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Marianna Csörnyei,* Eötvös University, Department of Analysis, Múzeum Krt. 6-8, H-1088 Budapest, Hungary, e-mail: `csornyei@cs.elte.hu`

ON THE AFFINE SHARPNESS OF HEART'S DENSITY THEOREM

Abstract

Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint intervals tending to 0 from the right, such that putting $I_n = (a_n, b_n)$ we have $\liminf \frac{|I_n|}{b_n} = 0$. We prove that for every Cantor set C there exists some set X of positive measure, such that for a.e. $x \in X$ there exists a $c \in C$ for which $(x + cI_n) \cap X = \emptyset$ for infinitely many n.

V. Aversa and D. Preiss proved in [1] that there exists a sequence of pairwise disjoint positive intervals $\{I_n\}_{n=1}^{\infty}$ tending to 0, such that putting $I_n = (a_n, b_n)$ we have

$$\liminf \frac{|I_n|}{b_n} = 0, \tag{1}$$

but the differentiation system given by the translations of this system of intervals has the density property; that is, for every measurable set X we have

$$\frac{\lambda((x + I_n) \cap X)}{|I_n|} \to 1 \quad (\text{a.e.}\, x \in X).$$

In the same paper it is proved that the differentiation system given by the affine images of this system has the density property if and only if (1) does not hold. Actually their proof shows that for every sequence of pairwise disjoint positive intervals tending to 0 for which (1) holds, there exists some set X of positive measure, such that for a.e. $x \in X$ there exists a $c \in (0, 1)$ for which $(x + cI_n) \cap X = \emptyset$ for infinitely many n.

In this paper, applying entirely different methods, we prove that the same statement holds even if the multipliers c are taken only from a given Cantor set, rather than from $(0, 1)$.

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Remark. This implies immediately that if the multipliers c are taken from a Borel set B, then the statement holds if and only if B is uncountable.

D. Borwein and S. Z. Ditor proved in [2] that there exists a set X of positive measure and a sequence d_n of positive numbers tending to 0 such that $x + d_n \notin X$ for infinitely many n.

A stronger result is proved in [3]. It is proved that for a sequence of pairwise disjoint positive intervals $\{I_n\}_{n=1}^{\infty} = \{I_1^1, I_2^1, ..., I_{n_1}^1,\ I_1^2, I_2^2, ..., I_{n_2}^2, ...\}$ tending to 0, where $I_1^j, ..., I_{n_j}^j$ are of the same length and $\{n_j\}_{j=1}^{\infty}$ is not bounded, there exists a Cantor set C of positive measure such that for a.e. $x \in C$ the interval $x + I_n$ is disjoint from C for infinitely many n. As a generalization of this result now we prove the following lemma.

Lemma. *Let* $\{I_{n_1 n_2 \dots n_{k+1}} : n_1 = 1,\ 1 \le n_{k+1} \le m_{n_1 n_2 \dots n_k},\ k = 0, 1, ...\}$ *be a system of open intervals, such that*

(i) $I_{n_1 n_2 \dots n_k j}$ $(1 \le j \le m_{n_1 n_2 \dots n_k})$ *are pairwise disjoint intervals of the same length;*

(ii) in every sequence $I_{n_1}, I_{n_1 n_2}, I_{n_1 n_2 n_3}, ...$ *the intervals are pairwise disjoint and with lengths that tend to* 0*;*

(iii) For every interval $I_{n_1 n_2 \dots n_k}$ *we have* $\sup_{l_1 l_2 \dots} m_{n_1 n_2 \dots n_k l_1 l_2 \dots l_r} = \infty$.

Then there exists a Cantor set C *of positive measure such that for a.e.* $x \in C$ *there exists a sequence* $n_1, n_2, ...$ *such that for infinitely many* k *the intervals* $x + I_{n_1 n_2 \dots n_k}$ *are disjoint from* C.

PROOF. We construct our Cantor set C by a perfect scheme; that is, $C = \cap C_n$, where C_n is the union of non-overlapping closed intervals each "associated to" one of the intervals $I_{n_1 n_2 \dots n_k}$ of the same length, and $C_1 \supset C_2 \supset C_3 \supset ....$

For ease of notation we put $l_{n_1 n_2 \dots n_k} \stackrel{\text{def}}{=} |I_{n_1 n_2 \dots n_k}|$, and let $k_{n_1 n_2 \dots n_k}$ be the least positive integer for which $(0, k_{n_1 n_2 \dots n_k} \cdot l_{n_1 n_2 \dots n_k j})$ covers the union of intervals $I_{n_1 n_2 \dots n_k j}$ $(1 \le j \le m_{n_1 n_2 \dots n_k})$.

