-time fractional advection-dispersion equation by Yıldırım and Koçak [11], fractional Zakharov-Kuznetsov equations by Yıldırım and Gülkanat [12], boundary value problems by He [13], integro-differential equation by El-Shahed [14], non-Newtonian flow by Siddiqui et al. [15], fractional PDEs in fluid mechanics by Yıldırım [16], fractional Schrödinger equation [17, 18] and nonlinear fractional predator-prey model [19] by HPM, linear PDEs of fractional order by He [20], Momani, and Odibat [21], and so forth. In 2009, Tian and Gao [22] studied the proof of the existence of the attractor for the one-dimensional viscous Fornberg-Whitham equation. Abidi and Omrani [23] have solved the Fornberg-Whitham equation by the homotopy analysis method. Recently, Gupta and Singh [24] have used homotopy perturbation method to numerical solution of fractional Fornberg-Whitham Equation.

The goal of this paper is to extend the two-dimensional differential transform method to solve fractional Fornberg-Whitham equation.

This paper is organized as follows.

In Section 2, we are giving definitions related to the fractional calculus theory briefly. To show in efficiency of this method, we give the implementation of the DTM for the Fornberg-Whitham equation and numerical results in Sections 3 and 4. The conclusions are then given in the final Section 5.

2. Basic Definitions

Here are some basic definitions and properties of the fractional calculus theory which can be found in [5, 6, 25, 26].

Definition 2.1. A real function $f(x),\ x>0$, in the space C_μ, $\mu \in R$ if there exists a real number $p>\mu$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$ and it is said to be in the space if $f^{(m)}\in C_\mu,\ m\in N$.

Definition 2.2. The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f\in C_\mu,\ \mu\geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \text{for } \alpha>0,\ x>0 \text{ and } J^0 f(x)=f(x). \tag{2.1}$$

The properties of the operator J^α can be found in Jang et al. [25].

Table 1: Operations of the two-dimensional differential transform.

Original function	Transformed function
$u(x,y) = f(x,y) \mp g(x,y)$	$U_{\alpha,\beta}(k,h) = F_{\alpha,\beta}(k,h) \mp G(k,h)$
$u(x,y) = \xi f(x,y)$	$U_{\alpha,\beta}(k,h) = \xi F_{\alpha,\beta}(k,h)$
$u(x,y) = \partial f(x,y)/\partial x$	$U_{\alpha,\beta}(k,h) = (k+1)F(k+1,h)$
$u(x,y) = D^{\alpha}_{*x_0} f(x,y), 0 < \alpha \le 1$	$U_{\alpha,\beta}(k,h) = (\Gamma(\alpha(k+1)+1)/\Gamma(\alpha k+1))F_{\alpha,\beta}(k+1,h)$
$u(x,y) = D^{\alpha}_{*y_0} f(x,y), 0 < \alpha \le 1$	$U_{\alpha,\beta}(k,h) = (\Gamma(\alpha(h+1)+1)/\Gamma(\alpha h+1))F_{\alpha,\beta}(k,h+1)$
$u(x,y) = (x-x_0)^{m\alpha}(y-y_0)^{n\beta}$	$U_{\alpha,\beta}(k,h) = \delta(k-m,h-n) = \begin{cases} 1, & k=r, h=s \\ 0, & \text{otherwise} \end{cases}$
$u(x,y) = f(x,y)g(x,y)$	$U_{\alpha,\beta}(k,h) = \sum_{m=0}^{k}\sum_{n=0}^{h} F_{\alpha,\beta}(m,h-n)G_{\alpha,\beta}(k-m,n)$
$u(x,y) = f(x,y)g(x,y)h(x,y)$	$U_{\alpha,\beta}(k,h) = \sum_{k_4=0}^{k}\sum_{k_3=0}^{k-k_4}\sum_{k_2=0}^{h}\sum_{k_1=0}^{h-k_2} F_{\alpha,\beta}(k_4,h-k_2-k_1)$ $G_{\alpha,\beta}(k_3,k_2)H_{\alpha,\beta}(k-k_4-k_3,k_1)$

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo [6] sense is defined as

$$D^{\alpha}_{*} f(x) = J^{(m-\alpha)} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{(m-\alpha-1)} f^{(m)}(t)dt, \quad \text{for } m-1 < \alpha < m,\ m \in N,\ x > 0,\ f \in C^{n}_{-1}. \tag{2.2}$$

The unknown function $f = f(x,t)$ is assumed to be a casual function of fractional derivatives (i.e., vanishing for $\alpha < 0$) taken in Caputo sense as follows.

