SEQUENCES OF COVERINGS

A. H. STONE

1. Introduction. The metrisable spaces S for which S' (the set of limit points of S) is compact, can be characterized as those uniformisable spaces for which the finest uniformity (compatible with the topology) is metrisable (see [5], [1], where further characterizations are given). B. T. Levshenko has shown [4] that they also coincide with the regular spaces in which every point-finite covering[1] can be refined by one of a fixed sequence of point-finite coverings, and that "point-finite" can be replaced throughout by "star-finite" or "locally finite". We shall extend these results (Theorem 2) and obtain an analogue for uniform spaces (Theorem 3). The proofs depend on a criterion for metrisability (Theorem 1) which may be of independent interest since, though not really new in content, it is particularly simple in form.

NOTATION. If $\mathscr{U}$ is a covering of a space S, and $A \subset S$, the star $St(A, \mathscr{U})$ of A in $\mathscr{U}$ is $\bigcup \{U \mid U \in \mathscr{U}, A \cap U \neq \phi\}$. When A is a 1-point set (x), we abbreviate $St((x), \mathscr{U})$ to $St(x, \mathscr{U})$. The covering by the sets $St(U, \mathscr{U})$, $U \in \mathscr{U}$, is denoted by $St(\mathscr{U})$. A covering $\mathscr{U}$ will be called "almost discrete" if only finitely many pairs U, V of sets of $\mathscr{U}$ intersect; such a covering is clearly star-finite (in fact star-bounded) and so locally finite.

2. Metrisation criterion.

THEOREM 1. *A necessary and sufficient condition that a T_0 space S be metrisable is that S have a sequence of coverings $\mathscr{U}_n$, $n = 1, 2, \cdots$, such that, for each $x \in S$, the stars $St(G, \mathscr{U}_n)$ of the open sets $G \ni x$ form a basis for the neighborhoods of x.*

The condition is trivially necessary. To prove it sufficient, we observe first that S is developable—i.e., the stars $St(x, \mathscr{U}_n)$ form a basis for the neighborhoods of each $x \in S$. It follows that S is T_1; for if x, y are distinct points of S, one of them, say x, has a neighborhood $St(x, \mathscr{U}_n)$ not containing y, and then $St(y, \mathscr{U}_n)$ does not contain x. We next show that S is collectionwise normal (see [2]). We may assume that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ (by replacing each $\mathscr{U}_n$ by the "intersection" of the coverings $\mathscr{U}_1, \cdots, \mathscr{U}_n$). Let A_λ $(\lambda \in \Lambda)$ be a discrete collection of closed subsets of S, and for each n and λ put

$$H_{n\lambda} = \bigcup \{U \mid U \in \mathscr{U}_n,\ St(U, \mathscr{U}_n) \text{ meets } A_\lambda \text{ but not } A_\mu \text{ if } \mu \neq \lambda\}\ ,$$

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[1] Throughout this paper, "covering" means "open covering."

Let $P_{n\lambda} = \bigcup \{H_{m\mu} \mid m \leqq n, \mu \neq \lambda\}$, $K_{n\lambda} = H_{n\lambda} - \overline{P}_{n\lambda}$, $H_\lambda = \bigcup \{H_{n\lambda} \mid n = 1, 2, \cdots\}$, $K_\lambda = \bigcup \{K_{n\lambda} \mid n = 1, 2, \cdots\}$; these sets are all open. It is easy to verify that $K_\lambda \cap K_\mu = \phi$ if $\lambda \neq \mu$, that $A_\lambda \subset H_\lambda$, and that $A_\lambda \cap \overline{P}_{n\lambda} = \phi$; hence $A_\lambda \subset K_\lambda$ where the sets K_λ are disjoint and open, as required.

As Bing has proved [**2**, Th. 10] that every developable collectionwise normal T_1 space is metrisable, the theorem follows. Alternatively Theorem 1 could be deduced from a general theorem of Nagata [**6**], or from a theorem of F. B. Jones [**3**].

3. THEOREM 2. *The following statements about a regular T_1 space S are equivalent*:

(1) *S is metrisable and S' is compact,*

(2) *S has a sequence of coverings $\mathscr{G}_n$ $(n = 1, 2, \cdots)$ such that each finite covering of S is refined by some $\mathscr{G}_n$,*

(3) *S has a sequence of almost discrete coverings $\mathscr{G}_n$ $(n = 1, 2, \cdots)$ such that each covering of S is refined by some $\mathscr{G}_n$.*