Let $C_1 = [0, l_1]$, and we say that this interval is associated to I_1. Suppose we have already defined C_{n-1}, and let J be one of its intervals, associated to the interval $I_{n_1 n_2 \dots n_k}$. We choose an interval $I_{n_1 n_2 \dots n_k l_1 l_2 \dots l_r}$, for which putting $\bar{n} \stackrel{\text{def}}{=} n_1 n_2 \dots n_k$, $\bar{l} \stackrel{\text{def}}{=} n_1 n_2 \dots n_k l_1 l_2 \dots l_r$ and $l^* \stackrel{\text{def}}{=} l_{\bar{l} j}$ we have

$$\frac{k_{\bar{l}} l^*}{l_{\bar{n}}} < \frac{1}{n^2} \tag{2}$$

and $m_{\bar{l}} > n^2$. This implies $k_{\bar{l}} > n^2$.

Let F be the set of intervals $((i-1)l^*, il^*)$ where i is a positive integer and this interval intersects some of the intervals $I_{\bar{l}j}$. Now, F has at least n^2 elements. We choose n elements, say with right endpoints $i_1 l^*, ..., i_n l^*$. Let $G \stackrel{\text{def}}{=} \{i_1, ..., i_n\}$, and by the greedy algorithm we choose integers $0 < m_1, ..., m_q \le k_{\bar{l}}$ for which the sets $G - m_i$ are pairwise disjoint (that is, we may choose $m_1 = 1$, let m_2 be the least integer for which $G - m_1$ and $G - m_2$ are disjoint, etc., etc.) Each set $G - m_i$ consists of n intervals. Thus $\bigcup_{i=1}^{q}(G - m_i)$ intersects $G - m$ for at most qn^2 different values of m. We stop the algorithm when $\frac{k_{\bar{l}}}{n^2} \le q \le \frac{k_{\bar{l}}}{n^2} + 1$. (We know $k_{\bar{l}} > n^2$).

Let

$$A \stackrel{\text{def}}{=} \bigcup_{i=m_p-2, m_p-1, m_p, m_p+1,\ 1 \le p \le q} ((i-1)l^*, il^*),$$

and let

$$B \stackrel{\text{def}}{=} \bigcup_{i=m_p - i_t, 1 \le p \le q, 1 \le t \le n} [(i-1)l^*, il^*] \setminus A.$$

Now, for every $((i-1)l^*, il^*) \subset B$ there exists an i_t such that

$$x + ((i_t - 2)l^*, (i_t + 1)l^*) \subset A;$$

that is, there is a j such that $x + I_{\bar{l}j} \subset A$.

The measure of A is "small", and the measure of B is "large":

$$\frac{k_{\bar{l}} l^*}{n^2} \le q l^* \le \lambda(A) \le 4 q l^* \le 8 \frac{k_{\bar{l}} l^*}{n^2}, \tag{3}$$

$$\lambda(B) \ge nql^* - \lambda(A) \ge \frac{k_{\bar{l}} l^*}{n} - 8 \frac{k_{\bar{l}} l^*}{n^2}, \tag{4}$$

and $A \subset (-2l^*, (k_{\bar{l}}+1)l^*) \subset (-2k_{\bar{l}}l^*, 2k_{\bar{l}}l^*)$, $B \subset [-k_{\bar{l}}l^*, k_{\bar{l}}l^*] \subset (-2k_{\bar{l}}l^*, 2k_{\bar{l}}l^*)$.

Now we construct C_n. For the interval $J = [a, b]$ associated to $I_{n_1 n_2 \ldots n_k}$ we consider the subintervals $[a + (4s-4)k_{\bar{l}}l^*, a + 4sk_{\bar{l}}l^*]$ $(s = 1, 2, ...)$, and we delete the remainder of the interval $(a + 4sk_{\bar{l}}l^*, b)$ of length less than $4k_{\bar{l}}l^*$, and we delete the sets $a + (4s-2)k_{\bar{l}}l^* + A \subset a + [(4s-4)k_{\bar{l}}l^*, a + 4sk_{\bar{l}}l^*]$, and possible isolated points. For every interval $I = a + (4s-2)k_{\bar{l}}l^* + ((i-1)l^*, il^*)$, where $((i-1)l^*, il^*) \subset B$, there is an interval $I_{\bar{l}j}$ for which $x + I_{\bar{l}j}$ lies in the deleted set $x + (4s-2)k_{\bar{l}}l^* + A$. The interval $a + (4s-2)k_{\bar{l}}l^* + [(i-1)l^*, il^*]$ is said to be associated to one of these intervals $I_{\bar{l}j}$, and the other non-deleted intervals of form $a + (4s-2)k_{\bar{l}}l^* + [(i-1)l^*, il^*]$ are said to be associated to an interval $I_{\bar{l}j}$ where j is chosen arbitrarily. This process yields a Cantor set C.