Definition 2.4. For m as the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D^{\alpha}_{*t} f(x,t) = \frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m f(x,\tau)}{\partial t^m} d\tau, & m-1 < \alpha < m, \\ \dfrac{\partial^m f(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases} \tag{2.3}$$

3. Two-Dimensional Differential Transformation Method

DTM is an analytic method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The method is well addressed by Odibat and Momani [26]. The proposed method is based on the combination of the classical two-dimensional DTM and generalized Taylor's Table 1 formula. Consider a function of two variables $u(x,y)$ and suppose that it can be represented as a product of two single-variable functions, that is, $u(x,y) = f(x)g(y)$. The basic definitions and fundamental operations of the two-dimensional differential transform of the function are expressed as follows [25–38]. Two-dimensional differential transform of $u(x,y)$ can be represented as:

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(k)(y-y_0)^{h\beta} \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}, \end{aligned} \tag{3.1}$$

where $0 < \alpha,\ \beta \leq 1, U_{\alpha,\beta}(k,h) = F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of $u(x,y)$. The generalized two-dimensional differential transform of the function $u(x,y)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^{k}\left(D^{\beta}_{*y_0}\right)^{h} u(x,y)\right]_{(x_0,y_0)}, \tag{3.2}$$

where $(D^{\alpha}_{*x_0})^{k} = \underbrace{D^{\alpha}_{*x_0} D^{\alpha}_{*x_0} \cdots D^{\alpha}_{*x_0}}_{k}$.

In case of $\alpha = 1$, and $\beta = 1$, the generalized two-dimensional differential transform (3.2) reduces to the classical two-dimensional differential transform.

From the above definitions, it can be found that the concept of two-dimensional differential transform is derived from two-dimensional differential transform which is obtained from two-dimensional Taylor series expansion.

4. The DTM Applied to Fractional Fornberg-Whitham Equation

In this section, we will research the solution of fractional Fornberg-Whitham equation, which has been widely examined in the literature. We described the implementation of the DTM for the fractional Fornberg-Whitham equation in detail. To solve (1.1)–(1.3), according to DTM, (1.2)-(1.3) with initial condition become

$$u(x,0) = e^{x/2}, \qquad u_t(x,0) = -\frac{2}{3}e^{x/2}, \tag{4.1}$$

with boundary conditions

$$u(0,t) = e^{-2t/3}, \qquad u_x(0,t) = \frac{1}{2}e^{-2t/3}. \tag{4.2}$$

Applying the differential transform of (1.1), (4.1), and (4.2), then

$$\frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)}U_{\alpha,1}(k,h+1)-(k+1)(k+2)(h+1)U_{\alpha,1}(k+2,h+1)+(k+1)U_{\alpha,1}(k+1,h)$$
$$-\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)(k-r+3)U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r+3,s)$$
$$+\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r+1,s)$$
$$-3\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)(r+1)U_{\alpha,1}(r+1,h-s)U_{\alpha,1}(k-r+2,s)=0. \tag{4.3}$$

$$U_{\alpha,1}(k,0)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{2}\right)^{k},\qquad U_{\alpha,1}(k,1)=-\frac{2}{3}U_{\alpha,1}(k,0),$$
$$U_{\alpha,1}(0,h)=\sum_{h=0}^{\infty}\frac{1}{h!}\left(-\frac{2t}{3}\right)^{h},\qquad U_{\alpha,1}(1,h)=\frac{1}{2}U_{\alpha,1}(0,h). \tag{4.4}$$

Substituting (4.3) into (4.4), we obtain the closed form solution as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,1}(k,h)x^{k}t^{h\alpha}$$
$$=\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\cdots\right)\left(1+\frac{(-2t/3)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(-2t/3)^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(-2t/3)^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right). \tag{4.5}$$

As $\alpha=1$, this series has the closed form $e^{(x/2-2t/3)}$, which is an exact solution of the classical gas dynamics equation.

The graphs of exact and DTM solutions belonging to examples examined above are shown in Figure 1. It can be deduced that DTM solution corresponds to the exact solutions.

Both the exact results and the approximate solutions obtained for the DTM approximations are plotted in Figure 1. There are no visible differences in the two solutions of each pair of diagrams.