The implication (3) $\rightarrow$ (2) is trivial. To prove (2) $\rightarrow$ (1), we first show that, assuming (2), S is metrisable. Given $x \in U$ where U is open in S, there is an open set V such that $x \in V$ and $\overline{V} \subset U$. The finite covering $\mathscr{F} = \{V, U - (x), S - \overline{V}\}$ of S has a refinement $\mathscr{G}_n$, and $x \in$ some $G^0 \in \mathscr{G}_n$; then $G^0 \subset V$, the only set of $\mathscr{F}$ which contains x. If $G^1 \in \mathscr{G}_n$ and meets G^0, it follows that $G^1 \subset V \cup (U - (x)) = U$. Thus $St(G^0, \mathscr{G}_n) \subset U$, so Theorem 1 applies and S is metrisable. Let ρ be a metric for S; we construct another, σ, for which each $\mathscr{G}_n$ is uniform. We do this by successively constructing coverings $\mathscr{U}_1, \mathscr{U}_2, \cdots$, such that $St(\mathscr{U}_{n+1})$ refines $\mathscr{U}_n$, $\mathscr{U}_n$ refines $\mathscr{G}_n$, and $\mathscr{U}_n$ consists of sets of ρ-diameters $< 1/n$. By [**7**, p. 51] there is a corresponding pseudo-metric σ for which each $\mathscr{U}_n$, and so each $\mathscr{G}_n$, is uniform; and as $\sigma(x, y) = 0$ implies $\rho(x, y) = 0$ here, σ is a metric. Condition (2) shows that every finite covering of S is uniform in the metric σ; it follows ([**5**]; see also [**1**, Th. 1, (4) $\rightarrow$ (3)] that S' is compact (and *every* covering of S is uniform).

Finally, (1) $\rightarrow$ (3) by the argument in [**4**], which we sketch for completeness. For each $n = 1, 2, \cdots$, cover S' by a finite system of open sets G_{ni} $(i = 1, 2, \cdots, k_n)$ of diameters $< 1/n$, all meeting S', and adjoin the 1-point sets (x) for each $x \in S - \bigcup \{G_{ni} \mid i = 1, \cdots, k_n\}$ to produce an almost discrete covering $\mathscr{G}_n$ of S. It is easy to see that every covering $\mathscr{U}$ of S is refined by $\mathscr{G}_n$ when n is large enough.

REMARK. To require that S be separable, in (1), would be equivalent to requiring that the coverings $\mathscr{G}_n$ be countable, in (2) and (3).

THEOREM 3. *The following statements about a completely regular T_1 space S are equivalent*:

(1) *S is metrisable,*

(2) *S has a uniformity in which every finite uniform covering is refined by some member of a fixed sequence of (not necessarily uniform) coverings* $\mathscr{G}_n$ *of* S,

(3) *S has a uniformity in which every uniform covering is refined by some member of a fixed sequence of locally finite uniform coverings* $\mathscr{G}_n$ *of* S.

To prove $(1) \rightarrow (3)$, we use the fact that S is paracompact to take $\mathscr{G}_n =$ a locally finite refinement of the covering of S by "spheres" of radius $1/n$. As $(3) \rightarrow (2)$ trivially, it remains to deduce (1) from (2). Given a neighborhood N of $x \in S$, there exists a uniform covering $\mathscr{U}$ such that $St(x, \mathscr{U}) \subset N$, and there exist uniform coverings $\mathscr{V}, \mathscr{W}$ such that $St(\mathscr{V})$ refines $\mathscr{U}$ and $St(\mathscr{W})$ refines $\mathscr{V}$. Let $x \in W_0 \in \mathscr{W}$ and $St(W_0, \mathscr{W}) \subset V \in \mathscr{V}$. Write $X = St(W_0, \mathscr{W})$, $Y = \bigcup \{W \mid W \in \mathscr{W}, x \notin W, W \text{ meets } V\}$, $Z = \bigcup \{W \mid W \in \mathscr{W}, W \cap V = \phi\}$. Then $\mathscr{F} = \{X, Y, Z\}$, being refined by $\mathscr{W}$, is a uniform covering of S. Some $\mathscr{G}_n$ refines $\mathscr{F}$; say $x \in G^0 \in \mathscr{G}_n$. Because $X \cap Z = \phi$, it follows by an argument similar to one used in the proof of Theorem 2 that $St(G^0, \mathscr{G}_n) \subset X \cup Y \subset St(V, \mathscr{W}) \subset St(V, \mathscr{V}) \subset St(x, \mathscr{U}) \subset N$; hence S is metrisable, by Theorem 1.

REMARK. The uniformities in (2) and (3) of Theorem 3 will be different in general; that in (3) will be metrisable, while that in (2) need not be. By Theorem 2, not every uniformity on S can arise in (2) or (3) (unless S' is compact), but I have not found any satisfactory description of those which do.

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UNIVERSITY OF MANCHESTER,
MANCHESTER, ENGLAND