According to (2) and (3) the set C constructed is of positive measure, and according to (4) the set of points of C contained in a set of the form $a + (4s-2)k_{\bar{l}}l^* + B$ only finitely often is of measure 0. $\square$

Theorem. *For a sequence of pairwise disjoint intervals $\{J_n\}_{n=1}^{\infty}$ and a Cantor set E condition (1) implies that there exists a Cantor set C of positive measure, such that for a.e. $x \in C$ there exists a $c \in E$ for which $x + cJ_n$ is disjoint from C for infinitely many n.*

PROOF. Let $J_n = (a_n, b_n)$ satisfy (1); that is, $\liminf \frac{b_n - a_n}{b_n} = 0$. Let $E = \cap E_n$ be a Cantor set, where $E_1 \supset E_2 \supset ...$ are the closed sets of a perfect scheme defining E, each is the union of finitely many closed intervals. We construct a system of intervals $I_{n_1 n_2 \ldots n_k}$ satisfying the conditions of the lemma, together with another system of closed intervals $L_{n_1 n_2 \ldots n_k} \subset E_k$ for which $L_{n_1 n_2 \ldots n_k} \supset L_{n_1 n_2 \ldots n_k n_{k+1}} \supset ...$, $\mathrm{int} L_{n_1 n_2 \ldots n_k} \cap E \neq \emptyset$, and for every $n_1 n_2 \ldots n_k$ there exists an interval J_n of our sequence, such that for every $c \in L_{n_1 n_2 \ldots n_k}$ we have $cJ_n \subset I_{n_1 n_2 \ldots n_k}$. Having these systems of intervals defined the theorem easily follows because by the Lemma there exists a Cantor set C of positive measure such that for a.e. $x \in C$ there exist a branch $I_{n_1}, I_{n_1 n_2} ...$ and infinitely many k for which $C \cap (x + I_{n_1 n_2 \ldots n_k}) = \emptyset$. Hence for $c \in \bigcap_k L_{n_1 n_2 \ldots n_k} \subset \bigcap_k E_k = E$ we have $C \cap (x + cJ_n) = \emptyset$ for infinitely many n.

Let $L_1 = [c_1, d_1] \subset E_1$ such that $(c_1, d_1) \cap E \neq \emptyset$, and put $I_1 \stackrel{\text{def}}{=} (c_1 a_1, d_1 b_1)$. Suppose we have already defined the intervals $I_{n_1 n_2 \ldots n_k}$ and $L_{n_1 n_2 \ldots n_k} = [c, d]$, where $(c, d) \cap E \neq \emptyset$.

We choose $\delta > 0$ and also points $\lambda^1, \lambda^2, ..., \lambda^k \in (c, d) \cap E$ such that $\lambda^1 < \lambda^2 < ... < \lambda^k$, $(\lambda^j - \delta, \lambda^j) \subset E_k$, and for each λ^j and for every $\varepsilon > 0$ we have $(\lambda^j - \varepsilon, \lambda^j) \cap E \neq \emptyset$. We can choose $J_n = (a_n, b_n) = (a, b)$ such that $(0, db) \cap I_{n_1 n_2 \ldots n_k} = \emptyset$, and putting

$$b^j \stackrel{\text{def}}{=} \lambda^j b,\ a^j \stackrel{\text{def}}{=} \lambda^j b - d(b - a)$$

the intervals $[a^1, b^1], [a^2, b^2], ..., [a^k, b^k]$ are pairwise disjoint subintervals of (ca, db) of length $d(b - a)$, and $\frac{a^j}{a} > \lambda^j - \delta$. Indeed, we observe that the two conditions $ca < a^1$, $b^j < a^{j+1}$ and $\frac{a^j}{a} > \lambda^j - \delta$ hold if and only if $\frac{|J_n|}{b_n} < \min(\frac{\lambda^1 - c}{d - c}, \frac{\lambda^{j+1} - \lambda^j}{d}, \frac{d - \lambda^j + \delta}{\delta})$.

Let

$$I_{n_1 n_2 \ldots n_k j} \stackrel{\text{def}}{=} [a^j, b^j]\ (j = 1, 2, ..., m_{n_1 n_2 \ldots n_k} \stackrel{\text{def}}{=} k)$$

and

$$L_{n_1 n_2 \ldots n_k j} \stackrel{\text{def}}{=} [\frac{a^j}{a}, \frac{b^j}{b}].$$

We need

$$c < \frac{a^1}{a} < \frac{b^1}{b} < \frac{a^2}{a} < ... < \frac{b^k}{b} < d.$$

The inequalities $c < \frac{a^1}{a}$ and $\frac{b^j}{b} < d$ follow immediately from $ca < a^1$ and $\lambda^j \in (c,d)$. Since $\lambda^j = \frac{b_j}{b} < d$, the length of the interval $\frac{b^j}{b}[a,b] = [\frac{b^j}{b}a, b^j]$ is less than $d(b-a)$, that is, $a^j < \frac{b^j}{b}a$. Finally, $a < b$ and $b^{j-1} < a^j$ implies $\frac{b^{j-1}}{b} < \frac{a^j}{a}$.

Now, $L_{n_1 n_2 \ldots n_k j} = [\frac{a^j}{a}, \lambda^j] \subset [\lambda^j - \delta, \lambda^j] \subset E_k$ and $\operatorname{int} L_{n_1 n_2 \ldots n_k j} \cap E \neq \emptyset$. From the definition of $L_{n_1 n_2 \ldots n_k j}$ it follows that, for every $c \in L_{n_1 n_2 \ldots n_k j}$ we have $cJ_n = c(a,b) \subset I_{n_1 n_2 \ldots n_k j}$ and conditions (i)-(iii) are easily verified. $\square$

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