5. Conclusions

In this paper, the applicability of the fractional differential transformation method to the solution of fractional Fornberg-Whitham equation with a number of initial and boundary values has been proved. DTM can be applied to many complicated linear and strongly nonlinear partial differential equations and does not require linearization, discretization, or perturbation. The obtained results indicate that this method is powerful and meaningful for solving the nonlinear fractional Fornberg-Whitham type differential equations.

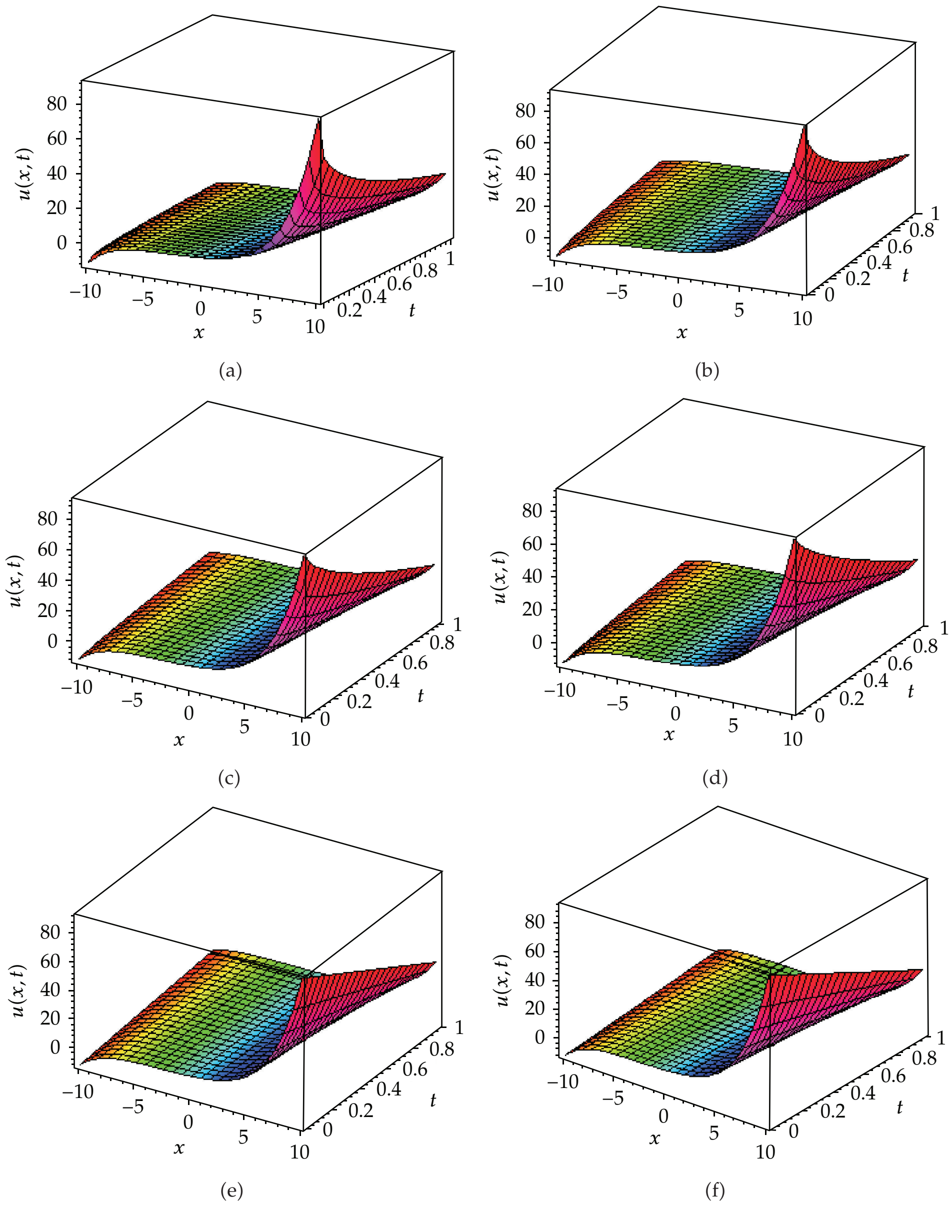

Figure 1: The surface shows the solution $u(x,t)$ for (1.1): (a) $\alpha = 1/3$, (b) $\alpha = 1/2$, (c) $\alpha = 2/3$, (d) $\alpha = 3/4$, (e) $\alpha = 1$, and (f) exact solution.

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